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ON SOME INTEGRAL INEQUALITIES RELATED WITH THE MULTICOMMODITY INVENTORY MODEL

The aim of this paper is to establish a relation among sets of solutions of some integral inequalities. The result allows us to reduce a minimization problem for a multicommodity inventory model to a similar problem for a one-commodity model, which is solved by Arrow and Karlin in [1] (see also Modigliani and Hohn [4], Blikle and Łoś [2]).

1. Theorem on integral inequalities. Let T be a fixed positive real number or ∞. All functions considered in this paper are defined on $[0, T)$ and take on real values. For the fixed function f let $N(f)$ denote the set of its discontinuity points.

Let $\boldsymbol{K}$ be the class of all functions satisfying the following conditions:

(K1) f is continuous on the left over $(0, T)$ and continuous on the right at 0.

(K2) f has a finite limit on the right at each point.

(K3) $N(f)$ is a nowhere-dense subset of $[0, T)$.

Condition (K1) implies that the set $N(f)$ is at most countable.

$\boldsymbol{K}$ is an algebra of functions with usual addition and multiplication.

By (K1) and (K2), a function from $\boldsymbol{K}$ is locally bounded, and so locally integrable.

Let r_1 and r_2 be fixed non-negative functions from $\boldsymbol{K}$. We examine sets of functions $u \in \boldsymbol{K}$ satisfying for all $t \in [0, T)$ the following systems (I) and (II) of the inequalities:

$$\int_0^t u_1(s)\,ds \geqslant \int_0^t r_1(s)\,ds, \tag{I.1}$$

$$\int_0^t u_2(s)\,ds \geqslant \int_0^t r_2(s)\,ds, \tag{I.2}$$

$$u_1(t) \geqslant 0, \tag{I.3}$$

$$u_2(t) \geqslant 0, \tag{I.4}$$

$$\text{(II.1)} \qquad \int_0^t u(s)\,ds \geqslant \int_0^t \big(r_1(s)+r_2(s)\big)\,ds,$$

$$\text{(II.2)} \qquad u(t) \geqslant 0.$$

THEOREM 1. (A) *If* $u_1^*, u_2^* \in \boldsymbol{K}$ *satisfy* (I), *then* $u^* = u_1^*+u_2^*$ *satisfies* (II) (*and belongs to* $\boldsymbol{K}$).

(B) *If* $u^* \in \boldsymbol{K}$ *satisfies* (II), *then there exist* $u_1^*, u_2^* \in \boldsymbol{K}$ *satisfying* (I) *and such that* $u^* = u_1^*+u_2^*$.

Proof. Result (A) is obvious. Result (B) is also obvious provided that for all $t \in [0, T)$

$$\text{(a)} \qquad u^*(t)-r_1(t) \geqslant 0,$$

or

$$\text{(b)} \qquad u^*(t)-r_2(t) \geqslant 0,$$

since then it is sufficient to put $u_1^* = r_1$, $u_2^* = u^*-r_1$ to (a), and $u_1^* = u^*-r_2$, $u_2^* = r_2$ to (b).

Therefore, we consider the case where neither (a) nor (b) is satisfied.

Let $u^* \in \boldsymbol{K}$ satisfy (II) and write

$$S = \{t \in [0, T):\ u^*(t) < r_1(t)\}.$$

Since (a) is assumed to be false, S is non-empty. Since u^*-r_1 is continuous on the left, the set S can be written as

$$S = \bigcup_{i\in I} \Delta(a_i, b_i),$$

where $\Delta(a_i, b_i)$ denotes the interval (open or one-sided closed) with ends a_i and b_i, $a_i < b_i$, the set I is at most countable, and intervals $\Delta(a_i, b_i)$ for $i \in I$ are disjoint. We assume that I is the set of positive integers or a segment of this set. By (II.1) it is clear that $a_i > 0$ for all $i \in I$. Let us write, for $t \in [0, T)$,

$$u_1^{(0)}(t) = r_1(t) \quad \text{and} \quad u_2^{(0)}(t) = u^*(t)-r_1(t).$$

It is obvious that $u_1^{(0)}$, $u_2^{(0)}$ satisfy (I.1)-(I.2), but they do not satisfy (I.4) for $t \in S$. The functions u_1^*, u_2^* satisfying (B) are obtained by improving $u_1^{(0)}$, $u_2^{(0)}$ step by step on intervals $\Delta(a_i, b_i)$ for $i = 1, 2, \ldots$ If the set I is finite, the good functions u_1^*, u_2^* can be obtained after consideration of all $\Delta(a_i, b_i)$ for $i \in I$. For infinite I, it is necessary to take the limit of the sequence of improved functions.

Consider the interval (a_1, b_1). Since

$$\int_{a_1}^{b_1} \big(u^*(s)-r_1(s)\big)\,ds < 0$$

and

$$\int_0^{b_1} \big(u^*(s)-r_1(s)\big)\,ds \geqslant \int_0^{b_1} r_2(s)\,ds \geqslant 0,$$

there exists a point $c_1 \in (0, a_1)$ such that

$$\int_{c_1}^{b_1} \big(u^*(s)-r_1(s)\big)\,ds = 0$$

and

$$\int_t^{b_1} \big(u^*(s)-r_1(s)\big)\,ds < 0 \qquad \text{for } c_1 < t < b_1.$$

Put

$$u_1^{(1)}(t) = \begin{cases} u^*(t) & \text{for } t \in (c_1, b_1], \\ r_1(t) = u_1^{(0)}(t) & \text{for } t \in [0, T)\setminus(c_1, b_1], \end{cases}$$

$$u_2^{(1)}(t) = \begin{cases} 0 & \text{for } t \in (c_1, b_1], \\ u^*(t)-r_1(t) = u_2^{(0)}(t) & \text{for } t \in [0, T)\setminus(c_1, b_1]. \end{cases}$$

We show that $u_1^{(1)}, u_2^{(1)}$ satisfy (I.1)-(I.3), but it is clear that $u_2^{(1)}$ satisfies (I.4) for $t \notin (a_i, b_i)$, $i = 2, 3, \ldots$ (thus $u_2^{(1)}$ is better than $u_2^{(0)}$ in the sense that it satisfies (I.4) over a larger set than $u_2^{(0)}$ does).

Verification of (I.1) and (I.3) is straightforward.

Consider inequality (I.2).

If $t \leqslant c_1$, then (I.2) is obvious. If $c_1 < t \leqslant b_1$, then

$$\int_0^t r_2(s)\,ds \leqslant \int_0^{b_1} r_2(s)\,ds \leqslant \int_0^{b_1} \big(u^*(s)-r_1(s)\big)\,ds$$
$$= \int_0^{c_1} \big(u^*(s)-r_1(s)\big)\,ds = \int_0^t u_2^{(1)}(s)\,ds.$$

Finally, if $t > b_1$, then

$$\int_0^t u_2^{(1)}(s)\,ds = \int_0^{c_1} \big(u^*(s)-r_1(s)\big)\,ds + \int_{b_1}^t \big(u^*(s)-r_1(s)\big)\,ds$$
$$= \int_0^t \big(u^*(s)-r_1(s)\big)\,ds \geqslant \int_0^t r_2(s)\,ds.$$

Notice also that for $t \in (c_1, b_1)$ we have

$$\int_{c_1}^t \big(u^*(s)-r_1(s)\big)\,ds = \int_{c_1}^{b_1} \big(u^*(s)-r_1(s)\big)\,ds - \int_t^{b_1} \big(u^*(s)-r_1(s)\big)\,ds$$
$$= -\int_t^{b_1} \big(u^*(s)-r_1(s)\big)\,ds \geqslant 0.$$

Hence it follows at once that if

$$(a_i, b_i)\cap(c_1, b_1) \neq \emptyset \quad \text{for some } i \in I,$$

then $(a_i, b_i) \subset (c_1, b_1)$. Let us suppose that we have constructed functions $u_1^{(k)}, u_2^{(k)} \in \boldsymbol{K}$ satisfying

$(\mathrm{i})_k$ $u^* = u_1^{(k)} + u_2^{(k)}$;

$(\mathrm{ii})_k$ $u_1^{(k)}$ satisfies (I.1) and (I.3), and $u_2^{(k)}$ satisfies (I.2);

$(\mathrm{iii})_k$ $u_2^{(k)}(t) \geqslant 0$ for $t \in [0, T)\setminus \bigcup_{i>k} (a_i, b_i)$;

$(\mathrm{iv})_k$ if $u_2^{(k)}(t_0) \geqslant 0$ for some $t_0 \in (a_i, b_i)$, then $u_2^{(k)}(t) \geqslant 0$ for all $t \in (a_i, b_i)$.

Such functions have been constructed above for $k = 1$. Suppose that $k > 1$ and consider the interval (a_{k+1}, b_{k+1}). If $u_2^{(k)}$ is non-negative over (a_{k+1}, b_{k+1}), we put $u_1^{(k+1)} = u_1^{(k)}$ and $u_2^{(k+1)} = u_2^{(k)}$. Thus conditions $(\mathrm{i})_{k+1}$-$(\mathrm{iv})_{k+1}$ are obviously satisfied. If $u_2^{(k)}$ is negative at a point from (a_{k+1}, b_{k+1}), then, by $(\mathrm{iv})_k$, it is negative over all (a_{k+1}, b_{k+1}). Thus

$$\int_{a_{k+1}}^{b_{k+1}} u_2^{(k)}(s)\,ds = \int_{a_{k+1}}^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds < 0.$$

On the other hand, in view of (I.2), we have

$$0 \leqslant \int_0^{b_{k+1}} u_2^{(k)}(s)\,ds = \int_0^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds$$

and, therefore, in a similar manner as for $k = 0$ we assert that there exists $c_{k+1} \in (0, b_{k+1})$ satisfying

$$(1) \qquad \int_{c_{k+1}}^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds = 0$$

and

$$\int_t^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds < 0 \quad \text{for } c_{k+1} < t < b_{k+1}.$$

Put

$$u_1^{(k+1)}(t) = \begin{cases} u^*(t) & \text{for } t \in (c_{k+1}, b_{k+1}], \\ u_1^k(t) & \text{for } t \in [0, T)\setminus(c_{k+1}, b_{k+1}], \end{cases}$$

$$u_2^{(k+1)}(t) = \begin{cases} 0 & \text{for } t \in (c_{k+1}, b_{k+1}], \\ u_2^k(t) & \text{for } t \in [0, T)\setminus(c_{k+1}, b_{k+1}]. \end{cases}$$

Clearly, $u_1^{(k+1)}, u_2^{(k+1)}$ belong to $\boldsymbol{K}$, satisfy $(\mathrm{i})_{k+1}$ and $u_1^{(k+1)} \geqslant 0$.
We want to check (I.1) and (I.2).
If $0 \leqslant t \leqslant c_{k+1}$, then $(\mathrm{ii})_k$ implies $(\mathrm{ii})_{k+1}$, since

$$u_1^{(k+1)}(t) = u_1^{(k)}(t) \quad \text{and} \quad u_2^{(k+1)} = u_2^{(k)}.$$

For $c_{k+1} < t \leqslant b_{k+1}$ we have

$$\begin{aligned}\int_0^t u_1^{(k+1)}(s)\,ds &= \int_0^{c_{k+1}} u_1^{(k)}(s)\,ds + \int_{c_{k+1}}^t u^*(s)\,ds \\ &= \int_0^t u_1^{(k)}(s)\,ds + \int_{c_{k+1}}^t \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &= \int_0^t u_1^{(k)}(s)\,ds - \int_t^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &\geqslant \int_0^t u_1^{(k)}(s)\,ds \geqslant \int_0^t r_1(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_0^t u_2^{(k+1)}(s)\,ds &= \int_0^{c_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &= \int_0^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds = \int_0^{b_{k+1}} u_2^{(k)}(s)\,ds \\ &\geqslant \int_0^{b_{k+1}} r_2(s)\,ds \geqslant \int_0^t r_2(s)\,ds.\end{aligned}$$

If $t > b_{k+1}$, then

$$\begin{aligned}\int_0^t u_1^{(k+1)}(s)\,ds &= \int_0^{c_{k+1}} u_1^{(k)}(s)\,ds + \int_{c_{k+1}}^{b_{k+1}} u^*(s)\,ds + \int_{b_{k+1}}^t u_1^{(k)}(s)\,ds \\ &= \int_0^t u_1^{(k)}(s)\,ds + \int_{c_{k+1}}^{b_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &= \int_0^t u_1^{(k)}(s)\,ds \geqslant \int_0^t r_1(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_0^t u_2^{(k+1)}(s)\,ds &= \int_0^{c_{k+1}} \big(u^*(s) - u_1^{(k)}(s)\big)\,ds + \int_{b_{k+1}}^t \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &= \int_0^t \big(u^*(s) - u_1^{(k)}(s)\big)\,ds \\ &= \int_0^t u_2^{(k)}(s)\,ds \geqslant \int_0^t r_2(s)\,ds.\end{aligned}$$

Thus we have proved $(\mathrm{ii})_{k+1}$.
Condition $(\mathrm{iii})_{k+1}$ is satisfied, since $u_2^{(k+1)} = 0$ over (a_{k+1}, b_{k+1}).

For the proof of $(\mathrm{iv})_{k+1}$ notice that there exists no $\varepsilon > 0$ such that

$$u_2^{(k)}(t) \leqslant 0 \quad \text{for } c_{k+1} < t < c_{k+1}+\varepsilon < b_{k+1}, \tag{2}$$

since, otherwise,

$$\int_{c_{k+1}}^{b_{k+1}} \big(u^*(s)-u_1^{(k)}(s)\big)\,ds = \int_{c_{k+1}}^{b_{k+1}} u_2^{(k)}(s)\,ds = \int_{c_{k+1}}^{c_{k+1}+\varepsilon} u_2^{(k)}(s)\,ds + \int_{c_{k+1}+\varepsilon}^{b_{k+1}} u_2^{(k)}(s)\,ds < 0,$$

which contradicts (1). Thus, by $(\mathrm{iv})_k$, for every $i \in I$ we have one of the following cases:

(α) $a_i < b_i < c_{k+1}$,

(β) $c_{k+1} < a_i < b_i \leqslant b_{k+1}$,

(γ) $b_{k+1} < a_i < b_i$.

In cases (α) and (γ), $(\mathrm{iv})_{k+1}$ is true by $(\mathrm{iv})_k$ (and by the equality $u_2^{(k+1)} = u_2^{(k)}$ over (a_i, b_i)). In case (β) we have $u_2^{(k+1)} = 0$ over (a_i, b_i), and thus $(\mathrm{iv})_{k+1}$ is also true.

Hence we have proved that for every $k \in I$ it is possible to construct functions $u_1^{(k)}, u_2^{(k)}$ satisfying $(\mathrm{i})_k$-$(\mathrm{iv})_k$. Since we showed that there exists no $\varepsilon > 0$ satisfying (2), we conclude now that $c_k \notin (c_i, b_i)$ for $i < k$. Thus, for a fixed $k \in I$, we have $u_2^{(i)}(t) = 0$ for $t \in (c_k, b_k)$, $i \geqslant k$. We also have the inclusions

$$N(u_2^{(i+1)}) \subset N(u_2^{(i)}) \cup \{c_{i+1}\} \cup \{b_{i+1}\}$$

and

$$N(u_2^{(0)}) \subset N(u^*) \cup N(r_1).$$

Let

$$Z = \bigcup_{i \in I} (c_i, b_i) \quad \text{and} \quad V = N(u^*) \cup N(r_1) \cup \mathrm{Fr}(Z).$$

Since Z is open, $\mathrm{Fr}(Z)$ is nowhere dense (cf. [3], Chapter 1, § 8), and so is V. Put

$$u_2^*(t) = \begin{cases} 0 & \text{for } t \in Z \setminus \overline{V}, \\ u^*(t) - r_1(t) & \text{for } t \in [0, T) \setminus (Z \cup \overline{V}). \end{cases}$$

(For $t \in \overline{V}$, u_2^* will be defined later.)

The set $[0, T) \setminus \overline{V}$ is open and dense in $[0, T)$. Therefore, it is the countable sum of open intervals. Since u_2^* is over each such interval equal to $u^* - r_1$ or to 0, it is continuous and has both one-sided limits at the ends. We claim that u_2^* has both one-sided limits at every point $t_0 \in \overline{V}$. To prove this we need only to show that for any monotonic sequence $\{t_n\} \subset [0, T) \setminus \overline{V}$ converging to t_0 the sequence $\{u_2^*(t_n)\}$ is convergent. But the convergence of the last sequence is not sure only if $\{t_n\}$ has two infinite subsequences $\{t_n'\}$ and $\{t_n''\}$ satisfying

1° $t'_n \in Z \setminus \overline{V}$,

2° $t''_n \in [0, T) \setminus (Z \cup \overline{V})$,

3° $u_2^*(t''_n) > 0$ and $\limsup u_2^*(t_n) = \limsup u_2^*(t''_n)$,

4° $|t_0 - t''_{n-1}| > |t_0 - t'_n| > |t_0 - t''_n| > |t_0 - t'_{n+1}|$.

Condition 4° means that the sequences $\{t'_n\}$ and $\{t''_n\}$ interlace one with another, and if we have sequences satisfying 1°-3°, then by induction it is easy to construct from them subsequences satisfying 4°. By 4°, there exists a sequence $\{i_n\} \subset I$ such that

$$|t_0 - t'_n| < |t_0 - b_{i_n}| < |t_0 - t''_n|.$$

Now we obtain

$$\begin{aligned} 0 \leqslant \liminf u_2^*(t_n) &\leqslant \limsup u_2^*(t_n) = \limsup u_2^*(t''_n) \\ &= \lim\big(u^*(t''_n) - r_1(t''_n)\big) = \lim\big(u^*(b_{i_n}) - r_1(b_{i_n})\big) \leqslant 0. \end{aligned}$$

Thus the sequence $\{u_2^*(t_n)\}$ is convergent to 0, and so we have proved that at every point $t_0 \in \overline{V}$ the function u_2^* has both one-sided limits equal to 0. Hence u_2^* may be extended, in the unique manner, to the functions continuous on the left defined over the whole interval $[0, T)$. This extended function will also be denoted by u_2^*. Since $N(u_2^*) \subset \overline{V}$, we have $u_2^* \in \boldsymbol{K}$. Put $u_1^* = u^* - u_2^*$.

For the proof of assertion (B) we have to check that u_1^* and u_2^* satisfy (I).

Since $S \subset Z$, we obtain $u_1^* \geqslant 0$ and $u_2^* \geqslant 0$. If the set I is finite, $I = \{1, 2, \ldots, k\}$, then $u_1^* = u_1^{(k)}$, $u_2^* = u_2^{(k)}$ and, by (ii)$_k$ and (iii)$_k$, system (I) is satisfied. If I is infinite, then for $t \in [0, T) \setminus V$ we have

$$u_2^*(t) = \lim_k u_2^{(k)}(t) \quad \text{and} \quad u_1^*(t) = \lim_k u_1^*(t).$$

Thus

$$\begin{aligned} \int_0^t u_i^*(s)\,ds &= \int_0^t \lim u_i^{(k)}(s)\,ds = \lim \int_0^t u_i^{(k)}(s)\,ds \\ &\geqslant \lim \int_0^t r_i(s)\,ds = \int_0^t r_i(s)\,ds \quad \text{for } i = 1, 2,\ t \in [0, T) \end{aligned}$$

(the interchange of the limit and of the integral is possible, since $0 \leqslant |u_i^{(k)}(s)| \leqslant u^*(s) + r_1(s) \leqslant \text{const}$ for $s \in [0, t]$). The proof is complete.

2. Applications. We now apply Theorem 1 to a minimization problem connected with a multicommodity inventory model.

Let $c\colon [0, T) \to R$ be a given non-negative, increasing and strictly convex function. Next, let $a_1, \ldots, a_n$ and $h_1, \ldots, h_n$ be positive, and $y_1, \ldots, y_n$ non-negative real numbers. Finally, let $\bar{r}_1, \ldots, \bar{r}_n$ be non-negative functions belonging to $C^1 \subset \boldsymbol{K}$.

Consider the following problem:

(P1) Minimize

$$I[u_1, \ldots, u_n] = \int\limits_0^T \Big\{c\Big(\sum_{i=1}^n a_i u_i(t)\Big) + \sum_{i=1}^n h_i\Big[y_i + \int\limits_0^t \big(u_i(s) - \bar{r}_i(s)\big)ds\Big]\Big\}dt$$

under the constraints

$$u_i \in \boldsymbol{K} \quad \text{for } i = 1, \ldots, n,$$

$$y_i + \int\limits_0^t \big(u_i(s) - \bar{r}_i(s)\big)ds \geqslant 0 \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n,$$

$$u_i(t) \geqslant 0 \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n.$$

The function u_i may be considered as the production intensity of the i-th commodity for which the demand intensity is $\bar{r}_i$, and the marginal cost of holding a unit of the commodity for a unit of time is h_i. Then $I[u_1, \ldots, u_n]$ is the global (production and inventory) cost incurred during the time period $[0, T)$ under the condition that the demand is met at any time. Problem (P1) for $n = 1$ has been solved by Arrow and Karlin in [1].

In this paper we show that, by Theorem 1, the problem for $n > 1$ may be reduced to the case of $n = 1$ provided that

$$\frac{h_1}{a_1} = \ldots = \frac{h_n}{a_n} = h \geqslant 0. \tag{3}$$

Notice first that, without loss of generality of (P1), we may always assume $a_1 = \ldots = a_n = 1$. Then assumption (3) takes the form

$$h_1 = \ldots = h_n = h. \tag{4}$$

Let

$$t_i = \sup\Big\{t \in [0, T):\ \int\limits_0^t \bar{r}_i(s)ds < y_i\Big\}$$

(here we assume that $\sup\emptyset = 0$) and

$$r_i(t) = \begin{cases} 0 & \text{for } t \in [0, t_i], \\ \bar{r}_i(t) & \text{for } t \in (t_i, T). \end{cases}$$

It is easy to check that, since (4) holds, problem (P1) may be rewritten as follows:

(P2) Minimize

$$I[u_1, \ldots, u_n] = \bar{I}[u] = \int\limits_0^T \Big[c\big(u(t)\big) + h\int\limits_0^t \Big(u(s) - \sum_{i=1}^n r_i(s)\Big)ds\Big]dt,$$

where $u = u_1 + \ldots + u_n$, under the constraints

$$u_i \in \boldsymbol{K} \quad \text{for } i = 1, \ldots, n,$$

$$\int_0^t u_i(s)\,ds \geqslant \int_0^t r_i(s)\,ds \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n,$$

$$u_i(t) \geqslant 0 \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n.$$

Consider the following auxiliary problem:

(P3) Minimize $\bar{I}[u]$ under the constraints $u \in \boldsymbol{K}$ and

$$\int_0^t u(s)\,ds \geqslant \int_0^t \big(r_1(s) + \ldots + r_n(s)\big)\,ds \quad \text{for all } t \in [0, T),$$

$$u(t) \geqslant 0 \quad \text{for all } t \in [0, T).$$

Let $u^* \in \boldsymbol{K}$ be the solution of (P3) (see [1]). Using Theorem 1 $n-1$ times we can present u^* in the form

$$u^* = \sum_{i=1}^{n} u_i^*,$$

where $u_i^* \in \boldsymbol{K}$ for $i = 1, \ldots, n$, and

$$\int_0^t u_i^*(s)\,ds \geqslant \int_0^t r_i(s)\,ds \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n,$$

$$u_i^*(t) \geqslant 0 \quad \text{for all } t \in [0, T),\ i = 1, \ldots, n.$$

It is evident that the vector function $[u_1^*, \ldots, u_n^*]$ is the solution of problem (P2).

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O PEWNYCH NIERÓWNOŚCIACH CAŁKOWYCH ZWIĄZANYCH Z WIELOTOWAROWYM MODELEM ZAPASÓW

STRESZCZENIE

W pracy zbadano zależności między rozwiązaniami dwu układów nierówności całkowych (twierdzenie 1). Następnie zależności te zastosowano do rozwiązania problemu minimalizacji kosztów produkcji i magazynowania w n-towarowym ciągłym modelu zapasów. Problem ten, w jednotowarowym przypadku, jest dokładnie opisany i rozwiązany w [1].