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# THE EFFECT OF RAPID ROTATION ON THE OVERSTABLE MODE OF CONVECTION IN A VISCOELASTIC FLUID LAYER

**0. Summary.** The purpose of the present paper is to derive analytic asymptotic expressions for the critical Rayleigh numbers ($R_c$), wave numbers ($a_c$) and frequencies ($\sigma_c$) for the onset of convection as overstability in an uniformly rotating *viscoelastic* fluid layer, bounded by two free surfaces, in the limit of large Taylor numbers ($T$). The results obtained using these expressions are in excellent agreement with the numerical results given in [2]. The analytic expression for the critical Rayleigh number so obtained, further confirms the destabilizing effect of rotation on the overstable mode of convection in a *viscoelastic* fluid ayer [2] in contrast with its stabilizing effect on a *viscous* fluid.

**1. Introduction.** When a horizontal layer of a *viscous* fluid is heated from below, the system remains stable to small disturbances for values of the Rayleigh number $R$ (defined later) smaller than a critical value $R_c$. When $R > R_c$, the system becomes unstable and convection sets in the form of a regular cellular pattern. The critical value $R_c$ depends, of course, on the boundary conditions. This problem of thermal instability in a horizontal layer of a viscous fluid was first studied by Bénard [1], experimentally. Theoretically it was first studied by Rayleigh [22] and later by Jeffreys [14] and Pellew and Southwell [21]. A comprehensive account of the contributions of many authors who have subsequently studied this problem has been given in the monograph by Chandrasekhar [10].

The effect of a magnetic field on the onset of convection in a viscous fluid layer was studied theoretically by Chandrasekhar [6] and experimentally by Nakagawa (see [15] and [16]). The effect of rotation on this stability problem was investigated theoretically by Chandrasekhar [7], Chandrasekhar and Elbert [11] and experimentally by Nakagawa and Frenzen [19] and Fultz, Nakagawa and Frenzen [12]. The simultaneous effect of rotation and magnetic field has also been investigated theoretically by Chandrasekhar (see [8] and [9]) and experimentally by Nakagawa (see [17] and [18]). It has been found in all these cases that in certain ranges of the governing parameters the fluid layer becomes *overstable*, that is, instability sets in via oscillations of increasing amplitude. In the greater part of the range overstability takes place before steady convection. Overstability is possible in the presence of constraining effects of rotation and/or a magnetic field because these effects give an elastic-like behaviour to the fluid enabling it thereby to sustain appropriate modes of wave propagation. It is, therefore, expected that if we study the problem of stability in a *viscoelastic* fluid layer, it can become overstable due to heating from below alone.

Recently, Vest and Arpaci [23] have investigated the stability of a horizontal layer of a viscoelastic fluid heated from below. They have evaluated the conditions under which thermally induced overstability occurs in a Maxwell fluid. They found that elasticity has a destabilizing influence both in the sense that oscillatory convection can occur at a lower critical Rayleigh number than stationary convection, and that $R_c$ for overstability decreases with the increase in elasticity. Green [13] has carried out another study of overstability in a viscoelastic fluid layer heated from below. Analysing the case of a two time constant model due to Oldroyd [20], he has discussed the problem for the case of two free boundaries. In both these investigations the effect of rotation or magnetic field has *not* been included.

Since both rotation and/or magnetic field impart an elastic-like behaviour to the fluid, it would, therefore, be of interest to include the effect of rotation and/or magnetic field on the stability of a viscoelastic fluid layer heated from below. Moreover, the *simultaneous* effect of a uniform rotation and a uniform magnetic field on the onset of thermal instability in a viscoelastic fluid layer is of particular interest in geomagnetism and geophysics.

In an attempt to solve this rather complicated problem the author has considered together with P. K. Bhatia the separate effects of rotation and magnetic field on the overstable mode of convection in a viscoelastic fluid layer (Maxwell fluid) bounded by two (i) free, (ii) rigid isothermal and non-deformable boundaries.

We have shown that, for both types of boundaries, the effect of rotation is destabilizing [2] while the effect of the magnetic field is stabilizing (see [3]-[5]). The effect of elasticity was found to be destabilizing for both cases. Two *new* results were thus found:

(a) rotation has a destabilizing influence, in contrast with its stabilizing influence on an ordinary viscous fluid;

(b) rotation and magnetic field have conflicting influences, in contrast with their stabilizing influence on an oridnary viscous fluid.

These results were obtained by numerical computation for a wide range of parameters. Clearly, as it is impossible to perform the computation for all the values of the rotation or magnetic field parameters, it is necessary to derive analytic asymptotic expressions in the limit as these parameters tend to infinity.

Consequently, the subject matter of this paper is to derive the *asymptotic behaviour of the critical constants for the onset of overstability in the very rapid rotation limit.* The corresponding problem for the magnetic field will be considered in the near future.

**2. Formulation.** Consider an infinite horizontal layer of a viscoelastic fluid of depth $d$ rotating uniformly about a vertical axis with angular velocity $\Omega$. Suppose that the fluid is heated uniformly from below. Assume that the viscoelastic nature of the fluid is described by the Maxwell constitutive relation [23]

$$p_{ij} + t_0 \frac{d}{dt} p_{ij} = \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

where $p_{ij}$ is the viscous stress tensor, $v_i(u, v, w)$ is the velocity vector, $t_0$ is the Maxwell relaxation time, and $\mu$ is the coefficient of viscosity.

Following [2], the linearized perturbation equations governing small perturbations are

$$(D^2 - a^2 - P\sigma)\Theta = -\left(\frac{\beta d^2}{\varkappa}\right) W,$$

$$(1 + \Gamma\sigma)\left[\sigma Z - \left(\frac{2\Omega d}{\nu}\right) DW\right] = (D^2 - a^2) Z,$$

$$(D^2 - a^2)[D^2 - a^2 - \sigma(1 + \Gamma\sigma)]W - \left(\frac{2\Omega d^3}{\nu}\right)(1 + \Gamma\sigma) DZ = (1 + \Gamma\sigma) a^2 \left(\frac{g\alpha d^2}{\nu}\right)\Theta. \tag{1}$$

Here $D \equiv d/dz$ ($z$-axis taken vertically upwards), $P = \nu/\varkappa$ is the Prandtl number, $\Gamma = t_0 \nu / d^2$ is the elastic parameter, $\sigma$ is growth rate, $a$ is the wave number, $\Theta$ is the perturbation in temperature, $W$ is the $z$-component of $v_i$, $Z$ is the $z$-component of vorticity, while $g$, $\alpha$, $\beta$, $\varkappa$ and $\nu$ denote the gravity, coefficient of volume expansion, adverse temperature gradient, thermometric conductivity and kinematic viscosity respectively. Further, we have scaled the physical variables using $d$, $d^2/\nu$, $\nu/d$ and $\beta d$ as length, time, velocity and temperature scale factors, respectively.

**3. Asymptotic solution for large Taylor number.** As pointed out earlier, we suppose that the fluid is confined between two free, isothermal and non-deformable boundaries. Hence the boundary conditions to be satisfied are [2]

$$W = D^2W = \Theta = DZ = 0 \quad \text{at } z = 0, 1. \tag{2}$$

Applying the operator

$$(D^2 - a^2 - P\sigma)[D^2 - a^2 - \sigma(1+\Gamma\sigma)]$$

to equation (1), we can eliminate $\Theta$ and $Z$ to obtain an equation in $W$,

$$(D^2 - a^2 - P\sigma)[\{D^2 - a^2 - \sigma(1+\Gamma\sigma)\}^2(D^2 - a^2) + T(1+\Gamma\sigma)^2D^2]W = -Ra^2(1+\Gamma\sigma)[D^2 - a^2 - \sigma(1+\Gamma\sigma)]W, \tag{3}$$

where $R$ is the *Rayleigh number* defined by

$$R = \frac{g\alpha\beta d^4}{\varkappa\nu},$$

and $T$ is the *Taylor number* given by

$$T = \frac{4\Omega^2}{\nu^2}d^4.$$

Equation (3) together with the boundary conditions (2) constitutes an eigenvalue equation of the eighth order. Furthermore, a close examination of the equations reveals that also $D^4W = D^6W = D^8W = 0$. By differentiating equation (3) an even number of times, we can conclude that all the even order derivatives of $W$ must vanish for both $z = 0, 1$. These considerations suggest that the proper solutions for $W$ belonging to the lowest mode must be

$$W = \text{const} \cdot \sin \pi z.$$

Substituting this solution for $W$ in (3), we obtain the characteristic equation

$$(\pi^2 + a^2 + P\sigma)[\{\pi^2 + a^2 + \sigma(1+\Gamma\sigma)\}^2(\pi^2 + a^2) + \pi^2(1+\Gamma\sigma)^2T] = Ra^2(1+\Gamma\sigma)[\pi^2 + a^2 + \sigma(1+\Gamma\sigma)]. \tag{4}$$

Remembering that $\sigma$ can be complex in this equation, we make the transformations $\sigma = i\sigma_1$, $x = a^2$ and $b = \pi^2 + a^2$. It is clear from equation (4) that, for an arbitrary $\sigma_1$, $R$ will be complex. But, from physical considerations, $R$ (Rayleigh number) must be real. Therefore, the condition that $R$ be real implies a relation between the real and imaginary parts of $\sigma_1$. But as we are interested in specifying the critical Rayleigh number for the onset of instability via a state of purely oscillatory motion, we suppose that $\sigma_1$ is real in the above-mentioned equations and try to obtain the conditions for such solutions to exist.

Assuming, then, that $\sigma_1$ is real, equation (4) can be separated into real and imaginary parts, both of which must vanish separately. This leads after some simplifications to the pair of equations

$$(5) \qquad R = \frac{b[b^2 + (\Gamma b - 1)P\sigma_1^2 - P\Gamma^2\sigma_1^4]}{x(1+\Gamma^2\sigma_1^2)} + \frac{\pi^2 T[b^2 + (1-\Gamma b)P\sigma_1^2 + P\Gamma^2\sigma_1^4]}{x[b^2 + (1-2\Gamma b)\sigma_1^2 + \Gamma^2\sigma_1^4]},$$

and the cubic equation in $\sigma_1^2$,

$$(6) \qquad A_0\sigma_1^6 + A_1\sigma_1^4 + A_2\sigma_1^2 + A_3 = 0,$$

where

$$\begin{aligned}
A_0 &= \Gamma^4 b,\\
A_1 &= -\Gamma^2 b(3\Gamma b - P - 2) - \pi^2\Gamma^4 T,\\
A_2 &= b(3\Gamma^2 b^2 - 2\Gamma P b - 3\Gamma b + P + 1) + \pi^2\Gamma^2(\Gamma b + P - 2)T,\\
A_3 &= -b^3(\Gamma b - P - 1) + \pi^2(\Gamma b + P - 1)T.
\end{aligned}$$

We now derive the asymptotic behaviour as $T \to \infty$. In this limit the roots of equation (6) can be represented as

$$(7) \qquad (\sigma_1^2)_i = T\left[(\sigma_1^2)_{i0} + \frac{(\sigma_1^2)_{i1}}{T} + \frac{(\sigma_1^2)_{i2}}{T^2} + \dots\right] \quad (i = 1, 2, 3),$$

where

$$(8) \quad \left\{\begin{aligned}
&(\sigma_1^2)_{10} = \frac{\pi^2}{b}, \quad (\sigma_1^2)_{20} = 0, \quad (\sigma_1^2)_{30} = 0,\\
&(\sigma_1^2)_{11} = \frac{2}{\Gamma^2}(\Gamma b - P), \quad (\sigma_1^2)_{21} = \frac{\Gamma b + P - 1}{\Gamma^2}, \quad (\sigma_1^2)_{31} = -\frac{1}{\Gamma^2},\\
&(\sigma_1^2)_{12} = -\frac{b}{\pi^2\Gamma^4}[(\Gamma b - P)^2 + 2(\Gamma b - P) + P^2],\\
&(\sigma_1^2)_{22} = \frac{2bP(\Gamma b + P^2 - P)}{\pi^2\Gamma^4(\Gamma b + P)},\\
&(\sigma_1^2)_{32} = \frac{b^2(\Gamma b - P)(\Gamma b + 2)}{\pi^2\Gamma^3(\Gamma b + P)}.
\end{aligned}\right.$$

Substituting for $(\sigma_1^2)_i$ from (7) and (8) into (5), we get, after some lengthy calculations,

$$R_i = T\left(R_{i0} + \frac{R_{i1}}{T} + \frac{R_{i2}}{T^2} + \dots\right) \quad (i = 1, 2, 3), \tag{9}$$

where

$$\begin{cases} R_{10} = 0, \quad R_{20} = \dfrac{\pi^2}{x}(\Gamma b + P), \quad R_{30} = \dfrac{2\pi^2(\Gamma b + P)}{x(\Gamma b + 2)}, \\ R_{11} = \dfrac{2P^2 b}{\Gamma^2 x}, \quad R_{21} = \dfrac{b[(\Gamma b + P)^2 - P^2(\Gamma b + P) - 2P^2\Gamma b]}{\Gamma^2(\Gamma b + P)x}, \\ R_{12} = \dfrac{2b^2}{\pi^2\Gamma^2 x}\left[b^2 + \dfrac{P^2(P-1)}{\Gamma^2}\right]. \end{cases} \tag{10}$$

It is clearly seen from equations (9) and (10) that, in the limit as $T \to \infty$, the solution $R_1$ corresponding to $(\sigma_1^2)_1$ has the lowest value. Consequently, the required asymptotic behaviour of $R$ and $\sigma_1^2$ are given by the equations

$$R = \frac{2P^2 b}{\Gamma^2 x} + \frac{2b^2}{\pi^2\Gamma^2 x}\left[b^2 + \frac{P^2(P-1)}{\Gamma^2}\right]T^{-1} + O(T^{-2}) \tag{11}$$

and

$$\sigma_1^2 = \frac{\pi^2}{b}T + \frac{2}{\Gamma^2}(\Gamma b - P) + O(T^{-1}). \tag{12}$$

The critical wave number $(a_c)$ can now be obtained by minimizing $R$ with respect to $x$. The required condition $dR/dx = 0$ becomes, after some simplifications,

$$x^4 + \frac{8}{3}\pi^2 x^3 + \left[2\pi^4 + \frac{P^2(P-1)}{3\Gamma^2}\right]x^2 - \frac{\pi^4}{3}\left[\pi^4 + \frac{P^2(P-1)}{\Gamma^2}\right] - \pi^4\left(\frac{P^2 T}{3}\right) = 0,$$

from which it follows that, for sufficiently large $T$, the critical wave number has the following asymptotic behaviour:

$$a_c^2 = \pi\left(\frac{P^2 T}{3}\right)^{1/4}\left[1 - \frac{2}{3}\pi\left(\frac{P^2 T}{3}\right)^{-1/4} - \right.$$
$$\left. - \frac{1}{12}\left\{-2\pi^2 + \frac{P^2(P-1)}{\pi^2\Gamma^2}\right\}\left(\frac{P^2 T}{3}\right)^{-1/2} + O(T^{-3/4})\right]. \tag{13}$$

Substituting for $x_c$ ($x_c = a_c^2$) from (13) into (11) and (12), we finally obtain

$$(14) \qquad R_c = \frac{2P^2}{\Gamma^2}\left[1 + \frac{4}{3}\pi\left(\frac{P^2T}{3}\right)^{-1/4} + \frac{4}{3}\pi^2\left(\frac{P^2T}{3}\right)^{-1/2} + O(T^{-3/4})\right]$$

and

$$(15) \qquad (\sigma_1^2)_c = \frac{3\pi}{P^2}\left(\frac{P^2T}{3}\right)^{3/4}\left[1 - \frac{1}{3}\pi\left(\frac{P^2T}{3}\right)^{-1/4} + \right.$$

$$\left. + \frac{1}{36}\left\{-2\pi^2 + \frac{24P^2}{\Gamma} + \frac{3P^2(P-3)}{\pi^2\Gamma^2}\right\}\left(\frac{P^2T}{3}\right)^{-1/2} + O(T^{-3/4})\right].$$

Equations (13), (14) and (15) are thus the required analytic asymptotic expressions giving the critical wave number ($a_c$), Rayleigh number ($R_c$) and frequency $(\sigma_1)_c$ for the onset of convection as overstability in the limit $T \to \infty$.

Finally we conclude with the following remarks:

(i) For large enough values of $T$, the critical constants for the onset of overstability obtained from these asymptotic expressions are in *excellent* agreement with the results obtained by the exact solution [2], illustrating thereby the correctness of both solutions. This agreement is typically illustrated for a particular set of results corresponding to $P = 100$, $\Gamma = 1$ and a few values of $T$ in the following table:

| $T$ | Exact solution | | | Asymptotic solution | | |
|---|---|---|---|---|---|---|
| | $a_c$ | $\log_{10}(\sigma_1)_c$ | $\log_{10} R_c$ | $a_c$ | $\log_{10}(\sigma_1)_c$ | $\log_{10} R_c$ |
| $10^6$ | 25.936 | 2.0966 | 4.3115 | 25.279 | 2.1072 | 4.3086 |
| $10^7$ | 35.962 | 2.4464 | 4.3058 | 35.698 | 2.4497 | 4.3053 |
| $10^8$ | 48.593 | 2.8120 | 4.3035 | 48.435 | 2.8134 | 4.3034 |
| $10^9$ | 65.015 | 3.1844 | 4.3024 | 64.953 | 3.1848 | 4.3024 |

(ii) It is apparent from equation (14) that the critical Rayleigh number ($R_c$) decreases as the Taylor number ($T$) increases showing thereby the destabilizing influence of rotation on the overstable mode of convection in a *viscoelastic* fluid layer. This result is in contrast with its stabilizing influence on a *viscous* fluid [10].

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## WPŁYW SZYBKIEGO OBROTU NA NADSTABILNĄ SKŁADOWĄ KONWEKCJI W WARSTWIE CIECZY LEPKOSPRĘŻYSTEJ

STRESZCZENIE

Celem niniejszej pracy jest wyprowadzenie wzorów asymptotycznych (dla dużych liczb Taylora $T$) na krytyczne wartości liczby Rayleigha $R_c$, liczby falowej $a_c$ i częstości $\sigma_c$ w przypadku nadstabilnej konwekcji w jednostajnie obracającej się lepkosprężystej warstwie cieczy, ograniczonej dwiema powierzchniami swobodnymi. Wartości otrzymane z tych wzorów świetnie zgadzają się z wynikami numerycznymi [2]. Znalezione wyrażenie dla krytycznej wartości liczby Rayleigha potwierdza również destabilizujący wpływ obrotu na stabilną składową konwekcji w warstwie cieczy lepkosprężystej [2], w przeciwieństwie do jego stabilizującego wpływu na ciecz lepką.