ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVI, 2 (1978)

J. GRABOWSKI (Wrocław)

FORMULATION AND SOLUTION OF THE SEQUENCING PROBLEM WITH PARALLEL MACHINES

1. INTRODUCTION

In many branches of industry (metallurgy, machine construction) the production process is characterized by a flow of elements in the technological sequence. These elements are processed on successive machines. The processing times of individual operations are different for different elements and machines. Therefore, the problem arises to determine such a sequence of operations that the total time of performing all operations is minimal.

In this paper this problem with parallel machines is presented and solved. We assume that the duration of operations may be different for different machines from the set of parallel machines. Therefore, the problem considered is more general than those examined in the papers [4], [5], [8], and [9]. The problem in which the set of parallel machines contains exactly one machine was presented in the papers [1]-[3], [5], and [7].

The mathematical model of the sequencing problem is formulated by using disjunctive graphs. The notions from [6] are generalized and some new ones are introduced. A number of properties are proved which allow us to construct a new, relatively effective algorithm. These problems may be modelled and solved by using zero-one linear programming [8]. The efficiency of algorithms which solve the zero-one programming problems is very low, so one can solve only sequencing problems which are of small size. Besides, to construct a zero-one programming model, the considered period time must be divided into a certain number of time periods. The greater the number of these periods, the better the accuracy of the solution. On the other hand, a greater number of periods gives a greater size of problems, so it is difficult to obtain a solution. For the models which are presented in the papers [4] and [5], in each iteration of the algorithm the coefficient of internal stability has to be computed, so it increases the computation time. The mathematical models of sequencing problems which are constructed by using disjunctive graphs have not these weaknesses.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let $N = \{1, 2, \ldots, n\}$ be the set of operations (number of operations) which should be carried out by using the set of various machine types $B = \{1, 2, \ldots, b\}$, and let $Q = \{1, 2, \ldots, q\}$ be the set of these types. Further, let $B^k \subset B$ be the set of machines of type $k \in Q$, $|B^k| = r_k$, and let the following relations be satisfied:

$$\bigcup_{k\in Q} B^k = B, \quad B^k \cap B^l = \emptyset \quad (k, l \in Q,\ k \neq l).$$

Let $N^k \subset N$ be the set of operations which should be carried out by using machines B^k of type $k \in Q$, and let the following relations hold:

$$\bigcup_{k\in Q} N^k = N, \quad N^k \cap N^l = \emptyset \quad (k, l \in Q,\ k \neq l).$$

Assume that each operation $j \in N^k$ is carried out by using exactly one machine $p \in B^k$, $k \in Q$. We assume also that the durations of individual operations are fixed and different for different machines. Therefore, the problem arises to determine the allocation of machines to the individual operations and to determine such a sequence of operations that the total time of all operations is minimal.

Let $RT \subset N \times N$ be the set of relations expressing the technological requirements of the operation order. Thus $\langle i, j\rangle \in RT$ means that the operation i must be carried out prior to the operation j. Let $N_x \subset N$ be the set of operations which have no predecessors,

$$N_x = \{j \in N \mid \forall i \in N \wedge \langle i, j\rangle \notin RT\},$$

and let $N_y \subset N$ be the set of operations which have no successors,

$$N_y = \{j \in N \mid \forall i \in N \wedge \langle j, i\rangle \notin RT\}.$$

Let us introduce the following notation:

$$x_{jp} = \begin{cases} 1 & \text{if the operation } j \text{ is carried out by using the machine } p, \\ 0 & \text{otherwise;} \end{cases}$$

t_j^x — the starting time of the operation j;

t_j^y — the finishing time of the operation j;

c_{jp} — the duration of the operation j which is carried out by using the machine p ($c_{jp} > 0$, $p \in B^k$, $j \in N^k$, $k \in Q$);

$t_0 = 0$ and t_z — the starting and the finishing times of all operations, respectively.

We formulate the sequencing problem as follows:

Find t_z, t_j^x, t_j^y, x_{jp} $(p \in B^k,\ j \in N^k,\ k \in Q)$ satisfying

$$t_z = \min, \tag{1}$$

$$t_j^y - t_j^x \geqslant \sum_{p=1}^{r_k} x_{jp} c_{jp} \quad (j \in N^k,\ k \in Q), \tag{2}$$

$$t_j^x - t_i^y \geqslant 0 \quad (\langle i, j\rangle \in RT), \tag{3}$$

$$t_j^x - t_0 \geqslant 0 \quad (j \in N_x), \tag{4}$$

$$t_z - t_j^y \geqslant 0 \quad (j \in N_y), \tag{5}$$

$$t_z, t_j^y, t_j^x \geqslant 0 \quad (j \in N), \tag{6}$$

$$(x_{jp} = 1) \wedge (x_{ip} = 1) \Rightarrow (t_j^x - t_i^y \geqslant 0) \vee (t_i^x - t_j^y \geqslant 0) \quad (p \in B^k, i, j \in N^k,\ i \neq j,\ k \in Q), \tag{7}$$

$$\sum_{p=1}^{r_k} x_{jp} = 1 \quad (j \in N^k,\ k \in Q), \tag{8}$$

$$x_{jp} \in \{0, 1\} \quad (p \in B^k,\ j \in N^k,\ k \in Q). \tag{9}$$

Conditions (1)-(9) constitute the problem which is called ***Problem P***. Constraint (2) requires that the difference between the starting and finishing times of a given operation which is carried out by using a given machine is not less than the duration of this operation (see constraint (8)). Condition (3) gives the required technological order of operations, and (4) and (5) state the condition that t_0 and t_z are the starting and finishing times, respectively, of all operations. Two different operations $j, i \in N^k$ which are to be carried out by using the same machine cannot be carried out at the same time, but one of them must be finished before the second is started. This is expressed by constraint (7) which is called the *implicit condition*. Constraint (8) ensures that each operation can be carried out by using exactly one machine.

Problem P is known as a mixed integer programming problem with the logical condition (7). Let $(\bar{t}, \bar{x})$ be a feasible solution to P, where $(\bar{t})$ is the vector of variables t_j^x, t_j^y, t_z $(j \in N)$, and $(\bar{x})$ is the vector of variables x_{jp} $(p \in B^k,\ k \in Q)$.

Let us consider a particular problem for which $B^k = \{1\}$, $r_k = 1$, $j \in N^k$, $k \in Q$. Then there exists exactly one machine of each type. Since $r_k = 1$ for each $k \in Q$, it follows from (8) that every variable is equal to 1, and instead of constraint (2) we obtain

$$t_j^y - t_j^x \geqslant c_{j1} = c_j \quad (j \in N^k,\ k \in Q). \tag{10}$$

Notice that the predecessor of the implicit condition (7) for each pair of operations $i, j \in N^k$ $(i \neq j,\ k \in Q)$ is satisfied. Therefore, the successor of this condition must also be satisfied, so we have

(11) $$(t_j^x - t_i^y \geqslant 0) \vee (t_i^x - t_j^y \geqslant 0) \quad (i, j \in N^k,\ i \neq j,\ k \in Q).$$

Constraints (11) are disjunctions each of which requires that two operations are not carried out on the same machine at the same time. Conditions (1), (3)-(6), (10), and (11) constitute the problem which is called *Problem* P_r. This problem has been formulated in [6] and [7]. In this formulation we have a disjunctive graph

$$\bar{D} = \langle A, U; V \rangle,$$

where

(12) $$D = \langle A, U \rangle$$

is the graph.

The set of nodes equals

$$A = X \cup Y \cup \{0\} \cup \{z\},$$

where

$$X = \bigcup_{j \in N} \{x_j\}, \qquad Y = \bigcup_{j \in N} \{y_j\},$$

and the nodes $x_j, y_j, 0, z$ are associated with the variables t_j^x, t_j^y, t_0, t_z, respectively.

The set of arcs equals

(13) $$U = U_1 \cup U_2 \cup U_3 \cup U_4,$$

where

$$U_1 = \bigcup_{j \in N} \{\langle x_j, y_j \rangle\}, \qquad U_2 = \bigcup_{\langle i,j \rangle \in RT} \{\langle y_i, x_j \rangle\},$$

$$U_3 = \bigcup_{j \in N} \{\langle 0, x_j \rangle\}, \qquad U_4 = \bigcup_{j \in N} \{\langle x_j, z \rangle\},$$

and the arcs $\langle x_j, y_j \rangle$, $\langle y_i, x_j \rangle$, $\langle 0, x_j \rangle$, $\langle y_j, z \rangle$ are associated with constraints (10), (3), (4), (5), respectively. The set V contains all disjunctive arcs of the form $[\langle y_j, x_i \rangle, \langle y_i, x_j \rangle]$ which are associated with constraints (11). An example of the disjunctive graph for $N^1 = \{1, 4, 6\}$, $N^2 = \{2, 5, 7\}$, $N^3 = \{3, 8\}$ is shown in Fig. 1, where the disjunctive arcs are plotted by dashed lines.

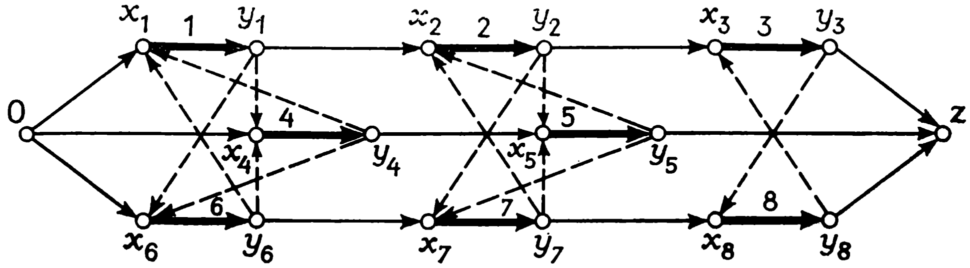

Fig. 1

The disjunctive graph $\bar{D}$ has the following properties:

PROPERTY 1. The graph $D = \langle A, U\rangle$ has the source node 0 and the sink node z and for each $x \in A - [\{0\} \cup \{z\}]$ there exists a path from the node 0 to the node x and from x to z. The graph D has no circuits.

PROPERTY 2. (a) In D there exist two subsets of nodes $X \subset A$ and $Y \subset A$ such that

$$\Gamma(X) = Y, \quad \Gamma^{-1}(Y) = X, \tag{14}$$

$$\forall_{x\in X} \exists!_{y\in Y}(y = \Gamma x) \wedge \forall_{y\in Y} \exists!_{x\in X}(x = \Gamma^{-1}y),$$

and each arc $\langle x, y\rangle$ $(x \in X, y \in Y)$ has the length $c(x, y) > 0$.

Condition (14) states that each node of the set X has exactly one successor in D, this successor belonging to Y, and that each node of the set Y has exactly one predecessor in D, this predecessor belonging to X.

(b) There exists a set of indices Q generating partions of X and Y such that

$$\bigcup_{k\in Q} X^k = X, \quad \bigcup_{k\in Q} Y^k = Y,$$

$$X^k \cap X^l = \emptyset, \quad Y^k \cup Y^l = \emptyset \quad (k, l \in Q, k \neq l), \quad \Gamma X^k = Y^k \quad (k \in Q).$$

It follows from condition (14) that $\Gamma^{-1}(Y^k) = X^k$ for $k \in Q$.

PROPERTY 3. The set V of disjunctive arcs with zero length is

$$V = \bigcup_{k\in Q} V^k,$$

$$V^k = \{\langle y, x\rangle \in Y^k \times X^k \mid [\mathrm{War}(y, x) = 0 \vee \mathrm{War}(x, y) = 0]\}, \tag{15}$$

where $\mathrm{War}(x, y)$ is the statement function with arguments $x, y \in A$ defined by

$$\mathrm{War}(x, y) = (\text{there exists a path from the node } x \text{ to } y \text{ in the graph } D).$$

Now we return to Problem P. It follows from (9) that we can replace constraint (8) by

$$(x_{j1} = 1) \dot{\vee} (x_{j2} = 1) \ \dot{\vee} \ \ldots \ \dot{\vee} \ (x_{jr_k} = 1) \quad (j \in N^k, k \in Q). \tag{16}$$

Hence, for each variable x_{jp} which is equal to 1, the variables x_{js} $(s = 1, 2, \ldots, r_k;\ s \neq p)$ are equal to 0 and constraint (2) takes the form

$$t_j^y - t_j^x \geqslant c_{jp} \quad (j \in N^k, p \in B^k, k \in Q). \tag{17}$$

Therefore, to each variable which is equal to 1 we can assign exactly one constraint of form (17), whereas to each constraint (16) we can assign the constraint of the disjunctive form

$$(t_j^y - t_j^x \geqslant c_{j1}) \dot{\vee} (t_j^y - t_j^x \geqslant c_{j2}) \dot{\vee} \ \ldots \ \dot{\vee} (t_j^y - t_j^x \geqslant c_{jr_k}) \quad (j \in N^k, \ k \in Q). \tag{18}$$

Further, to each constraint of the disjunctive form (18) we can assign the set of disjunctive arcs in $\bar{D}$,

(19) $$[\langle x_j, y_j\rangle_1 \dot{\vee} \langle x_j, y_j\rangle_2 \dot{\vee} \ldots \dot{\vee} \langle x_j, y_j\rangle_{r_k}] \quad (j \in N^k,\ k \in Q),$$

where $\langle x_j, y_j\rangle_p$ has the length c_{jp}, whereas to each constraint (7) we can assign the implicit condition of disjunctive arcs:

(20) $$[\langle x_j, y_j\rangle_p \wedge \langle x_i, y_i\rangle_p] \Rightarrow (\langle y_i, x_j\rangle \vee \langle y_j, x_i\rangle) \quad (i, j \in N^k,\ i \neq j,\ k \in Q).$$

Let

$$U^0 = U_2 \cup U_3 \cup U_4 = U - U_1$$

be the set of arcs of form (13) of the graph $D = \langle A, U\rangle$ without the set of arcs U_1, and let

$$D^0 = \langle A, U^0\rangle$$

be the partial graph of D. The set of arcs U_1 is the set of operations in Problem P_r. In Problem P, the set of operations is determined by a certain subset of disjunctive arcs of form (19). Let V^0 denote the set of all those arcs. Then the disjunctive graph in Problem P can be determined by

$$\bar{D}^0 = \langle A, U^0; V; V^0\rangle.$$

An example of the disjunctive graph from Fig. 1 for which $|B^1| = 3$, $|B^2| = 2$, and $|B^3| = 1$ is shown in Fig. 2.

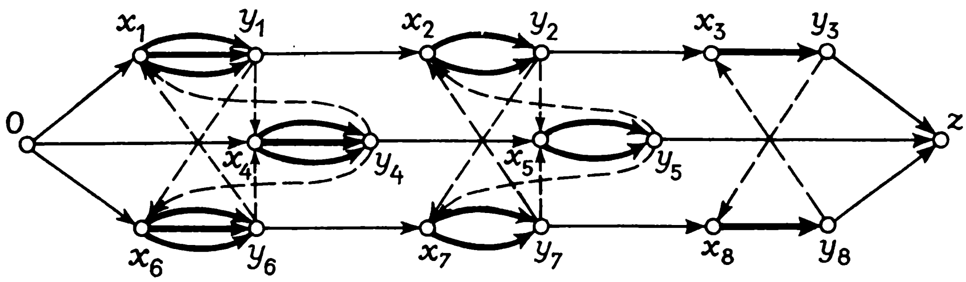

Fig. 2

The graph $\bar{D}^0$ has the following additional property:

PROPERTY 4. The set V^0 of disjunctive arcs is

(21) $$V^0 = \bigcup_{k \in Q} \bigcup_{j \in N^k} V^{jk},$$

where

(22) $$V^{jk} = \bigcup_{p \in B^k} \{\langle x_j, y_j\rangle_p \in Y \times Y \mid y_j = \Gamma x_j \text{ in the graph } D\} \quad (j \in N^k,\ k \in Q),$$

and each arc $\langle x_j, y_j\rangle_p$ has the length $c_{jp} > 0$.

It follows from Property 3 and from condition (14) that for each arc $\langle y, x\rangle \in V$ there exists exactly one arc $\langle u, v\rangle \in V$ such that

(23) $$(u = \Gamma x) \wedge (v = \Gamma^{-1} y).$$

Two disjunctive arcs $[\langle y, x\rangle, \langle u, v\rangle]$ which satisfy condition (23) are called a *disjunctive pair*. Any arc from a disjunctive pair is called the *complement* of the other arc of this pair. Replacement of an arc by its complement is called *complementing*.

The subset of the set V containing at most one arc from each disjunctive pair is called a *selection*. A selection containing exactly one arc from each pair is called *complete*.

Let $R_s = \{S_1, S_2, \ldots, S_p\}$ be the family of all selections (not necessarily complete). It follows from Property 4 that for each arc $\langle y, x\rangle \in V^0$ there exist exactly $r'_k = |r^k| - 1 = r_k - 1$ arcs $\langle u, v\rangle \in V^0$ such that

$$(u = x) \wedge (v = y), \tag{24}$$

and these arcs may have different lengths. All disjunctive arcs (the number of r_k) which satisfy condition (24) are called a *disjunctive set*. Any arc from a disjunctive set is called a *cousin* of every other arc of this set. Replacement of an arc by its cousin is also called *complementing*.

The subset of V^0 containing at most one arc from each disjunctive set is called a *selection of sets*. A selection of sets containing exactly one arc from each disjunctive set is called *complete*.

Let $R_s^0 = \{S_1^0, S_2^0, \ldots, S_p^0\}$ be the family of all complete selections of sets.

It follows from conditions (15) and (22) that for each arc $\langle y, x\rangle \in V$ there exist two disjunctive sets V^{jk} and V^{ik} $(j \neq i)$ such that for every arc $\langle a, b\rangle \in V^{jk}$ and for every arc $\langle c, d\rangle \in V^{ik}$ the following condition is satisfied:

$$(b = y) \wedge (c = x).$$

The sets V^{jk} and V^{ik} (and the arcs from V^{jk} and V^{ik}) are called *adjacent sets* (and *adjacent arcs*) to the disjunctive arc $\langle y, x\rangle \in V$.

We write the arcs from the set V in the form $\langle y_j, x_i\rangle$ $(j, i \in N)$ (see (20)), whereas those from the set V^0 in the form $\langle x_j, y_j\rangle$ $(j \in N)$ (see (19) and (21)). The arcs $\langle x_j, y_j\rangle$ and $\langle x_i, y_i\rangle$ are adjacent to the disjunctive arcs $\langle y_j, x_i\rangle$.

Each selection $S_p^0 \in R_s^0$ generates a conjunctive graph

$$D_p^0 = \langle A, U^0 \cup S_p^0\rangle. \tag{25}$$

Let

$$R_D^0 = \{D_p^0 = \langle A, U^0 \cup S_p^0\rangle\}$$

be the family of graphs of form (25). It can easily be seen that the structure of each graph $D_p^0 \in R_D^0$ is identical with that of the graph of form (12). Besides, for each selection $S_p^0 \in R_s^0$ we have $U_1 = S_p^0$ and each graph of the family R_D^0 has Properties 1 and 2. If we assume $|B^k| = 1$ for each $k \in Q$, then $|R_D^0| = 1$ and the graph $D_p^0 \in R_D^0$ is of form (12) for Problem P_r.

It can easily be seen that each graph $D_p^0 \in R_D^0$ has no circuits. Each selection $S_r \in R_s$ and each selection of sets $S_p^0 \in R_s^0$ generates a conjunctive graph

(26) $$D_{rp}^0 = \langle A, U^0 \cup S_r \cup S_p^0 \rangle .$$

Let

$$R_{Dd}^0 = \{D_{rp}^0 = \langle A, U^0 \cup S_r \cup S_p^0 \rangle\}$$

be the family of graphs of form (26), and let

$$R_{Dd}^{0\prime} = \{D_{rp}^0 \in R_{Dd}^0 \mid D_{rp}^0 \text{ has no circuits}\}$$

be the family of graphs without circuits. Further, let

$$R_{Ddi}^{0\prime} = \{D_{rp}^0 \in R_{Dd}^{0\prime} \mid D_{rp}^0 \text{ satisfies condition (20)}\}$$

be the family of graphs for which the implicit condition (20) is satisfied.

The critical path of the graph $D_{00}^0 \in R_{Ddi}^{0\prime}$ is called a *minimaximal path with the implicit condition* (20) in the disjunctive graph $\bar{D}^0 = \langle A, U^0, V; V^0 \rangle$, and the associated selection S_0 and the selection of sets S_0^0 are called *optimal* if

(27) $$L_{00} = \min_{D_{rp}^0 \in R_{Ddi}^{0\prime}} L_{rp},$$

where L_{rp} is the length of the critical path in D_{rp}^0. Since the graph D_{rp}^0 has no circuits, the critical path exists in D_{rp}^0.

THEOREM 1. *Problem* P *is equivalent to the problem of finding the minimaximal path* L_{00} (27) *with the implicit condition* (20) *and optimal selections* S_0 *and* S_0^0 *in the disjunctive graph* $\bar{D}^0 = \langle A, U^0, V; V^0 \rangle$. *The length* L_{00} *of the critical path* C_{00} *in* D_{00}^0 *is the optimal value of* t_z, *while the optimal values of the vector* $(\bar{t})$ *can be obtained by applying the critical path method to the graph* D_{00}^0.

Proof. Let $(\bar{x}_r)$ be a certain feasible solution to Problem P. Let us denote by Z_{rp} the problem (1)-(6), (8), (9) which is obtained if $(\bar{x}) = (\bar{x}_r)$ and if to this problem we add at most one constraint from each disjunctive pair of the successor of the implicit condition (7). Let $Z = \{Z_{11}, Z_{21}, \ldots, Z_{m1}, \ldots, Z_{ml}\}$ be the set of all problems Z_{rp} for every feasible solution of the vector $(\bar{x})$, which can be obtained from Problem P. A graph $D_{rp}^0 \in R_{Dd}^0$ corresponds to a problem $Z_{rp} \in Z$. The graph D_{rp}^0 having circuits or for which the implicit condition (20) is not satisfied corresponds to the problem $Z_{rp} \in Z$ with no feasible solution. The conditions set by the theorem for the graph D_{rp}^0 translate, therefore, the requirement of finding, among all problems $Z_{rp} \in Z$, a problem Z_{00} having a feasible solution and such that its optimal solution is minimal over the set of solutions to all problems $Z_{rp} \in Z$. But this is exactly what solving the Problem P amounts to.

Similarly as Theorem 1 in [6], we can prove the following

THEOREM 2. *Let C_{rp} be the set of arcs of the critical path in $D^0_{rp} \in R^{0\prime}_{Dd}$. Then any graph D^0_{sp} obtained by complementing any arc $\langle y, x\rangle \in S_r \cap C_{rp}$ has no circuits.*

For any $D^0_{rp} \in R^{0\prime}_{Dd}$ and any node $x \in A$ we adopt some notation. The longest path from the node 0 to any node $x \in A - \{0\}$ is

$$L_{rp}(0, x) = \max_{y \in \Gamma^{-1}x} [L_{rp}(0, y) + c(y, x)]. \tag{28}$$

The longest path from any node $x \in A - \{z\}$ to the node z is

$$L_{rp}(x, z) = \max_{y \in \Gamma x} [L_r(y, z) + c(x, y)]. \tag{28a}$$

Let x_i and x_j be nodes for which the right-hand sides of (28) and (28a) take their maximal values. For $\Gamma^{-1}x - \{x_i\} \neq \emptyset$ let

$$L'_{rp}(0, x) = \max_{y \in \Gamma^{-1}x - \{x_i\}} [L_{rp}(0, y) + c(y, x)],$$

and for $\Gamma x - \{x_j\} \neq \emptyset$ let

$$L'_{rp}(x, z) = \max_{y \in \Gamma x - \{x_j\}} [L_{rp}(y, z) + c(x, y)].$$

$L'_{rp}(0, x)$ is the longest path from the node 0 to the node x which does not contain the arc $\langle x_i, x\rangle$, and $L'_{rp}(x, z)$ is the longest path from the node x to the node z which does not contain the arc $\langle x, x_j\rangle$.

Let

$$\begin{aligned} \alpha_{rp}(0, x) &= L_{rp}(0, x) - L'_{rp}(0, x) \quad \text{for } x \in X, \\ \beta_{rp}(y, z) &= L_{rp}(y, z) - L'_{rp}(y, z) \quad \text{for } y \in Y. \end{aligned} \tag{29}$$

Now we write

$$\begin{aligned} &\Delta_{rp}[(y, x), (u, v)] \\ &= \max[-\alpha_{rp}(0, x), -\beta_{rp}(y, z), c(x, u) + c(v, y) - \beta_{rp}(y, z) - \alpha_{rp}(0, x)]. \end{aligned} \tag{30}$$

Similarly as Theorem 2 in [6] we can prove the following

THEOREM 3. *Let $D^0_{rp} \in R^{0\prime}_{Dd}$ and let D^0_{sp} be the graph obtained from D^0_{rp} by complementing one arc $\langle y, x\rangle \in S_r \cap C_{rp}$, where C_{rp} is the longest path in D^0_{rp}. Then*

$$L_{sp}(0, z) \geqslant L_{rp}(0, z) + \Delta_{rp}[(y, x), (u, v)].$$

Therefore, by complementing the arc $\langle y, x\rangle \in C_{rp} \cap S_r$, the lower bound of the longest path C_{sp} in D^0_{sp} is $L_{rp}(0, z) + \Delta_{rp}[(y, x), (u, v)]$.

3. ALGORITHM

The minimaximal path with the implicit condition (20) of the disjunctive graph $\bar{D}^0$ is obtained by generating a sequence of circuit graphs $D^0_{rp} \in R^{0'}_{Did}$, finding the critical path for each D^0_{rp}, and testing its feasibility.

Let S^0_1 be a complete initial selection of sets such that $S^0_1 \in R^0_s$. It can easily be seen that the graph $D^0_1 = \langle A, U^0 \cup S^0_1\rangle$ has no circuits. Further, let S_1 be a complete initial selection such that $S_1 \in R_s$ and the graph $D^0_{11} = \langle A, U^0 \cup S_1 \cup S^0_1\rangle$ has no circuits. The disjunctive arcs $\langle a, b\rangle \in V$ and $\langle c, d\rangle \in V^0$ are called *normal* if $\langle a, b\rangle \in S_1$ and $\langle c, d\rangle \in S^0_1$. The disjunctive arcs $\langle a, b\rangle \in V - S_1$ and $\langle c, d\rangle \in V^0 - S^0_1$ are called *reverse*. Each arc $\langle y, x\rangle \in S^0_1$ has $r_k - 1$ reverse arcs in the set $V^0 - S^0_1$. Moreover, we assume that

$$(31) \quad \forall [i \in N^k \wedge \langle x_i, y_i\rangle \in S^0_1]\{\forall \langle x, y\rangle \in V^{jk}\ [c(x, y) \geqslant c(x_i, y_i)]\}, \quad k \in Q.$$

Starting with the graph $D^0_{11} = \langle A, U^0 \cup S_1 \cup S^0_1\rangle$, we generate a sequence of graphs $D^0_{rp} = \langle A, U^0 \cup S_r \cup S^0_p\rangle \in R^{0'}_{Ddi}$ $(S_r \in R_s, S^0_p \in R^0_s)$.

For the graph D^0_{rp}, the implicit condition (20) must be satisfied. Hence the selection S_r is not necessarily complete. Thus, for each pair of disjunctive arcs of the selection S_r one of the following possibilities holds:

(a) it is a normal arc,

(b) it is a reverse arc,

(c) it is neither a normal nor a reverse arc from a disjunctive pair.

If (c) holds, we have an *empty arc* which is denoted by $\overline{\langle y, x\rangle}$ or $\overline{\langle u, v\rangle}$. Replacement of an arc by an empty arc is called *eliminating*. The set of empty arcs $\bar{S}_r$ such that $S_r \cup \bar{S}_r$ is a complete selection is called an *eliminated selection*.

Each graph D^0_{sq} is obtained from a certain graph D^0_{rq} or D^0_{sp} of the sequence by complementing (or eliminating) one normal arc from the selection S_r or by complementing one normal arc from the selection of sets S^0_p. Each arc from S^0_p is complemented on $r_k - 1$ cousins. In case where the graph D^0_{sq} is obtained by eliminating arcs from the selection S_r, the implicit condition (20) must be satisfied for this graph. The process of generating is presented in the form of a solution tree H. Each node in H corresponds to a graph D^0_{rp}. Each node D^0_{sp} or D^0_{rq} is obtained from D^0_{rp} by complementing (or eliminating) one disjunctive arc from $\langle y, x\rangle \in S_r$ or $\langle y, x\rangle \in S^0_p$. Then the arc $\langle D^0_{rp}, D^0_{sp}\rangle$ in H represents the complement $\langle u, v\rangle \in S_s$ (or an empty arc $\overline{\langle y, x\rangle} \in \bar{S}_s$) of $\langle y, x\rangle \in S_r$, and the arc

$\langle D^0_{rp}, D^0_{rq}\rangle$ in H represents the complement $\langle u, v\rangle \in S^0_q$ of $\langle y, x\rangle \in S^0_p$. We say that D^0_{rp} is the *predecessor* of D^0_{sp} or D^0_{rq} (and D^0_{sp} is the *successor* of D^0_{rp}) if there is a path in H between D^0_{rp} and D^0_{sp} or D^0_{rq}. The initial graph $D^0_{11} = \langle A, U \cup S_1 \cup S^0_1\rangle$ is the root in the solution tree H. The generation of a new branch in H is connected with the choice of a certain normal arc for complementing (or eliminating) from S_r or for complementing from S^0_p. This choice is called the *operation of choice*. For each graph D^0_{rp} from the sequence we perform:

(a) an operation of testing to check the critical path and the possibility of generation of a graph $D^0_{sq} \in R'_{Ddi}$ with a critical path smaller than that already found;

(b) an operation of testing to check the implicit condition (20).

If the result of testing (a) is negative, we abandon the considered graph D^0_{sp} or D^0_{rq} and backtrack the tree H to the predecessor D^0_{rp} from which the graph D^0_{sp} or D^0_{rq} was generated. Since the process of generating the graph D^0_{rp} is started from the complete selection S_1, a negative result of checking the implicit condition (20) may appear if we perform the operation of eliminating an arc from S_r. If a new graph D^0_{sp} is obtained from D^0_{rp} by complementing (or eliminating) a normal arc $\langle y, x\rangle \in S_r$, we temporarily fix a reverse arc $\langle u, v\rangle \in S_s$ (or an empty arc $\overline{\langle u, v\rangle} \in \bar{S}_r$) in D^0_{sp}. This arc cannot be complemented (or eliminated) in any successor D^0_{sp} in H. However, if we backtrack the first time a reverse arc (or an empty arc) of a certain normal arc to D^0_{rp}, we momentarily fix this reverse arc (or an empty arc). If we backtrack the second time, we constantly fix this normal arc. The normal arc such that its reverse arc (or an empty arc) is momentarily fixed can be complemented (or eliminated) if we need to perform the second operation of eliminating (or complementing) which has not been performed yet for this normal arc. So, for each graph D^0_{rp}, we temporarily or constantly fix a subset $F_r \subset S_r$ and momentarily fix a set F^t_r of disjunctive arcs. The reverse arcs in F_r are temporarily fixed and represent the path from the root to D^0_{rp} in H. Each normal arc in F_r is constantly fixed and represents two arcs (reverse and empty) which have been abandoned during the backtracking process. Each momentarily fixed arc in F^t_r is a reverse (or empty) arc which has been abandoned during the backtracking process. The set F^t_r contains all arcs belonging to the path from the root to D^0_{rp} in H. No arc from the set F_r can be complemented and eliminated in any successors D^0_{tu} of D^0_{rp}. If a new graph D^0_{rq} is obtained from the graph D^0_{tp} by complementing a normal arc $\langle y, x\rangle \in S^0_p$, we temporarily fix a reverse arc $\langle u, v\rangle \in S^0_q$ in D^0_{rq}. This arc cannot be complemented in any successors of D^0_{rq} in H. However, if we backtrack the n-th time (where $n < r'_k = r_k - 1$) a reverse arc of a certain normal arc to D^0_{rp}, we momentarily fix this reverse arc. If we backtrack the r'_k-th time, we constantly fix this normal arc. A normal arc such that its reverse arc is momentarily fixed can be complemented if we need to perform another operation of complementing which has not been performed yet for this normal arc. For each graph D^0_{rp} we temporarily or constantly fix a subset $F^0_p \subset S^0_p$ and momentarily fix a set F^{t0}_r of disjunctive arcs. The comments relative to F^0_p and F^{t0}_p are analogous to those relative to F_r and F^t_r, respectively.

3.1. Operation of testing the critical path. The basic task of the operation of testing is the computation of the lower bound of the critical path for every possible successor $D^0_{sq} \in R^{0'}_{Ddi}$ generated from the graph D^0_{rp}. The arcs from the sets F_r are fixed in any successors of D^0_{rp}. It follows from (31) that the arcs from the selection of sets S^0_p do not decrease the length. Let

$$D(F_r \cup S^0_p) = \langle A, U^0 \cup F_r \cup S^0_p \rangle \tag{32}$$

be the graph from the sets F_r and S^0_p, and let $L(F_r \cup S^0_p)$ be the length of a critical path in this graph. Let L^* be the length of the shortest critical path found so far. Then if

$$L(F_r \cup S^0_p) \geqslant L^*,$$

we can reject the graph and all its successors. The value L^* is the upper bound of the length of the minimaximal path with the implicit condition (20) in $\bar{D}^0$.

3.2. Operation of choice. The purpose of the operation of choice is to point out the normal arc for complementing (or eliminating) and to generate the successor in H. The arcs

$$E_r = S_r - F_r \quad \text{and} \quad E^0_p = S^0_p - F^0_p$$

are free. We complement (or eliminate) only arcs of these sets which belong to the current critical path, i.e.

$$K_r = E_r \cap C_{rp} \tag{33}$$

and

$$K^0_p = E^0_p \cap C_{rp}. \tag{34}$$

Let K'_r be the set of reverse and empty arcs with normal arcs belonging to the set K_r. The set K'_r is called the *set of candidates*. We want to choose a normal arc the complementing of which generates a successor with the possibly shortest critical path. This is especially important for the operation of testing. The choice criterion for an arc of K'_r is the expression $\Delta_{rp}[(y, x), (u, v)]$ defined by (30). By Theorem 2, the arc with the least value of $\Delta_{rp}[(y, x), (u, v)]$ should be chosen for complementing.

To choose a normal arc for eliminating, we introduce the formula

$$(35)\qquad \Delta_{rp}[(y,x),\overline{(u,v)}] = \max[-\alpha_{rp}(0,u), -\beta_{rp}(v,z)],$$

where $\alpha_{rp}(0,u)$ and $\beta_{rp}(v,z)$ are defined by (29).

THEOREM 4. *Let $D^0_{rp} \in R^{0'}_{Dd}$ and let D^0_{sp} be the graph obtained from D^0_{rp} by eliminating one arc $\langle y, x\rangle \in C_{rp} \cap S_r$, where C_{rp} is the longest path in D^0_{rp}. Then*

$$L_{rp}(0,z) \geqslant L_{sp}(0,z) \geqslant L_{rp}(0,z) + \Delta_{rp}[(y,x),\overline{(u,v)}].$$

Proof. Since D^0_{sp} is the partial graph of D^0_{rp}, the first inequality is true. In order to prove the second inequality, we note that $L_{rp}(0,z) + \Delta_{rp}[(y,x),\overline{(u,v)}]$ is a particular case of the expression on the right-hand side of the inequality from Theorem 3. Therefore, by eliminating the arc $\langle y,x\rangle \in C_{rp} \cap S_r$, the lower bound of the longest path C_{sp} in D^0_{sp} is $L_{rp}(0,z) + \Delta_{rp}[(y,x),\overline{(u,v)}]$, which completes the proof.

Let $K^{0'}_p$ be the set of reverse arcs with normal arcs belonging to the set K^0_p. The set $K^{0'}_p$ is also called the *set of candidates*. We want to choose a normal arc the complementing of which generates a successor with the possibly shortest critical path. To choose a normal arc for complementing, we introduce the formula

$$(36)\qquad \Delta^0_{rp}[(x,y),(u,v)] = c(u,v) - c(x,y).$$

THEOREM 5. *Let $D^0_{rp} \in R^{0'}_{Dd}$ and let D^0_{rq} be the graph obtained from D^0_{rp} by complementing one arc $\langle x,y\rangle \in C_{rp} \cap S^0_p$, where C_{rp} is the longest path in D^0_{rp}. Then*

$$L_{rq}(0,z) \geqslant L_{rp}(0,z) + \Delta^0_{rp}[(x,y),(u,v)].$$

Proof. Note that for each $\langle y,x\rangle \in C_{rp} \cap S^0_p$ we have

$$\begin{aligned} L_{rp}(0,z) + \Delta^0_{rp}[(x,y),(u,v)] &= L_{rp}(0,z) + c(u,v) - c(x,y) \\ &= L_{rp}(0,x) + c(x,y) + L_{rp}(y,z) + c(u,v) - c(x,y) \\ &= L_{rp}(0,x) + c(u,v) + L_{rp}(y,z). \end{aligned}$$

Since the arc $\langle u,v\rangle$ is a cousin of the arc $\langle x,y\rangle$, it follows from (24) that $u = x$ and $v = y$. Thus we have

$$L_{rp}(0,x) + c(u,v) + L_{rp}(y,z) = L_{rp}(0,u) + c(u,v) + L_{rp}(v,z).$$

It can easily be observed that $L_{rp}(0,u) + c(u,v) + L_{rp}(v,z)$ is the length of a path in D^0_{rq}. Since the longest path in D^0_{rq} cannot be shorter than any other path, we have

$$L_{rq}(0,z) \geqslant L_{rp}(0,z) + \Delta^0_{rp}[(x,y),(u,v)].$$

Therefore, by complementing the arc $\langle x, y\rangle \in C_{rp}\cap S_p^0$, the lower bound of the longest path C_{rq} in D_{rq}^0 is $L_{rp}(0, z) + \Delta_{rp}^0[(x, y), (u, v)]$, which completes the proof.

3.3. Operation of testing the implicit condition. The implicit condition cannot be satisfied when we perform the operation of eliminating an arc from K_r. Each arc $\langle y_i, x_j\rangle \in K_r$ has two adjacent arcs $\langle x_j, y_j\rangle$ and $\langle x_i, y_i\rangle$ which belong to the set S_p^0. In the process of generating the sequence of graphs D_{rp}^0, the adjacent arcs may be complemented, so the logical result of the implicit condition may be changed. Therefore, we can check this condition and eliminate the arc $\langle y_j, x_i\rangle \in K_r$ the adjacent arcs of which are temporarily or constantly fixed. Let

$$R_r = \{\langle y_j, x_i\rangle \in K_r' \mid \langle y_j, x_i\rangle \text{ is an empty arc}\}$$

and

$$R_r' = K_r' - R_r$$

be the sets of empty and reverse arcs, respectively, on which we can replace arcs from the set K_r. Further, let

$$R_{rp} = \{\langle y_j, x_i\rangle \in R_r \mid (\langle x_j, y_j\rangle \in F_p^0) \wedge (\langle x_i, y_i\rangle \in F_p^0)\}$$

be the set of empty candidates the adjacent arcs of which are temporarily or constantly fixed. The normal arc from K_r can be eliminated if its empty arc belongs to the set R_{rp}.

3.4. Algorithm. We start with

$$D_{11}^0 = \langle A, U^0\cup S_1\cup S_1^0\rangle, \quad F_1 = \emptyset, \quad F_1^0 = \emptyset, \quad F_1^t = \emptyset,$$
$$F_1^{t0} = \emptyset, \quad L^* = \infty.$$

The graph D_{11} represents the root of the solution tree H.

Let $D_{rp}^0 = \langle A, U^0\cup S_p\cup S_p^0\rangle$ be the current graph and let $F_r, F_p^0, F_r^t, F_p^{t0}$ be the current sets of temporarily, constantly and momentarily fixed disjunctive arcs in the $(r+p)$-th iteration of the algorithm.

Step 1 (test step). Compute the lower bound $L(F_r\cup S_p^0)$ of the graph defined by (32). If $L(F_r\cup S_p^0) \geqslant L^*$, then go to Step 4. Otherwise, go to Step 2.

Step 2 (evaluation step). Compute $L_{rp}(0, z)$ defined by (28). If $L_{rp}(0, z) < L^*$, then set $L^* = L_{rp}(0, z)$.

Identify a critical path C_{rp} and the sets K_r and K_p^0 defined by (33) and (34), respectively. If $K_r = \emptyset$ and $K_p^0 = \emptyset$, then go to Step 4. Otherwise, identify the sets of candidates K_r' and $K_p^{0\prime}$. Nex perform

$$K_r' = K_r' - F_r^t \quad \text{and} \quad K_p^{0\prime} = K_p^{0\prime} - F_p^{t0}.$$

If $K'_r = \emptyset$ and $K_p^{0'} = \emptyset$, then go to Step 4. Otherwise, for any arcs $\langle u, v\rangle \in K'_r$ and $\overline{\langle u, v\rangle} \in K'_r$ compute $\Delta_{rp}[(y, x), (u, v)]$ and $\Delta_{rp}[(y, x), (u, v)]$ defined by (30) and (35), respectively. For each arc $\langle x, y\rangle \in K_p^{0'}$ compute $\Delta_{rp}^0[(x, y), (u, v)]$ defined by (36). Then go to Step 3.

Step 3 (forward step). If $K'_r = \emptyset$ and $K_p^{0'} = \emptyset$, then go to Step 4. Otherwise, identify the sets R_r, R_{rp}, and R'_r. Next, choose $\langle u, v\rangle \in R'_r \cup R_{rp}$ or $\langle u, v\rangle \in K_p^{0'}$ (perhaps an empty arc) such that

$$\delta_{rp}[(y, x), (u, v)]$$
$$= \min\{ \min_{\langle c,d\rangle \in R'_r \cup R_{rp}} \Delta_{rp}[(a, b), (c, d)], \min_{\langle c,d\rangle \in K_p^{0'}} \Delta_{rp}^0[(a, b), (c, d)]\}.$$

If an empty arc $\overline{\langle u, v\rangle}$ is chosen and $\overline{\langle u, v\rangle} \in R_{rp}$, then check the implicit condition (20) as follows. Let $\langle y, x\rangle = \langle y_j, x_i\rangle$ be the normal arc of the arc $\overline{\langle u, v\rangle}$ and let $\langle y_j, x_i\rangle$ have adjacent arcs of the forms $\langle x_j, y_j\rangle_p$ and $\langle x_i, y_i\rangle_w$ $(i, j \in N^k,\ p, w \in B^k,\ k \in Q)$. If $p = w$, do not eliminate these normal arcs, perform

$$K'_r = K'_r - \{\overline{\langle u, v\rangle}\},$$

and go to the beginning of Step 3. If $p \neq w$, then generate a new graph D_{sp}^0 by eliminating the normal arc $\langle y, x\rangle$ and perform

$$U_{sp}^0 = U_{rp}^0 - \{\langle y, x\rangle\}.$$

If the reverse arc $\langle u, v\rangle$ is chosen and $\langle u, v\rangle \in R'_r$, then generate a new graph D_{sp}^0 by complementing the normal arc $\langle y, x\rangle$ and perform

$$U_{sp}^0 = [U_{rp}^0 - \{\langle y, x\rangle\}] \cup [\{\langle u, v\rangle\}],$$
$$F_s = F_r \cup \{\langle u, v\rangle\}, \quad F_s^t = F_r^t.$$

Simultaneously, add to the solution tree H a new node D_{sp}^0 and a new arc $\langle D_{rp}^0, D_{sp}^0\rangle$ associated with an empty arc $\overline{\langle u, v\rangle}$ or a reverse arc $\langle u, v\rangle$ of the disjunctive graph $\bar{D}^0$. Go to Step 1.

If the reverse arc $\langle u, v\rangle$ is chosen and $\langle u, v\rangle \in K_p^{0'}$, then generate a new graph D_{rq}^0 by complementing the normal arc and perform

$$U_{rq}^0 = [U_{rp}^0 - \{\langle y, x\rangle\}] \cup [\{\langle u, v\rangle\}],$$
$$F_q^0 = F_p^0 \cup \{\langle u, v\rangle\}, \quad F_q^{t0} = F_p^{t0}.$$

Simultaneously, add to the solution tree H a new node D_{rq}^0 and a new arc $\langle D_{rp}^0, D_{sq}^0\rangle$ associated with the reverse arc $\langle u, v\rangle$. Then go to Step 1.

Step 4 (backtracking step). Backtrack to the predecessor D_{kp}^0 or D_{rl}^0 of D_{rp}^0 in H. If D_{rp}^0 has no predecessor, then the algorithm terminates, the selection S_* and the selection of sets S_*^0 associated with the current L^* are optimal, and the longest path in D_* is minimaximal in $\bar{D}^0$. Otherwise, drop the data for D_{rp}^0 and update the data for D_{kp}^0 or D_{rl}^0 as follows.

If D_{rp}^0 is generated by an empty arc $\overline{\langle u, v\rangle} \in K_k'$, then perform

$$K_k' = K_k' - \{\overline{\langle u, v\rangle}\}, \quad F_k^t = F_k^t \cup \{\overline{\langle u, v\rangle}\}.$$

If D_{rp}^0 is generated by a reverse arc $\langle u, v\rangle \in K_k'$, then perform

$$K_k' = K_k' - \{\langle u, v\rangle\}, \quad F_k^t = F_k^t - \{\langle u, v\rangle\}.$$

Besides, if the backtracking is performed the second time while considering the normal arc $\langle y, x\rangle$, then perform

$$F_k = F_r \cup \{\langle y, x\rangle\}, \quad F_k^t = F_k^t - [\{\overline{\langle u, v\rangle}\} \cup \{\langle u, v\rangle\}].$$

If the graph D_{rp}^0 is generated by the reverse arc $\langle u, v\rangle \in K_r^{0\prime}$, then perform

$$K_l' = K_l' - \{\langle u, v\rangle\}, \quad F_l^{t0} = F_l^{t0} \cup \{\langle u, v\rangle\}.$$

If the backtracking is performed the r_k'-th time while considering the normal arc $\langle y, x\rangle$, then perform

$$F_l^0 = F_l^0 \cup \{\langle y, x\rangle\}, \quad F_l^{t0} = F_l^{t0} - [\bigcup_{w=1}^{r_k'} \{\langle u, v\rangle\}_w],$$

where $\langle u, v\rangle_w$ is the reverse arc of $\langle y, x\rangle$. Go to Step 3.

THEOREM 6. *The algorithm consisting of Steps* 1, 2, 3, *and* 4 *above finds a minimaximal path with the implicit condition* (20) *in* $\bar{D}^0$ *in a finite number of steps.*

Proof. It follows from Theorem 3 that each graph D_{rp}^0 generated by the algorithm has no circuits and has a critical path. Since the family $R_{Ddi}^{0\prime}$ is finite, we have to prove that

(a) each graph of the family $R_{Ddi}^{0\prime}$ is explicitly or implicitly examined,

(b) no graph is examined twice (i.e., the algorithm cannot be cyclic).

(a) The process of generating the graphs of the family R_{Ddi}^d is presented in the form of a solution tree H. The node D_{rp}^0 in H is abandoned (with all of its successors) when the result of testing in Step 1 is negative or when all successors of D_{rp}^0 are examined. In the first case, no successors of D_{rp}^0 in H can have critical paths shorter than the lower bound $L(F_r \cup S_p^0)$, since arcs from the set F_r of the graph D_{rp}^0 are fixed in all of its successors, and arcs from the set S_p^0 do not decrease the length. Besides, we generate only graphs for which the implicit condition is satisfied (see Step 3).

(b) For each node D^0_{rp} in H we have fixed sets F_r and F^0_p. The backtracking may be performed from D^0_{rp} to the predecessor D^0_{tp} or D^0_{ru} in H. In the first case, if we backtrack the second time while considering the normal arc, then this arc belongs to the set F_t. In the second case, the same holds when we backtrack the r'_k-th time. Hence, after backtracking from D^o_{rp}, neither this node in H nor any of its successors in H may be generated the second time. If we backtrack the first time from D^0_{rp} in the first case (or the n-th time ($n < r'_k$) in the second case), then the set F^t_t (F^{t0}_u) contains longer arcs than those on which backtracking has been performed. Since the sets of candidates of all successors D^0_{tp} and D^0_{ru} are reduced by the sets F^t_t (F^{t0}_u) (see Step 2), the node D^0_{rp} in H cannot be generated the second time and neither can any of its successors in H.

References

[1] E. Balas, *An additive algorithm for solving linear programs with zero-one variables*, Opns. Res. 13 (1965), p. 517-546.

[2] — *Finding a minimaximal path in a disjunctive PERT network*, International Symposium on the Theory of Graphs, Rome 1966.

[3] — *Discrete programming by the filter method*, Opns. Res. 15 (1967), p. 915-967.

[4] — *Project scheduling with resource constraints*, Report No. 185, Carnegie - Mellon University, Pittsburgh 1970.

[5] S. Gorenstein, *An algorithm for project (job) sequencing with resource constraints*, Opns. Res. 20 (1972), p. 835-850.

[6] J. Grabowski, *A new formulation and solution of the sequencing problem: mathematical model*, Zastosow. Matem. 15 (1976), p. 325-343.

[7] — *A new formulation and solution of the sequencing problem: algorithm*, ibidem 15 (1977), p. 463-474.

[8] Z. Jankowska-Zorychta, *Modele sekwencyjne i ich zastosowanie w planowaniu optymalnej organizacji w dyskretnych procesach produkcyjnych*, Prace CO PAN, Warszawa 1973.

[9] L. Schrage, *Solving resource-constrained network problem by implicit enumeration, Nonpreemptive case*, Opns. Res. 18 (1970), p. 263-278.

INSTITUTE OF TECHNICAL CYBERNETICS
WROCŁAW TECHNICAL UNIVERSITY
50-370 WROCŁAW

Received on 20.12.1976

J. GRABOWSKI (Wrocław)

SFORMUŁOWANIE I ROZWIĄZANIE ZAGADNIENIA KOLEJNOŚCIOWEGO Z RÓWNOLEGŁYMI MASZYNAMI

STRESZCZENIE

Praca poświęcona jest zagadnieniom wyznaczenia optymalnej kolejności wykonywania operacji w procesie produkcyjnym z maszynami równoległymi. Przedstawiono zagadnienie tego typu, prowadzące do nowej konstrukcji grafu dysjunktywnego. Algorytm rozwiązania tego zagadnienia oparty jest na metodzie podziału i ograniczeń.