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K. SZAJOWSKI (Wrocław)

THE NUMBER OF ORTHOGONAL LATIN SQUARES

1. Introduction. A *Latin square* of order n ($n \geqslant 2$) may be defined as an arrangement of n distinct elements in an $(n \times n)$-matrix in such a way that each element occurs in every row and in every column. Two Latin squares are said to be *orthogonal* if for every element a of one square and every element b of the other one there exists exactly one pair of integers i, j such that the element a stands in the i-th row and in the j-th column of the first square, and the element b stands at the same place in the second square. Latin squares of the same order are said to be *mutually orthogonal* if every two of them are orthogonal.

Let $N(n)$ denote the largest value of an integer k for which there exists a set of k mutually orthogonal Latin squares of order n. The following theorems are well known (see [5] and [12]):

THEOREM 1. *If* $n \geqslant 2$, *then* $1 \leqslant N(n) \leqslant n-1$.

THEOREM 2. *If* n *is a prime power, then* $N(n) = n-1$.

THEOREM 3. $N(n \cdot m) \geqslant \min\{N(n), N(m)\}$.

THEOREM 4. *If* $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, *where* p_i *are distinct primes, and* a_i *— positive integers, then*

$$N(n) \geqslant \min_{1 \leqslant i \leqslant r} (p_i^{a_i} - 1).$$

If $N(n) = n-1$, then we say that there exists a *complete* of orthogonal Latin squares of order n. We do not know whether a complete of orthogonal Latin squares exists when n is not a prime or a prime power.

Euler conjectured that $N(n) = 1$ for $n \equiv 2 \pmod 4$. Tarry [13] showed, by a systematic enumeration, that this is true for $n = 6$. In 1959 Parker [9] showed that $N(21) \geqslant 3$. This result was very important, since it showed that in theorem 4 the sign of inequality was essential.

Bose and Shrikhande [1] found the first counterexample to Euler's conjecture, a pair of orthogonal Latin squares of order 22. Parker [9] found a pair of order 10. In 1960, these three authors [3] proved that $N(n) \geqslant 2$ if $n > 6$. It demolished Euler's conjecture. Chowla et al. [4]

proved that $N(n) \geqslant \frac{1}{3} n^{1/91}$ for a sufficiently large n. This established the fact that $N(n)$ tends to infinity with n.

Hanani [5] has shown that $N(n) \geqslant 3$ if $n > 51$, $N(n) \geqslant 5$ if $n > 62$ and $N(n) \geqslant 29$ if $n > 34\,115\,553$. For even n, $N(n) \geqslant 29$ if $n > 2\,733\,666$.

Wilson [14] proved that $N(n) \geqslant 6$ if $n > 76$, except for $n = 82$ and 90. He also improved the result due to Hanani and showed that $N(n) \geqslant 3$ if $n > 46$, and $N(n) \geqslant 4$ if $n > 60$.

We assume that any two Latin squares of order 1 are orthogonal and there is only one Latin square of order 0, but it is orthogonal to itself. This convention yields $N(0) = N(1) = \infty$.

Most of the above-mentioned results were obtained by using relations between orthogonal Latin squares and pairwise balanced designs. In this paper connections of that kind will be also applied.

The aim of this paper is to prove that, for odd n, $N(n) \geqslant 7$ if $n > 303$, except for $n = 469$, 427, 335, and $N(n) \geqslant 15$ if $n > 54\,047$.

2. T-systems, semi-T-systems and orthogonal Latin squares. We remind first some definitions and lemmas. In what follows $|X|$ stands for the cardinality of a set X.

Definition 1 (see [7], [8] and [14]). Let $W_1, W_2, \ldots, W_k$ be a given class of k mutually disjoint m-element sets (called *groups*). If it is possible to form a system of m^2 k-tuples P_j (called *bloks*) in such a way that

(a) $\quad |W_i \cap P_j| = 1 \quad$ for all $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m^2$,

(b) $\quad |P_l \cap P_j| \leqslant 1 \quad$ for all $l, j = 1, 2, \ldots, m^2$; $l \neq j$,

then we denote the system of P_j by $T[k, m]$ and call it a *T-system.*

Lemma 1 (see [7]). *Let a T-system $T[k, n]$ be given and let $a \in W_i$, $b \in W_j$, $i < j$. Then in $T[k, n]$ there exists exactly one k-tuple containing both elements a and b.*

Lemma 2 (see [10] and [14]). *A set of $k-2$ mutually orthogonal Latin squares of order n exists iff there exists a T-system $T[k, n]$.*

Definition 2 (see [11]). Let $W_1, W_2, \ldots, W_k$ be a given class of k mutually disjoint sets (called *groups*) such that $|W_i| = m$ for $i = 1, 2, \ldots, k-1$ and $|W_k| = m-u$. If it is possible to form a system of m^2 sets P_j (called *blocks*) such that $|P_j| = k$ for $j = 1, 2, \ldots, m(m-u)$ and $|P_j| = k-1$ for $j = m(m-u)+1, \ldots, m$, and

(α) $\quad |W_i \cap P_j| = 1 \quad$ for all $i = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots, m^2$,

(α') $$|W_k \cap P_j| = \begin{cases} 1 & \text{for } j = 1, 2, \ldots, m(m-u), \\ 0 & \text{for } j = m(m-u)+1, \ldots, m^2, \end{cases}$$

(β) $\quad |P_l \cap P_j| \leqslant 1 \quad$ for all $l, j = 1, 2, \ldots, m^2$; $l \neq j$

are satisfied, then we denote the system of P_j by $ST[k, m, u]$ and call it a *semi-T-system.*

LEMMA 3 (see [11]). *If a T-system $T[k, m]$ exists and $0 \leqslant u \leqslant m$, then a semi-T-system $ST[k, m, u]$ exists.*

Applying the construction due to Wilson [14], we can prove the following

THEOREM 5. *The existence of $ST[k, m, u]$ implies that*

$$N(km-u) \geqslant \min\{N(k), N(k-1), N(m)-1, N(m-u)\}.$$

Proof. Let $l = 2+\min\{N(k), N(k-1), N(m)-1, N(m-u)\}$. From lemma 1 it follows the existence of $T[l, k]$, $T[l, k-1]$, $T[l+1, m]$, $T[l, m-u]$. For the proof we construct $T[l, k\cdot m-u]$.

Let be given a class of sets W_i, $i = 1, 2, \ldots, l+1$, such that $|W_i| = m$ for $i = 1, 2, \ldots, l$, and $|W_{l+1}| = m-u$. As a group we take

$$W_i^* = (W_i \times K)\cup(\{i\}\times W_{l+1}), \quad i = 1, 2, \ldots, l,$$

where K is a set of $k-1$ elements.

Blocks are obtained as follows:

(i) For each block $A \in ST[l+1, m, u]$ such that $|A\cap W_{l+1}| = 0$, we construct a $T[l, k-1]$ with groups $(A\cap W_i)\times K$, $i = 1, 2, \ldots, l$, and blocks P_A.

(ii) For each block $A \in ST[l+1, m, u]$ such that $|A\cap W_{l+1}| = 1$, we construct a $T[l, k]$ with groups $(A_0\cap W_i)\times K\cup(\{i\}\times(A\cap W_{l+1}))$, $i = 1, 2, \ldots, l$, where $A_0 = A\cap\bigcup_{i=1}^{l} W_i$, and blocks P_A.

Since $T[l, m-u]$ exists, we may form a system with groups $\{i\}\times W_{l+1}$, $i = 1, 2, \ldots, l$, and blocks P'_A. The block $\{i\}\times(A\cap W_{l+1})$, $i = 1, 2, \ldots, l$, for A containing an element of W_{l+1}, belongs to $T[l, m-u]$ and to a certain $T[l, k]$.

Let

$$P = \Big\{\bigcup_{A\in ST[l+1,m,u]} (P_A \setminus P'_A)\Big\}\cup P'_A.$$

It is easy to verify that the blocks P form the system $T[l, k\cdot m-u]$. This completes the proof.

Theorem 5 can be written in another form:

THEOREM 5a (see [14]). *If $0 \leqslant u \leqslant m$, then*

$$N(km+u) \geqslant \min\{N(k), N(k+1), N(m)-1, N(u)\}.$$

3. Seven mutually orthogonal Latin squares. The main aim of this paper is the proof of the following

THEOREM 6. *For odd n, $N(n) \geqslant 7$ if $n > 303$, except for $n = 469$, 427, 335.*

Proof. If $(n, 210) = 1$ (n and 210 are relatively prime), then it follows from theorem 4 that $N(n) \geqslant 10$, and thus for such numbers n theorem 6 is valid.

If $(n, 210) > 1$, we show that there are m and u such that $n = 9m - u$ and $ST[9, m, u]$ exists. In table 1 there are shown suitable m and u for $n \pmod{1890}$. In column 4 we examine those cases in which $(m, 210) > 1$ or $(u, 210) > 1$. In column 5 the maximum n is given for which either $N(n) < 7$ or it is not known whether $N(n) \geqslant 7$. The minimum number taken from column 5 implies the result.

4. Fifteen mutually orthogonal Latin squares.

THEOREM 7. *If n is odd and $n > 54\,047$, then $N(n) \geqslant 15$.*

Proof. The proof of this theorem is based on the method used by Hanani [6]. For more accurate evaluation we could had proceeded similarly as in the proof of theorem 6, but in this case the evaluations would be very extensive.

Let n be odd and let $n = 16m_1 + u_1$, $17 \leqslant u_1 \leqslant 47$, $P = 2\cdot 3\cdot 5$ and $Q = 7\cdot 11\cdot 13$. We show that for each $m_1 \pmod{P}$ and each u_1 suitable s_i and t can be found in such a way that

$$n = 16m + u, \quad m = m_1 - t - P\cdot s_i, \quad u = u_1 + 16t + 16\cdot P\cdot s_i,$$
$$(m, PQ) = 1, \quad (u, PQ) = 1,$$

which will imply, in view of theorem 5a, that $N(n) \geqslant 15$ for $m \geqslant u$.

For each $m_1 \pmod{P}$ and each u_1 we find a minimum non-negative number t such that $(u_1 + 16t, P) = 1$ and $(m_1 - t, P) = 1$. Let $m_2 = m_1 - t$ and $u_2 = u_1 + 16t$. For each u_2 we find four non-negative integers s_i, $i = 1, 2, 3, 4$, such that $(u_2 + 16\cdot P\cdot s_i, Q) = 1$ for each $i = 1, 2, 3, 4$ and no two of them are congruent modulo 7, 11, 13. By the definition of Q and s_i, for every u_2 and m_2 there exists an integer s_i with $(m_2 - P\cdot s_i, Q) = 1$. Using this s_i for every n, we obtain suitable

$$m = m_1 - t - P\cdot s_i \quad \text{and} \quad u = u_1 + 16t + 16\cdot P\cdot s_i.$$

In table 2 for every u_1 and every m_1 we show the suitable t, the set of s_i and the maximum value of u. The maximum value of u is obtained for $u_1 = 45$, $t = 16$, $s_4 = 6$. In this case $u = 3181$ and $N(n) \geqslant 15$ for $m \geqslant 3179$, i.e. for $n > 54\ 047$. This completes the proof of theorem 7.

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TABLE 1

n (mod 1890)	m (mod 210)	u	Proof of special class	Odd n for which $N(n) > 7$ is not proved
1	2	3	4	5
3	223	114		3
5	221	94		5
7	211	2		7
9	1	0		—
15	221	84		15
21	221	78		21
25	223	92	$n = 25 = 5^2$ (Th. 2)	—
27	221	72	$n = 27 = 3^3$ (Th. 2)	—
33	223	84		33
35	221	64		35
39	227	114		39
45	221	54		45
49	229	122	$n = 49 = 7^2$ (Th. 2)	—
51	221	48		51
55	229	116		55
57	221	42		57
63	223	54		63
65	221	34		65
69	221	30		69
75	221	24		75
77	221	22		77
81	223	36	$n = 81 = 3^4$ (Th. 2)	—
85	223	32		85
87	221	12		87
91	223	26		91
93	223	24		93
95	227	58		95
99	11	0		—
105	13	12		—
111	229	60		111
115	13	2		—
117	13	0		—
119	227	34		119
123	227	30		123
125	227	28	$n = 125 = 5^3$ (Th. 2)	—
129	229	42	$n = 129 = 9 \cdot 16 - 15$ (Th. 5)	—
133	229	38		133
135	227	18		135
141	229	30	$n = 141 = 9 \cdot 16 - 3$ (Th. 5)	—
145	241	134	$n = 145 = 9 \cdot 17 - 8$ (Th. 5)	—
147	17	6		—

tab. 1, contd.

1	2	3	4	5
151	229	20		151
153	17	0		—
155	233	52		155
159	239	102		159
161	233	46	$n = 161 = 9\cdot 19-10$ (Th. 5)	—
165	19	6		—
171	19	0		—
175	241	104		175
177	241	102		177
183	233	24		183
185	23	22		—
189	239	72		189
195	23	12		—
201	23	6		—
203	23	4		—
205	241	74		205
207	23	0		—
213	239	48	$n = 213 = 9\cdot 25-12$ (Th. 5)	—
215	239	46		215
217	241	62	$n = 217 = 9\cdot 25-8$ (Th. 5)	—
219	239	42	$n = 219 = 9\cdot 25-6$ (Th. 5)	—
225	241	54	$n = 225 = 3^2\cdot 5^2$ (Th. 4)	—
231	239	30		231
235	241	44	$n = 235 = 9\cdot 27-8$ (Th. 5)	—
237	241	42		237
243	29	18		—
245	29	16		—
249	29	12		—
255	29	6		—
259	31	20		—
261	29	0		—
265	31	14		—
267	31	12		—
273	247	60	$n = 273 = 17\cdot 17-16$ (Th. 5)	—
275	251	94	$n = 275 = 9\cdot 32-13$ (Th. 5)	—
279	31	0		—
285	247	48	$n = 285 = 9\cdot 32-3$ (Th. 5)	—
287	251	87	$n = 287 = 9\cdot 32-1$ (Th. 5)	
291	251	78		291
295	247	38		295
297	37	36		—
301	253	86		301
303	253	84		303
305	251	64	$n = 305 = 9\cdot 37-28$ (Th. 5)	—
309	37	24		—
315	37	18		—
321	253	66	$n = 321 = 9\cdot 37-12$ (Th. 5)	—
325	37	8		—

1	2	3	4	5
327	37	6		—
329	41	40		—
333	37	0		—
335	257	88		335
339	41	30		—
343	253	44	$n = 343 = 7^3$ (Th. 2)	—
345	41	24		—
351	41	18		—
355	43	32		—
357	41	12		—
363	43	24		—
365	41	4		—
369	41	0		—
371	263	106	$n = 371 = 9 \cdot 43 - 16$ (Th. 5)	—
375	43	12		—
381	43	6		—
385	43	2		—
387	43	0		—
393	47	30		—
395	47	28		—
399	47	24		—
405	47	18		—
411	263	66	$n = 411 = 9 \cdot 49 - 30$ (Th. 5)	—
413	47	10		—
415	271	134	$n = 415 = 9 \cdot 49 - 26$ (Th. 5)	—
417	47	6		—
423	47	0		—
425	53	52		—
427	271	122		427
429	269	102	$n = 429 = 9 \cdot 49 - 12$ (Th. 5)	—
435	53	42		—
441	53	36		—
445	271	104	$n = 445 = 17 \cdot 27 - 14$ (Th. 5)	—
447	53	30		—
453	53	24		—
455	53	22		—
459	269	72	$n = 459 = 3^3 \cdot 17$ (Th. 4)	—
465	53	12		—
469	271	80		469
471	53	6		—
475	271	74	$n = 475 = 5^2 \cdot 19$ (Th. 4)	—
477	53	0		—
483	59	48		—
485	59	46		—
489	59	42		—
495	59	36		—
497	281	142	$n = 497 = 9 \cdot 61 - 52$ (Th. 5)	—
501	59	30		—

tab. 1, contd.

1	2	3	4	5
505	61	44		—
507	61	42		—
511	61	38		—
513	59	18		—
515	59	16		—
519	59	12		—
525	59	6		—
531	59	0		—
535	61	14		—
537	67	66		—
539	281	100	$n = 539 = 7^2 \cdot 11$ (Th. 4)	—
543	283	114	$n = 543 = 9 \cdot 64 - 33$ (Th. 5)	—
545	281	94	$n = 545 = 9 \cdot 67 - 58$ (Th. 5)	—
549	61	0		—
553	67	50		—
555	67	48		—
561	283	96	$n = 561 = 9 \cdot 67 - 42$ (Th. 5)	—
565	67	38		—
567	67	36		—
573	67	30		—
575	293	172	$n = 575 = 5^2 \cdot 23$ (Th. 4)	—
579	67	24		—
581	71	58		—
585	71	54		—
591	71	48		—
595	67	8		—
597	67	6		—
603	67	0		—
605	71	34		—
609	71	30		—
615	71	24		—
621	71	18		—
623	293	124	$n = 623 = 17 \cdot 37 - 6$ (Th. 5)	—
625	73	32		—
627	71	12		—
633	79	78		—
635	71	4		—
637	73	20		—
639	71	0		—
645	73	12		—
651	73	6		—
655	73	2		—
657	73	0		—
663	79	48		—
665	83	82		—
669	79	42		—
675	79	36		—
679	79	32		—

1	2	3	4	5
681	83	66		—
685	79	26		—
687	83	60		—
693	79	18		—
695	83	52		—
699	79	12		—
705	79	6		—
707	83	40		—
711	79	0		—
715	307	158	$n = 715 = 9 \cdot 81 - 14$ (Th. 5)	—
717	83	30		—
721	313	206	$n = 721 = 9 \cdot 89 - 80$ (Th. 5)	—
723	83	24		—
725	83	22		—
729	89	72		—
735	83	12		—
741	89	60		—
745	307	128	$n = 745 = 9 \cdot 83 - 81$ (Th. 5)	—
747	83	0		—
749	89	52		—
753	89	48		—
755	89	46		—
759	89	42		—
763	307	110	$n = 763 = 17 \cdot 47 - 36$ (Th. 5)	—
765	89	36		—
771	89	30		—
775	307	98	$n = 775 = 5^2 \cdot 31$ (Th. 4)	—
777	97	96		—
783	89	18		—
785	89	16		—
789	97	84		—
791	89	10		—
795	89	6		—
801	89	0		—
803	317	160	$n = 803 = 9 \cdot 99 - 88$ (Th. 5)	—
805	97	68		—
807	97	66		—
813	97	60		—
815	317	148	$n = 815 = 9 \cdot 99 - 76$ (Th. 5)	—
819	97	54		—
825	101	84		—
831	101	78		—
833	317	130	$n = 833 = 9 \cdot 103 - 94$ (Th. 5)	—
835	97	38		—
837	97	36		—
843	97	30		—
845	101	64		—
847	97	26		—

tab. 1, contd.

1	2	3	4	5
849	97	24		—
851	101	58		—
855	97	18		—
861	101	48		—
865	97	8		—
867	101	42		—
873	97	0		—
875	101	34		—
879	101	30		—
885	103	42		—
889	109	92		—
891	101	18		—
895	103	32		—
897	101	12		—
903	103	24		—
905	101	4		—
909	101	0		—
915	107	48		—
917	107	46		—
921	103	6		—
925	103	2		—
927	103	0		—
931	109	50		—
933	109	48		—
935	107	28		—
939	107	24		—
945	107	18		—
951	109	30		—
955	109	26		—
957	107	6		—
959	107	4		—
963	107	0		—
965	113	52		—
969	109	12		—
973	109	8		—
975	109	6		—
981	109	0		—
985	121	104		—
987	113	30		—
993	113	24		—
995	341	184	$n = 995 = 17\cdot 61 - 14$ (Th. 5)	—
999	121	90		—
1001	113	16		—
1005	113	12		—
1011	113	6		—
1015	121	74		—
1017	113	0		—
1023	347	210	$n = 1023 = 9\cdot 125 - 102$ (Th. 5)	—

1	2	3	4	5
1025	341	154	$n = 1025 = 9 \cdot 127 - 118$ (Th. 5)	—
1029	121	60		—
1035	121	54		—
1041	121	48		—
1043	347	190	$n = 1043 = 9 \cdot 127 - 100$ (Th. 5)	—
1045	127	98		—
1047	121	42		—
1053	127	90		—
1055	347	178	$n = 1055 = 9 \cdot 127 - 88$ (Th. 5)	—
1057	121	32		—
1059	127	84		—
1065	121	24		—
1071	121	18		—
1075	121	14		—
1077	121	12		—
1083	127	60		—
1085	131	94		—
1089	121	0		—
1095	127	48		—
1099	127	44		—
1101	131	78		—
1105	127	38		—
1107	131	72		—
1113	127	30		—
1115	131	64		—
1119	127	24		—
1125	127	18		—
1127	131	52		—
1131	131	48		—
1135	139	116		—
1137	127	6		—
1141	139	110		—
1143	127	0		—
1145	131	34		—
1149	131	30		—
1155	131	24		—
1161	131	18		—
1165	139	86		—
1167	137	66		—
1169	131	10		—
1173	139	78		—
1175	131	4		—
1179	131	0		—
1183	139	68		—
1185	137	48		—
1191	139	60		—
1195	139	56		—
1197	137	36		—

tab. 1, contd.

1	2	3	4	5
1203	137	30		—
1205	137	28		—
1209	137	24		—
1211	143	76		—
1215	139	36		—
1221	139	30		—
1225	139	26		—
1227	137	6		—
1233	137	0		—
1235	149	106		—
1239	139	12		—
1245	143	42		—
1251	139	0		—
1253	143	34		—
1255	151	104		—
1257	143	30		—
1263	149	78		—
1265	143	22		—
1267	151	92		—
1269	151	90		—
1275	143	12		—
1281	143	6		—
1285	157	128		—
1287	143	0		—
1293	149	48		—
1295	149	46		—
1299	149	42		—
1305	149	36		—
1309	151	50		—
1311	151	48		—
1315	151	44		—
1317	151	42		—
1323	149	18		—
1325	377	178	$n = 1325 = 9\cdot 153-52$ (Th. 5)	—
1329	149	12		—
1335	149	6		—
1337	167	166		—
1341	149	0		—
1345	151	14		—
1351	151	8		—
1353	157	60		—
1355	167	148		—
1359	151	0		—
1365	157	48		—
1371	163	96		—
1375	163	92		—
1377	157	36		—
1379	167	124		—

1	2	3	4	5
1383	157	30		—
1385	173	172		—
1389	167	114		—
1293	157	20		—
1395	157	18		—
1401	163	66		—
1405	157	8		—
1407	157	6		—
1413	157	0		—
1415	167	88		—
1419	167	84		—
1421	173	136		—
1425	163	42		—
1431	163	36		—
1435	163	32		—
1437	167	66		—
1443	163	24		—
1445	167	58		—
1449	167	54		—
1455	163	12		—
1461	163	6		—
1463	167	40		—
1465	169	56		—
1467	163	0		—
1473	167	30		—
1475	167	28		—
1477	181	152		—
1479	167	24		—
1485	167	18		—
1491	169	30		—
1495	169	26		—
1497	173	60		—
1503	167	0		—
1505	173	52		—
1509	169	12		—
1515	169	6		—
1519	169	2		—
1521	169	0		—
1525	187	158		—
1527	173	30		—
1533	173	24		—
1535	173	22		—
1539	179	72		—
1545	179	66		—
1547	173	10		—
1551	173	6		—
1555	181	74		—
1557	173	0		—

tab. 1, contd.

1	2	3	4	5
1561	181	68		—
1563	179	48		—
1565	191	154		—
1569	179	42		—
1575	179	36		—
1581	179	30		—
1585	181	44		—
1587	181	42		—
1589	179	22		—
1593	187	90		—
1595	179	16		—
1599	179	12		—
1603	187	80		—
1605	179	6		—
1611	179	0		—
1615	181	14		—
1617	181	12		—
1623	187	60		—
1625	191	94		—
1629	181	0		—
1631	191	88		—
1635	187	48		—
1641	191	78		—
1645	187	38		—
1647	187	36		—
1653	187	30		—
1655	191	64		—
1659	187	24		—
1665	187	18		—
1671	191	48		—
1673	197	100		—
1675	187	8		—
1677	187	6		—
1683	187	0		—
1685	191	34		—
1687	193	50		—
1689	197	84		—
1695	191	24		—
1701	191	18		—
1705	199	86		—
1707	191	12		—
1713	193	24		—
1715	191	4		—
1719	191	0		—
1725	193	12		—
1729	199	62		—
1731	193	6		—
1735	193	2		—

1	2	3	4	5
1737	193	0		—
1743	197	30		—
1745	197	28		—
1749	197	24		—
1755	197	18		—
1757	197	16		—
1761	199	30		—
1765	199	26		—
1767	197	6		—
1771	199	20		—
1773	197	0		—
1775	209	106		—
1779	199	12		—
1785	199	6		—
1791	199	0		—
1795	211	104		—
1797	211	102		—
1799	209	82		—
1803	209	78		—
1805	221	184		—
1809	209	72		—
1813	223	194		—
1815	209	66		—
1821	209	60		—
1825	211	74		—
1827	211	72		—
1833	227	210		—
1835	209	46		—
1839	209	42		—
1841	209	40		—
1845	209	36		—
1851	209	30		—
1855	211	44		—
1857	211	42		—
1863	209	18		—
1865	209	16		—
1869	209	12		—
1875	211	24		—
1881	209	0		—
1883	227	160		—
1885	211	14		—
1887	211	12		—

TABLE 2

u_1	t	u_2	s_1	s_2	s_3	s_4	Maximum value of u
1	2	3	4	5	6	7	8
17	0	17	0	2	3	5	2417
	2	49	1	2	3	5	2449
	6	113	0	1	2	3	1553
	14	241	0	2	3	4	2161
19	0	19	0	1	3	5	2419
	4	83	0	1	3	4	2003
	10	179	0	1	2	3	1619
	12	211	0	1	2	4	2131
21	2	53	0	2	3	4	1973
	8	149	0	1	2	4	2069
	10	181	0	1	3	5	2581
	16	277	0	1	2	3	1717
23	0	23	0	1	2	4	1943
	6	119	1	3	4	6	2999
	8	151	0	1	3	4	2071
	14	247	1	2	5	6	3127
25	4	89	0	1	2	5	2489
	6	121	1	2	5	6	3001
	12	217	1	3	4	5	2617
	16	281	0	1	2	3	1721
27	2	59	0	2	3	4	1979
	4	91	1	2	3	4	2011
	10	187	1	2	3	6	3067
	14	251	0	1	3	6	3131
29	0	29	0	1	2	4	1949
	2	61	0	1	2	3	1501
	8	157	0	2	3	4	2077
	12	221	1	2	4	5	2621
31	0	31	0	2	3	4	1951
	6	127	0	1	2	3	1567
	10	191	0	2	4	5	2591
	16	287	2	4	5	6	3167
33	4	97	0	1	3	4	2017
	8	161	1	2	3	4	2081
	10	193	0	1	2	3	1633
	14	257	0	2	3	5	2657
35	2	67	0	1	4	5	2467
	6	131	0	2	3	5	2531
	8	163	0	1	2	4	2083
	12	227	0	2	3	4	2147

1	2	3	4	5	6	7	8
37	0	37	0	2	4	5	2437
	4	101	0	2	3	4	2021
	6	133	1	2	4	5	2533
	10	197	0	1	3	4	2117
39	2	71	0	1	2	3	1511
	4	103	0	2	3	5	2503
	8	167	0	1	3	4	2087
	10	199	0	2	5	6	3079
41	0	41	0	1	3	4	1961
	2	73	0	2	3	4	1993
	6	137	0	1	2	3	1577
	8	169	2	3	4	6	3049
43	0	43	0	1	2	3	1483
	4	107	0	1	4	5	2507
	6	139	0	1	3	4	2059
	10	203	1	2	3	5	2603
45	2	77	1	2	3	4	1997
	4	109	0	1	2	4	1549
	8	173	0	1	3	5	2573
	16	301	3	4	5	6	3181
47	0	47	0	1	2	3	1487
	2	79	0	2	4	5	2479
	6	143	2	3	4	5	2543
	14	271	0	1	2	3	1711

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K. SZAJOWSKI (Wrocław)

LICZBA ORTOGONALNYCH KWADRATÓW ŁACIŃSKICH

STRESZCZENIE

Niech $N(n)$ oznacza liczbę ortogonalnych kwadratów łacińskich rzędu n. W pracy pokazano, że $N(n) \geqslant 7$ dla nieparzystych $n > 303$ z wyjątkiem $n = 469, 427, 335$ oraz że $N(n) \geqslant 15$ dla nieparzystych $n > 54\,047$.
