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SIZE-ROBUSTNESS OF TESTS BASED ON ORDER STATISTICS AND SPACINGS FOR THE EXPONENTIAL DISTRIBUTION

Abstract. The paper deals with the concept of robustness given by Zieliński (see [7], [8]). Let a sample from the exponential distribution with an unknown scale parameter $\lambda > 0$ be given. We consider the problem of size-robustness of tests based on nonnegative linear combinations of order statistics (Section 2) and normalized spacings (Section 3) of the hypothesis $H\colon \lambda \leqslant \lambda_0$ against the alternative $K\colon \lambda > \lambda_0$. In Sections 1.1 and 1.2 we present some facts of stochastic and dispersive orderings which are used in the sequel. Section 1.3 defines the violations of the exponential model.

1. Preliminaries. Denote by $X_{k:n}$ and $D_{k:n}$, for $k = 1, \ldots, n$, the k-th order statistic and k-th normalized spacing of a sample $X_1, \ldots, X_n$ from the distribution function (d.f.) F, respectively, i.e.,

$$D_{k:n} = (n-k+1)(X_{k:n} - X_{k-1:n}), \quad \text{where } k = 1, \ldots, n,\ X_{0:n} = 0.$$

Let $F_{\boldsymbol{a}}$ and $\tilde{F}_{\boldsymbol{a}}$ denote the d.f.'s of $\sum_1^n a_k X_{k:n}$ and $\sum_1^n a_k D_{k:n}$, respectively. Let $f_{\boldsymbol{a}}$ ($\tilde{f}_{\boldsymbol{a}}$), if it exists, be the density function of $F_{\boldsymbol{a}}$ ($\tilde{F}_{\boldsymbol{a}}$). Then we use the following notation: $\bar{F} = 1 - F$, and F^{-1} for the inverse of F.

1.1. *Stochastic ordering.*

Definition 1. We say that the d.f. F is *stochastically less than* the d.f. G, written $F \leqslant_{\text{st}} G$, if and only if $F(x) \geqslant G(x)$ for every x.

Lemma 1. *If $F \leqslant_{\text{st}} G$, then $F_{\boldsymbol{a}} \leqslant_{\text{st}} G_{\boldsymbol{a}}$, where $\boldsymbol{a} = (a_1, \ldots, a_n) \geqslant \boldsymbol{0}$.*

For the proof see [6].

1.2. *Dispersive ordering.*

DEFINITION 2. The d.f. F is said to be *less dispersed than* the d.f. G, written $F \leqslant_{\mathrm{disp}} G$, if and only if

$$F^{-1}(v)-F^{-1}(u) \leqslant G^{-1}(v)-G^{-1}(u)$$

whenever $0 < u < v < 1$.

LEMMA 2. *Let F and G be absolutely continuous with corresponding densities f and g. Then $F \leqslant_{\mathrm{disp}} G$ if and only if*

$$gG^{-1}(v) \leqslant fF^{-1}(v) \quad \textit{for every } 0 < v < 1.$$

The lemma follows directly from Definition 2 (see, e.g., [5]).

LEMMA 3. *Let F and G be continuous, increasing and $F(0) = 0 = G(0)$. Then $F \leqslant_{\mathrm{disp}} G$ implies*

$$\tilde{F}_{\boldsymbol{a}} \leqslant_{\mathrm{st}} \tilde{G}_{\boldsymbol{a}}, \quad \textit{where } \boldsymbol{a} = (a_1, \ldots, a_n) \geqslant \boldsymbol{0}.$$

The lemma follows directly from the proof of Oja [4].

1.3. *A violation of the exponential distribution.* Consider the statistical model $(R_+^n, B_+^n, \{W_\lambda: \lambda > 0\}^n)$, where R_+ and B_+ are the positive half-line and the family of Borel subsets of R_+, respectively, W_λ is the exponential distribution with the d.f. $1-\exp(-x/\lambda)$, $x \geqslant 0$, $\lambda > 0$. Let us assume that the violation of the model is invariable with respect to a change of the scale parameter λ, i.e., a violation of the d.f. W into some d.f. F is equivalent to the condition that W_λ is violated into F_λ for every $\lambda > 0$, where $F_\lambda(\cdot) = F(\cdot/\lambda)$. Consequently, the violation of the statistical model is defined by the violation of the probabilistic model $(R_+^n, B_+^n, \{W\}^n)$.

Let G be a specified d.f. which satisfies the following conditions:

(i) $G(0) = 0$,

(ii) G has continuous density g,

(iii) r_G is nondecreasing, not constant and

$$\lim_{x\to\infty} r_G(x) = 1,$$

where r_G is the failure rate function of G, i.e., $r_G = g/\bar{G}$. It is well known that $G = 1-\exp(-R_G)$, where R_G denotes the hazard function of G, i.e.

$$R_G(x) = \int_0^x r_G(u)\,du, \quad x \geqslant 0.$$

Moreover, let H be a specified d.f. which satisfies conditions (i), (ii), (iii) and is increasing.

We assume that the violations of W are defined by mappings π_G and $\tilde{\pi}_H$ from $\{W\}$ into the family of all probability measures on (R_+, B_+) as follows:

$$\pi_G(W) = \{F:\ W \leqslant_{\mathrm{st}} F \leqslant_{\mathrm{st}} G\},$$

$$\tilde{\pi}_H(W) = \{F:\ F \text{ is continuous, } S_F = [0, \infty),\ W \leqslant_{\mathrm{disp}} F \leqslant_{\mathrm{disp}} H\},$$

where

$$S_F = \{x\colon F(x) \text{ is increasing}\}.$$

The violations generated by ordering relations in the set of d.f.'s have been introduced by Bartoszewicz [2].

Note that $W, G \in \pi_G(W)$ and $W, H \in \tilde{\pi}_H(W)$. This is a consequence of the fact that $W \leqslant_{\mathrm{st}} G$ if and only if

$$\int_0^x r_G(u)du \leqslant x \quad \text{for every } x \geqslant 0$$

and, according to Lemma 2, $W \leqslant_{\mathrm{disp}} H$ if and only if

$$hH^{-1}(v) \leqslant 1-v, \quad 0 < v < 1,$$

i.e., if $r_H \leqslant 1$. Obviously, if $G = H$, then $\tilde{\pi}_H(W) \subset \pi_G(W)$.

2. Tests based on order statistics. Let X be a sample of size n from W_λ, where $\lambda > 0$ is an unknown parameter of the scale. Consider the problem of testing the hypothesis $H\colon \lambda \leqslant \lambda_0$ against $K\colon \lambda > \lambda_0$, where λ_0 is a fixed positive number, at a significance level $\alpha \in (0, 1)$. Let us define the following class of tests:

$$P_\alpha^+ = \{\varphi_{\boldsymbol{a}}\colon \sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{W_\lambda}\varphi_{\boldsymbol{a}}(X) = \alpha,\ \sum_1^n a_i > 0,\ \boldsymbol{a} \geqslant \boldsymbol{0}\},$$

where

$$\varphi_{\boldsymbol{a}}(X) = I\big(\sum_1^n a_i X_{i:n} > c_\alpha(\boldsymbol{a})\big),$$

i.e., $\varphi_{\boldsymbol{a}}(X)$ is the indicator function of the set

$$\{\sum_1^n a_i X_{i:n} > c_\alpha(\boldsymbol{a})\}.$$

It is easy to see that the uniformly most powerful test $\varphi_{1/n}$ belongs to P_α^+. Under the assumptions of Section 1.3 let us define

$$R_{\boldsymbol{a}} = \sup_{F \in \pi_G(W)} \big(\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda}\varphi_{\boldsymbol{a}}(X)\big) - \inf_{F \in \pi_G(W)} \big(\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda}\varphi_{\boldsymbol{a}}(X)\big),$$

where $0 < \alpha < 1$, $\varphi_{\boldsymbol{a}} \in P_\alpha^+$.

If F runs through the set $\pi_G(W)$, then $R_{\boldsymbol{a}}$ is the oscillation of the maximal probability of the error of first kind and gives us a measure of robustness of the test $\varphi_{\boldsymbol{a}} \in P_\alpha^+$ with respect to its size, under the violation $\pi_G(W)$. It can be easily shown that $\varphi_{\boldsymbol{a}} \in P_\alpha^+$ implies

$$\varphi_{\boldsymbol{a}}(X) = I\big(\sum_1^n a_i X_{i:n} > \lambda_0 [W_{\boldsymbol{a}}]^{-1}(1-\alpha)\big)$$

and, consequently,

$$\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda} \varphi_{\boldsymbol{a}}(X) = P_F\left(\sum_1^n a_i X_{i:n} > [W_{\boldsymbol{a}}]^{-1}(1-\alpha)\right).$$

From Lemma 1 it follows that

$$R_{\boldsymbol{a}} = 1-\alpha-G_{\boldsymbol{a}}[W_{\boldsymbol{a}}]^{-1}(1-\alpha). \tag{1}$$

Hence

$$R_{\boldsymbol{a}} = \sup_{F \in \pi_G(W)} \left(\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda} \varphi_{\boldsymbol{a}}(X) - \alpha\right)$$

describes the maximal upper deviation of the real size of the test $\varphi_{\boldsymbol{a}} \in P_\alpha^+$ from its theoretical size. It should be mentioned that in fact P_α^+ consists of tests $\varphi_{\boldsymbol{a}}$ for which

$$A = \sum_1^n a_i = 1.$$

Really, the equality

$$[W_{\boldsymbol{a}}]^{-1}(1-\alpha) = A[W_{\boldsymbol{a}/A}]^{-1}(1-\alpha)$$

implies $\varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{a}/A}$.

We prove the following

THEOREM 1. *For every test $\varphi_{\boldsymbol{a}}$, where $\boldsymbol{a} \neq (\boldsymbol{0}, 1_n)$, there exists α_0, $0 < \alpha_0 < 1$, such that for every $0 < \alpha \leqslant \alpha_0$ the following holds:*

If $\varphi_{\boldsymbol{a}}$, $\varphi_{(\boldsymbol{0},1_n)} \in P_\alpha^+$, then $R_{(\boldsymbol{0},1_n)} < R_{\boldsymbol{a}}$.

2.1. *Technical results.* First we present some auxiliary lemmas. Let G satisfy conditions (i)–(iii) of Section 1.3. For simplicity, the failure rate function and the hazard function of G are denoted by r and R, respectively. We introduce some notation which are used in the sequel.

Let $\boldsymbol{a} \geqslant \boldsymbol{0}$ be the vector of size n such that

$$\sum_1^n a_i = 1.$$

(2) Let us define

$$a = \max(a_j\colon 1 \leqslant j \leqslant n), \qquad p = \operatorname{card}(j\colon a_j = a)$$

and let $d_1, \ldots, d_q \in \{a_1, \ldots, a_n\}$, $0 \leqslant q \leqslant [n/2]$ be all numbers such that

$$k_i = \operatorname{card}(j\colon a_j = d_i) \geqslant 2, \quad \text{where } d_1 < \ldots < d_q.$$

Let $\tilde{a}_1, \ldots, \tilde{a}_k$ be defined as follows:

$$\{\tilde{a}_1, \ldots, \tilde{a}_k\} = \{a_1, \ldots, a_n\} \setminus \{d_1, \ldots, d_q\}.$$

We have

$$\sum_1^q k_i + k = n.$$

(3) Let $m \in \{1, \ldots, n\}$ be a number such that

$$a_m > 0, \quad a_{m+1} = \ldots = a_n = 0.$$

Define

$$A_k = \sum_k^m a_i, \quad b_k = A_k/(n-k+1) \quad \text{for } k = 1, \ldots, m,$$

$$b = \max(b_j\colon 1 \leqslant j \leqslant m), \quad l = \operatorname{card}(j\colon b_j = b).$$

LEMMA 4. *Under the notation of* (3) *we have*

$$g_a(z) = \frac{n!}{a_m(n-m)!} \int_0^z \int_{u_1}^{D_1} \ldots \int_{u_{m-2}}^{D_{m-2}} \Big(\prod_1^{m-1} g(u_i)\Big) g(D_{m-1}) \times \exp\big(-(n-m)R(D_{m-1})\big) du_{m-1} \ldots du_1,$$

where

$$D_k = D_k(z, \boldsymbol{u}, \boldsymbol{a}) = \Big(z - \sum_1^k a_i u_i\Big)/A_{k+1} \quad \textit{for } k = 0, \ldots, m-1.$$

Proof. The equality follows from the differentiation of

$$G_a(z) = \frac{n!}{(n-m)!} \int_0^z \int_{u_1}^{D_1} \ldots \int_{u_{m-1}}^{D_{m-1}} \Big(\prod_1^m g(u_i)\Big) \exp\big(-(n-m)R(u_m)\big) du_m \ldots du_1.$$

LEMMA 5. *Let* $\boldsymbol{c} = (c_1, \ldots, c_k)$ *satisfy the following conditions*:
(1) $c_i \geqslant 0$.
(2) $c_k(k-i) > \sum_i^{k-1} c_j$ *for* $i = 1, \ldots, k-1$. *Let us define* $v = 1-r$, $H_1 = 1$,

$$H_k(\boldsymbol{c}) = \frac{\prod_1^{k-1} \Big(c_k(k-i) - \sum_i^{k-1} c_j\Big)}{c_k^{k-1}} \times \int_0^\infty \int_{u_1}^\infty \ldots \int_{u_{k-2}}^\infty \prod_1^{k-1} r(u_i) \exp\Big(u_i(c_i/c_k - 1) + \int_0^{u_i} v(s)\,ds\Big) du_{k-1} \ldots du_1,$$

where $k \geqslant 2$.

Then we have the following:
If there exists $l < k$ *such that* $c_l > 0$, *then*

$$H_k(\boldsymbol{c}) > 1 = H_k(\boldsymbol{0}, c_k).$$

Proof. Let $u_0 = 0$. Using the transformations

$$z_j = \frac{(k-j)c_k - \sum_j^{k-1} c_m}{c_k} (u_j - u_{j-1}) \quad \text{for } j = 1, \ldots, k-1$$

and assumptions (1), (2), we can express $H_k(c)$ in the form

$$H_k(c) = \int_0^\infty \dots \int_0^\infty \prod_1^{k-1} r(L_i)\exp\Big(-z_i + \int_0^{L_i} v(s)ds\Big)dz_{k-1}\dots dz_1,$$

where

$$L_i = L_i(z, c) = \sum_1^i \frac{c_k z_j}{(k-j)c_k - \sum_j^{k-1} c_m} \quad \text{for } i = 1, \dots, k-1.$$

From the fact that

$$\frac{c_k}{(k-j)c_k - \sum_j^{k-1} c_m} \geqslant \frac{1}{k-j} \quad \text{for } j = 1, \dots, k-1,$$

r and v are nondecreasing and nonnegative, respectively, it follows that

$$H_k(c) \geqslant \int_0^\infty \dots \int_0^\infty \prod_1^{k-1} r\big(L_i(z, 0, c_k)\big)\exp\Big(-z_i + \int_0^{L_i(z,0,c_k)} v(s)ds\Big)dz_{k-1}\dots dz_1$$

$$= H_k(0, c_k) = (k-1)! \int_0^\infty \dots \int_{u_{k-2}}^\infty \prod_1^{k-1} g(u_i)du_{k-1}\dots du_1 = 1.$$

Let us assume that there exists $l < k$ such that $c_l > 0$. From the fact that r is not constant and

$$\frac{c_k}{(k-l)c_k - \sum_l^{k-1} c_m} > \frac{1}{k-l}$$

the proof is completed.

Under the notation of (2) we have

LEMMA 6. *If $\tilde{w}_a$ is the density of $\tilde{W}_a$, then*

$$\lim_{z\to\infty} \frac{\tilde{w}_a(z)\exp(z/a)}{z^{p-1}} = \frac{a^{n-2p}}{(p-1)! \prod_{\{a_j \neq a\}} (a-a_j)}$$

Proof. Without loss of generality we assume that $a_j > 0$ for $j = 1, \dots, n$. Let

$$(4) \quad B(u, k) = \frac{\exp(-u/d_q)}{d_q(k_q-1)!}$$

$$\times \int_0^{M_1} \dots \int_0^{M_{q-1}} M_q^{k_q-1} \prod_1^{q-1} \frac{u_i^{k_i-1}}{(k_i-1)!}\exp\big(u_i(d_i/d_q-1)\big)du_{q-1}\dots du_1,$$

where

$$M_j = M_j(u, \boldsymbol{d}) = \Big(u - \sum_{1}^{j-1} d_i u_i\Big)/d_j \quad \text{for } j = 1, \ldots, q,$$

$\boldsymbol{k} = (k_1, \ldots, k_q)$ and $\boldsymbol{d} = (d_1, \ldots, d_q)$ are defined in (2). It is easy to see that $B(u, \boldsymbol{k})$ is the density of $\sum_1^q d_i V_i$, where $V_1, \ldots, V_q$ mean the independent random variables and V_i is distributed according to the gamma d.f. with the shape and the scale parameters which are equal to k_i and 1, respectively. It is well known that the normalized spacings from the sample from W are independent and distributed according to W. Consequently, for $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_k)$ defined in (2) we have

$$\tilde{W}_{\boldsymbol{a}}(z) = \int_0^z \tilde{W}_{\tilde{\boldsymbol{a}}}(z-u) B(u, \boldsymbol{k})\, du.$$

Likeš [3] has proved that

$$\tilde{W}_{\tilde{\boldsymbol{a}}}(z) = 1 - \sum_1^k \Big(\prod_{j \neq i} \frac{\tilde{a}_i}{(\tilde{a}_i - \tilde{a}_j)} \Big) \exp(-z/\tilde{a}_i).$$

We obtain

(5)

$$\tilde{w}_{\boldsymbol{a}}(z) = \begin{cases} \displaystyle\sum_1^k \frac{\tilde{a}_i^{k-2}}{\prod_{j\neq i} (\tilde{a}_i - \tilde{a}_j)} \exp(-z/\tilde{a}_i) & \text{for } k = n, \\ \displaystyle\sum_1^k \frac{\tilde{a}_i^{k-2}}{\prod_{j\neq i} (\tilde{a}_i - \tilde{a}_j)} \exp(-z/\tilde{a}_i) \int_0^z \exp(u/\tilde{a}_i) B(u, \boldsymbol{k})\, du & \text{for } 1 \leqslant k < n, \end{cases}$$

(6)
$$\tilde{w}_{\boldsymbol{a}}(z) = B(z, \boldsymbol{k}) \quad \text{for } k = 0,$$

where B is defined in (4).

We will distinguish two cases: $a > d_q$ and $a = d_q$.

If $a > d_q$, then $p = 1$ and there exists $j_0 \in \{1, \ldots, k\}$ such that $a = \tilde{a}_{j_0}$. From (5) we obtain

$$\lim_{z\to\infty} \tilde{w}_{\boldsymbol{a}}(z) \exp(z/a) = \begin{cases} \dfrac{\tilde{a}_{j_0}^{k-2}}{\prod_{j \neq j_0} (\tilde{a}_{j_0} - \tilde{a}_j)} \displaystyle\int_0^\infty \exp(u/\tilde{a}_{j_0}) B(u, \boldsymbol{k})\, du & \text{if } k < n, \\ \dfrac{\tilde{a}_{j_0}^{k-2}}{\prod_{j \neq j_0} (\tilde{a}_{j_0} - \tilde{a}_j)} & \text{if } k = n. \end{cases}$$

Using the transformations

$$\tilde{\tilde{d}}_i = d_i/(\tilde{a}_{j_0}-d_i), \qquad \tilde{u} = u/\tilde{a}_{j_0},$$
$$v_i = (\tilde{a}_{j_0}-d_i)u_i/\tilde{a}_{j_0} \qquad \text{for } i = 1, \ldots, q,$$

we have

$$\int_0^\infty \exp(u/\tilde{a}_{j_0})B(u, \mathbf{k})du = \tilde{a}_{j_0}^{n-k}/\prod_1^q(\tilde{a}_{j_0}-d_i)^{k_i}$$

and the proof of this case is completed.

Let us consider the case $a = d_q$. From (5) we obtain the equality

$$\lim_{z\to\infty} \frac{\tilde{w}_{\mathbf{a}}(z)\exp(z/a)}{z^{p-1}}$$
$$= \frac{a^{S_q-k_q+1}}{(k_q-1)!\prod_1^{q-1}(a-d_i)^{k_i}} \sum_1^k \frac{\tilde{a}_i^{k-1}}{(a-\tilde{a}_i)\prod_{j\neq i}(\tilde{a}_i-\tilde{a}_j)}, \qquad \text{where } S_q = \sum_1^{q-1} k_i.$$

It is easy to notice that

$$\sum_1^k \frac{\tilde{a}_i^{k-1}}{(a-\tilde{a}_i)\prod_{j\neq i}(\tilde{a}_i-\tilde{a}_j)} = \frac{1}{\prod_1^k(a-\tilde{a}_j)}\sum_1^k \tilde{a}_i^{k-1}\prod_{j\neq i}\frac{a-\tilde{a}_j}{\tilde{a}_i-\tilde{a}_j} = \frac{a^{k-1}}{\prod_1^k(a-\tilde{a}_j)}.$$

Consequently, from the fact that

$$\sum_1^q k_i + k = n, \qquad k_q = p,$$

the proof for the case $k \geqslant 1$ is complete. If $k = 0$, then the proof follows analogously from (6). Under the notation (3), from Lemma 6 and the obvious relation $W_{\mathbf{a}} = \tilde{W}_{\mathbf{b}}$ we obtain the following

LEMMA 7. *If $w_{\mathbf{a}}$ is the density of $W_{\mathbf{a}}$, then*

$$\lim_{z\to\infty} \frac{w_{\mathbf{a}}(z)\exp(z/b)}{z^{l-1}} = \frac{b^{m-2l}}{(l-1)!\prod_{\{b_j\neq b\}}(b-b_j)}.$$

Under the notation of Lemma 4 and (3) we have

LEMMA 8. *Let*

$$C_k(y, \mathbf{a}) = \int_0^y \ldots \int_{u_{k-2}}^{D_{k-2}} \left[\prod_1^{k-1} r(u_i)\exp\left(u_i\frac{a_i}{b_k}-u_i+\int_0^{u_i} v(s)ds\right)\right]$$
$$\times \exp\Big((n-k+1)\int_0^{D_{k-1}} v(s)ds - \int_0^{y/b_k} v(s)ds\Big)du_{k-1}\ldots du_1$$

for $k = 2, \ldots, m$ *and*

$$C_1(y, \boldsymbol{a}) = \exp\left(n\int_0^y v(s)ds - \int_0^{ny} v(s)ds\right), \quad \textit{where } y \geqslant 0,\ v = 1-r.$$

Then for the vector $\boldsymbol{a} \neq (\boldsymbol{0}, \boldsymbol{1}_n)$ *we have*

$$\underline{\lim}_{y\to\infty} \frac{C_m(y, \boldsymbol{a})}{y^{l-1}\exp\left(R(y/b_m) - R(y/b)\right)} > \frac{(n-m)!\,a_m b^{m-2l}}{n!(l-1)! \prod_{\{b_j \neq b\}} (b-b_j)}.$$

Proof. We show in detail the proof for the case $l = 1$. For the case $l > 1$ the proof follows analogously. We have the following obvious relations:

(a) $$D_{k-1}\left(y, u_1, \ldots, u_{k-2}, D_{k-2}(y, \boldsymbol{u}, \boldsymbol{a}), \boldsymbol{a}\right) = D_{k-2}(y, \boldsymbol{u}, \boldsymbol{a}) \quad \text{for } k = 2, \ldots, m,$$

(b) $$\sum_1^{k-2} u_i(a_i/b_k - 1) + (a_{k-1}/b_k - 1)D_{k-2}(y, \boldsymbol{u}, \boldsymbol{a}) = \sum_1^{k-2} u_i(a_i/b_{k-1} - 1) + y/b_k - y/b_{k-1}$$

for $k = 2, \ldots, m$,

(c) $$v(y/b_k) \leqslant v\left(D_{k-1}(y, \boldsymbol{u}, \boldsymbol{a})\right) \quad \text{for } k = 1, \ldots, m,$$

which imply

$$\frac{d}{dy}C_k(y, \boldsymbol{a}) \geqslant \frac{r(y)}{A_{k-1}} \exp\left(R(y/b_k) - R(y/b_{k-1})\right) C_{k-1}(y, \boldsymbol{a}).$$

Consequently, under the notation (3), for $k > t = \min(j\colon b_j = b)$ we have

$$\underline{\lim}_{y\to\infty} \frac{C_k(y, \boldsymbol{a})}{\exp\left(R(y/b_k) - R(y/b_t)\right)} \geqslant \frac{1}{A_{k-1}(1/b_k - 1/b_t)} \underline{\lim}_{y\to\infty} \frac{C_{k-1}(y, \boldsymbol{a})}{\exp\left(R(y/b_{k-1}) - R(y/b_t)\right)}.$$

Hence, by induction,

(7) $$\underline{\lim}_{y\to\infty} \frac{C_m(y, \boldsymbol{a})}{\exp\left(R(y/b_m) - R(y/b_t)\right)} \geqslant \frac{b_t^{m-t} \prod_{t+1}^{m} b_j}{\prod_{t}^{m-1} A_j \prod_{t+1}^{m} (b_t - b_j)} \underline{\lim}_{y\to\infty} C_t(y, \boldsymbol{a}).$$

Under the notation of Lemma 5 we obtain

$$\underline{\lim}_{y\to\infty} C_t(y, \boldsymbol{a}) = \frac{b_t^{t-1} H_t(a_1, \ldots, a_{t-1}, b_t)\exp\left((n-t)\int_0^\infty v(s)ds\right)}{\prod_1^{t-1} \left((t-j)b_t - \sum_j^{t-1} a_i\right)}.$$

From (3) it follows that the vector $(a_1, \ldots, a_{t-1}, b_t)$ satisfies assumptions (1) and (2) of Lemma 5. Consequently,

$$\varliminf_{y\to\infty} C_t(y, \boldsymbol{a}) > \frac{b_t^{t-1}}{\prod_1^{t-1}\big((t-j)b_t - \sum_j^{t-1} a_i\big)}$$

and after simple calculations on the right-hand side of (7) the proof is complete.

Proof of Theorem 1. Let us put for the vector $\boldsymbol{a} \neq (\boldsymbol{0}, 1_n)$:

$$y = [W_{\boldsymbol{a}}]^{-1}(1-\alpha), \quad x(y) = [W_{(0,1_n)}]^{-1}(1-\alpha),$$

i.e.,

$$1-\alpha = W_{\boldsymbol{a}}(y) = W_{(0,1_n)}(x(y)). \tag{8}$$

Write $S(y) = R_{(0,1_n)} - R_{\boldsymbol{a}}$. From (1) and (8) we obtain the equality

$$S(y) = G_{\boldsymbol{a}}(y) - G_{(0,1_n)}(x(y)).$$

It is easy to notice that

$$\lim_{y\to\infty} S(y) = 0.$$

If we are able to prove that there exists $y_0 > 0$ such that for every $y \geqslant y_0$ we have

$$\frac{d}{dy}S(y) > 0$$

and, consequently, $S(y) < 0$, then the proof will be completed. From the obvious equality $G_{(0,1_n)} = G^n$ it follows that

$$\frac{d}{dy}S(y) = G^{n-1}(x(y))g(x(y))\frac{d}{dy}x(y)\left(\frac{g_{\boldsymbol{a}}(y)}{G^{n-1}(x(y))g(x(y))\frac{d}{dy}x(y)} - n\right).$$

Thus it remains to prove the inequality

$$\varliminf_{y\to\infty} \frac{g_{\boldsymbol{a}}(y)}{nG^{n-1}(x(y))g(x(y))\frac{d}{dy}x(y)} > 1.$$

After the differentiation of (8) with respect to y we obtain

$$nW^{n-1}(x(y))\exp(-x(y))\frac{d}{dy}x(y) = w_{\boldsymbol{a}}(y). \tag{9}$$

Lemma 7 and (9) imply

$$(10)\qquad \lim_{y\to\infty} \frac{nW^{n-1}(x(y))\left(\frac{d}{dy}x(y)\right)\exp(y/b-x(y))}{y^{l-1}} = \frac{b^{m-2l}}{(l-1)!\prod_{\{b_j\neq b\}}(b-b_j)},$$

where $b, b_1, \ldots, b_m, l$ are defined in (3). Under the notation of Lemma 8, from Lemma 4, (10) and property (iii) of Section 1.3 it follows that

$$\varliminf_{y\to\infty} \frac{g_a(y)}{nG^{n-1}(x(y))g(x(y))\frac{d}{dy}x(y)}$$

$$= \varliminf_{y\to\infty} \frac{n!(l-1)!\left(\prod_{\{b_j\neq b\}}(b-b_j)\right)\exp\left(\int_{x(y)}^{y/b} v(s)ds\right)C_m(y, \boldsymbol{a})}{(n-m)!a_m b^{m-2l}y^{l-1}\exp(R(y/b_m)-R(y/b))}.$$

From (8) and from the proof of Lemma 6 it can be easily obtained

$$\lim_{y\to\infty} \frac{\exp(y/b-x(y))}{y^{l-1}} = \frac{b^{m-2l+1}}{n(l-1)!\prod_{\{b_j\neq b\}}(b-b_j)}.$$

Consequently,

$$\varliminf_{y\to\infty} \int_{x(y)}^{y/b} v(s)ds \geqslant 0.$$

Hence Lemma 8 completes the proof.

3. Tests based on normalized spacings. For the statistical problem of Section 2 we consider the following classes of tests:

$$\tilde{P}_\alpha^+ = \{\tilde{\varphi}_{\boldsymbol{a}}\colon \sup_{\lambda\leqslant\lambda_0} \mathrm{E}_{W_\lambda}\tilde{\varphi}_{\boldsymbol{a}}(X) = \alpha,\ \sum_1^n a_i > 0,\ \boldsymbol{a}\geqslant \boldsymbol{0}\},$$

$$\tilde{S}_\alpha^+ = \{\psi_i\colon \sup_{\lambda\leqslant\lambda_0} \mathrm{E}_{W_\lambda}\psi_i(X) = \alpha,\ i = 1, \ldots, n\},$$

where

$$\tilde{\varphi}_{\boldsymbol{a}}(X) = I\left(\sum_1^n a_i D_{i:n} > d_\alpha(\boldsymbol{a})\right) \quad\text{and}\quad \psi_i = \tilde{\varphi}_{(0,1_i)} \quad \text{for } i = 1, \ldots, n.$$

Obviously, $\tilde{S}_\alpha^+ \subset \tilde{P}_\alpha^+$. Moreover, if $\tilde{\varphi}_{\boldsymbol{a}} \in \tilde{P}_\alpha^+$, then

$$\tilde{\varphi}_{\boldsymbol{a}} = \tilde{\varphi}_{\boldsymbol{a}/\sum a_i}.$$

It is easy to note that the uniformly most powerful test $\tilde{\varphi}_{1/n}$ belongs to $\tilde{P}_\alpha^+$.

Under the assumptions of Section 1.3 let us consider the size-robustness function

$$\tilde{R}_{\boldsymbol{a}} = \sup_{F \in \tilde{\pi}_H(W)} (\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda} \tilde{\varphi}_{\boldsymbol{a}}(X)) - \inf_{F \in \tilde{\pi}_H(W)} (\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda} \tilde{\varphi}_{\boldsymbol{a}}(X)),$$

where $0 < \alpha < 1$, $\tilde{\varphi}_{\boldsymbol{a}} \in \tilde{P}_\alpha^+$. After some calculations we see from Lemma 3 that

$$(11) \qquad \tilde{R}_{\boldsymbol{a}} = \sup_{F \in \tilde{\pi}_H(W)} (\sup_{\lambda \leqslant \lambda_0} \mathrm{E}_{F_\lambda} \tilde{\varphi}_{\boldsymbol{a}}(X) - \alpha) = 1 - \alpha - \tilde{H}_{\boldsymbol{a}}[\tilde{W}_{\boldsymbol{a}}]^{-1}(1-\alpha).$$

We have the following

THEOREM 2. *Let* $0 < \alpha < 1$ *and* $\psi_i, \psi_j \in \tilde{S}_\alpha^+$. *If* $i < j$, *then*

$$\tilde{R}_{(0,1_j)} \leqslant \tilde{R}_{(0,1_i)}.$$

Proof. Note that the normalized spacings from the sample from W have the same distribution W. Thus the proof follows immediately from the fact that under the assumptions of Section 1.3 we have (see [1])

$$D_{n:n} \leqslant_{\mathrm{st}} \ldots \leqslant_{\mathrm{st}} D_{1:n},$$

where $D_{i:n}$ is the i-th normalized spacing from H.

We prove the following

THEOREM 3. *For every test* $\tilde{\varphi}_{\boldsymbol{a}}$, *where* $\boldsymbol{a} \neq (0, 1_n)$, *there exists* α_0, $0 < \alpha_0 < 1$, *such that for every* $0 < \alpha \leqslant \alpha_0$ *the following holds*:

If $\tilde{\varphi}_{\boldsymbol{a}}, \tilde{\varphi}_{(0,1_n)} \in \tilde{P}_\alpha^+$, *then* $\tilde{R}_{(0,1_n)} < \tilde{R}_{\boldsymbol{a}}$.

3.1. *Technical results.* Denote by $G_{k:n}$ the d.f. of the k-th order statistic from the sample of size n from G. We have

LEMMA 9. *Let G satisfy assumptions* (i)–(iii) *of Section* 1.3. *Let r denote the failure rate function of G, $v = 1 - r$ and $S(a_1) = 1$,*

$$S(\boldsymbol{a}) = \int_0^\infty \ldots \int_{u_{k-2}}^\infty \Big(\prod_1^{k-1} r(u_i)\Big)$$

$$\times \exp\Big(\sum_1^{k-1} (v_i/a_k - 1)u_i + (n-k+1) \int_0^{u_{k-1}} v(s)ds + \sum_1^{k-2} \int_0^{u_i} v(s)ds\Big) du_{k-1} \ldots du_1,$$

where $k \in \{2, \ldots, n\}$,

$$\boldsymbol{a} = (a_1, \ldots, a_k), \quad a_k > 0 \text{ and } a_j \geqslant 0,$$

$$v_j = (n-j+1)a_j - (n-j)a_{j+1} \quad \text{for } j = 1, \ldots, k-1.$$

Then

(i) *if* $a_j < a_k$ *for* $j = 1, \ldots, k-1$ *and* $\sum_1^{k-1} a_j > 0$, *then*

$$S(\boldsymbol{a}) > \frac{a_k^{k-1}}{\prod_1^{k-1} (a_k - a_i)} S(0, a_k),$$

(ii) $$\frac{n!}{(n-k+1)!}S(\mathbf{0}, a_k) = \int_0^\infty (\exp(-u)/\bar{G}(u))dG_{k-1:n}(u) \quad \text{for } k = 2, \ldots, n,$$

(iii) $$\int_0^\infty (\exp(-u)/\bar{G}(u))dG_{k-1:n}(u) > \int_0^\infty (\exp(-u)/\bar{G}(u))dG_{n-1:n}(u)$$

$$\text{for } k = 2, \ldots, n-1.$$

Proof. (i) If we put

$$z_i = (n-i+1)(a_k-a_i)(u_i-u_{i-1})/a_k$$

for $i = 1, \ldots, k-1$, where $u_0 = 0$, we get

$$S(\mathbf{a}) = \frac{a_k^{k-1}}{\prod_1^{k-1}(n-i+1)(a_k-a_i)} \int_0^\infty \ldots \int_0^\infty \Big(\prod_1^{k-1} r(l_i)\Big)$$

$$\times \exp\Big(-\sum_1^{k-1} z_i + (n-k+1)\int_0^{l_{k-1}} v(s)ds + \sum_1^{k-2}\int_0^{l_i} v(s)ds\Big)dz_{k-1}\ldots dz_1,$$

where

$$l_i = l_i(\mathbf{a}, z) = \sum_1^i \frac{a_k z_j}{(n-j+1)(a_k-a_j)}.$$

From the assumption it follows that

$$l_i(\mathbf{a}, z) \geqslant l_i(\mathbf{0}, a_k, z) \quad \text{for } i = 1, \ldots, k-2,\ z \geqslant \mathbf{0},$$

and

$$l_{k-1}(\mathbf{a}, z) > l_{k-1}(\mathbf{0}, a_k, z), \quad z > \mathbf{0}.$$

Consequently, by the properties of G the proof of this case is completed.

(ii) It is easy to note that

$$S(\mathbf{0}, a_k) = \int_0^\infty \ldots \int_{u_{k-2}}^\infty \exp(-u_{k-1})\bar{G}^{n-k}(u_{k-1})\prod_1^{k-1} g(u_i)du_{k-1}\ldots du_1.$$

Let us introduce the following notation:

$$P(u_j) = g(u_j)\int_{u_j}^\infty \ldots \int_{u_{k-2}}^\infty \exp(-u_{k-1})\bar{G}^{n-k}(u_{k-1})\prod_{j+1}^{k-1} g(u_i)du_{k-1}\ldots du_{j+1},$$

where $j = 2, \ldots, k-2$. Hence, after changing the order of integration, we get

$$S(\mathbf{0}, a_k) = \int_0^\infty\int_0^{u_2} P(u_2)g(u_1)du_1du_2 = \int_0^\infty P(u_2)G(u_2)du_2 = \ldots$$

$$= \frac{1}{(k-2)!}\int_0^\infty \exp(-u_{k-1})g(u_{k-1})\bar{G}^{n-k}(u_{k-1})G^{k-2}(u_{k-1})du_{k-1}.$$

Thus

$$\frac{n!}{(n-k+1)!}S(0, a_k) = \frac{n!}{(n-k+1)!(k-2)!} \times \int_0^\infty (\exp(-u)/\bar{G}(u))g(u)\bar{G}^{n-k+1}(u)G^{k-2}(u)du$$

$$= \int_0^\infty (\exp(-u)/\bar{G}(u))dG_{k-1:n}(u).$$

(iii) By the assumptions the function $\exp(-u)/\bar{G}(u)$ is nonincreasing. Thus the result follows from the well-known property of stochastic ordering and the fact that

$$\frac{2}{n}\Big(\int_0^\infty (\exp(-u)/\bar{G}(u))dG_{n-2:n}(u) - \int_0^\infty (\exp(-u)/\bar{G}(u))dG_{n-1:n}(u)\Big)$$

$$= (n-1)\int_0^\infty \exp(-u)g(u)G^{n-3}(u)\big((n-2)-nG(u)\big)du$$

$$= \int_0^\infty \exp(-u)G^{n-2}(u)\big((n-1)\bar{G}(u)-G(u)\big)du$$

$$> \int_0^\infty \exp(-u)G^{n-2}(u)\big((n-1)g(u)-G(u)\big)du = 0.$$

Let a and p be defined by (2) and $z = \min(j\colon a_j = a)$. Denote by $i_1, \ldots, i_j$ the natural numbers such that $1 \leqslant i_1 < \ldots < i_j \leqslant n$, $a_{i_k} > 0$ for $k = 1, \ldots, j$ and $a_i = 0$ for $i \neq i_1, \ldots, i_j$. Let us introduce the notation

$$\tilde{A}_l = \tilde{A}_l(y, \boldsymbol{u}, \boldsymbol{a}) = \begin{cases} \dfrac{y - \sum_1^l v_i u_i}{(n-l)a_{l+1}} & \text{if } l = i_1 - 1, \ldots, i_j - 1, \\ \infty & \text{if } l \neq i_1 - 1, \ldots, i_j - 1, \end{cases}$$

where $v_i = (n-i+1)a_i - (n-i)a_{i+1}$,

(12) $$V(y, \boldsymbol{a}) = y^{p-1}\exp\Big(y(1/a_{i_j} - 1/a) + \int_0^{y/a} v(s)ds\Big),$$

$$V_j(y, \boldsymbol{a}) = \frac{1}{(n-i_j+1)!a_{i_j}} \int_0^{\tilde{A}_0} \ldots \int_{u_{i_j-2}}^{\tilde{A}_{i_j-2}} \Big(\prod_1^{i_j-1} r(u_l)\Big) \times \exp\Big(\sum_1^{i_j-1}\Big((v_l/a_{i_l} - 1)u_l + \int_0^{u_l} v(s)ds\Big) + (n-i_j+1)\int_0^{\tilde{A}_{i_j-1}} v(s)ds\Big)d\boldsymbol{u}.$$

Under the notation of Lemma 9 we have

LEMMA 10. *If* $\boldsymbol{a} \neq (\boldsymbol{0}, 1_n)$, *then*

$$\varlimsup_{y\to\infty} \frac{V(y, \boldsymbol{a})}{V_j(y, \boldsymbol{a})} \leqslant \frac{(n-z+1)!(p-1)! \prod_{\{j>z, a_j\neq a\}} (a-a_j)}{a^{n-z+1-2p} S(a_1, \ldots, a_z)}$$

$$< \frac{n!(p-1)! \prod_{\{a_j\neq a\}} (a-a_j)}{a^{n-2p} \int_0^\infty (\exp(-u)/\bar{G}(u)) dG_{n-1:n}(u)}.$$

The proof of the first inequality is similar to that of Lemma 8 and is omitted. The second inequality follows immediately from Lemma 9.

Proof of Theorem 3. We will use the notation introduced in this section. Let $r = r_H$. Under the relation (11) for $0 < \alpha < 1$, $\tilde{R}_{(0,1_n)} < \tilde{R}_{\boldsymbol{a}}$ is equivalent to the inequality

$$\tilde{H}_{\boldsymbol{a}}[\tilde{W}_{\boldsymbol{a}}]^{-1}(1-\alpha) < \tilde{H}_{(0,1_n)}[W]^{-1}(1-\alpha).$$

Let $x = [W]^{-1}(1-\alpha)$ and $y = [\tilde{W}_{\boldsymbol{a}}]^{-1}(1-\alpha)$. Then

$$x = x(y) = -\ln\big(1-\tilde{W}_{\boldsymbol{a}}(y)\big), \quad \frac{d}{dy} x(y) = \exp\big(x(y)\big)\tilde{w}_{\boldsymbol{a}}(y). \tag{13}$$

Obviously, $\alpha \to 0+$ if and only if $y \to \infty$. Let the function b be defined as follows:

$$b(y) = \tilde{H}_{(0,1_n)}\big(x(y)\big) - \tilde{H}_{\boldsymbol{a}}(y).$$

It is easy to obtain

$$\tilde{H}_{(0,1_n)}\big(x(y)\big) = 1 - n(n-1) \int_0^\infty \bar{H}\big(x(y)+u\big)h(u)H^{n-2}(u)du,$$

$$\tilde{H}_{\boldsymbol{a}}(y) = n! \int_{C_{\boldsymbol{a}}} \ldots \int \prod_1^n h(u_i) du_n \ldots du_1,$$

where

$$C_{\boldsymbol{a}} = C_{\boldsymbol{a}}(y) = \Big\{(u_1, \ldots, u_n) \geqslant \boldsymbol{0}\colon u_{i-1} < u_i < \frac{y - \sum_1^{i-1} v_k u_k}{(n-i+1)a_i} \text{ if } i = i_1, \ldots, i_j,$$

$$u_{i-1} < u_i \text{ if } i \neq i_1, \ldots, i_j\Big\}.$$

The functions $\tilde{H}_{(0,1_n)}(x(y))$ and $\tilde{H}_a(y)$ are differentiable and their derivatives are given by the following formulas:

$$(14)\qquad \frac{d}{dy}\tilde{H}_{(0,1_n)}(x(y)) = \left(\frac{d}{dy}x(y)\right)n(n-1)\int_0^\infty h(x(y)+u)h(u)H^{n-2}(u)du,$$

$$\tilde{h}_a(y) = \frac{n!}{(n-i_j+1)a_{i_j}}\int_{\tilde{C}_a}\!\!\ldots\!\int h\left(\frac{y-\sum_1^{i_j-1} v_k u_k}{(n-i_j+1)a_{i_j}}\right)$$
$$\times \prod_{i\neq i_j} h(u_i)du_n\ldots du_{i_j+1}du_{i_j-1}\ldots du_1,$$

where

$$\tilde{C}_a = \tilde{C}_a(y) = \Bigg\{(u_1, \ldots, u_{i_j-1}, u_{i_j+1}, \ldots, u_n) \geqslant 0:$$

$$u_{i-1} < u_i < \frac{y-\sum_1^{i-1} v_k u_k}{(n-i+1)a_i} \text{ if } i = i_1, \ldots, i_{j-1},$$

$$u_{i-1} < u_i \text{ if } i \neq i_1, \ldots, i_{j-1}, i_j+1, \frac{y-\sum_1^{i_j-1} v_k u_k}{(n-i_j+1)a_{i_j}} < u_{i_j+1}\Bigg\}.$$

If we denote by K the function

$$K(y) = \int_0^\infty r(u)\exp(-2u+\int_0^{u+x(y)} v(s)ds+\int_0^u v(s)ds)$$
$$\times H^{n-2}(u)\exp(-\int_0^{y/a} v(s)ds)du,$$

we get

$$(15)\qquad \frac{d}{dy}\tilde{H}_{(0,1_n)}(x(y)) \leqslant n(n-1)\,\tilde{w}_a(y)K(y)\exp(\int_0^{y/a} v(s)ds),$$

$$\tilde{h}_a(y) \geqslant n!r(cy)\exp(-y/a_{i_j})V_j(y, \boldsymbol{a}),$$

where

$$c = 1/\max_{1\leqslant k\leqslant j}((n-i_k+1)a_{i_k}).$$

This follows from (14), the monotonicity of r and from the fact that $r \leqslant 1$. From (13) and Lemma 6 we obtain

$$\lim_{y\to\infty} \exp(y/a-x(y)) > 1.$$

Thus there exists $y_0 > 0$ such that $y/a > x(y)$ for every $y > y_0$. For $y > y_0$ let us define the functions K_1 and K_2 as follows:

$$K_1(y) = \int_0^{y/a-x(y)} r(u)\exp\Big(-2u+\int_0^u v(s)ds\Big)H^{n-2}(u)du,$$

$$K_2(y) = \int_{y/a-x(y)}^{\infty} r(u)\exp\Big(-2u+v(y/a)u+\int_0^u v(s)ds\Big)H^{n-2}(u)du.$$

It is easy to note that

(16) $$K(y) < K_1(y)+K_2(y) \quad \text{for } y > y_0$$

and

$$\lim_{y\to\infty}\big(K_1(y)+K_2(y)\big) = \frac{1}{(n-1)n}\int_0^{\infty}\big(\exp(-u)/\bar{H}(u)\big)dH_{n-1:n}(u).$$

From (15) we get

$$\frac{d}{dy}b(y) \leqslant \tilde{h}_a(y)\left(\frac{\tilde{w}_a(y)\exp(y/a)K(y)V(y,a)}{(n-2)!\,y^{p-1}r(cy)V_j(y,a)}-1\right).$$

Moreover, from (16), Lemmas 6, 9 and 10 we obtain

$$\overline{\lim_{y\to\infty}}\frac{\tilde{w}_a(y)\exp(y/a)K(y)V(y,a)}{(n-2)!\,y^{p-1}r(cy)V_j(y,a)} < 1.$$

Thus there exists $y_1 > 0$ such that

$$\frac{d}{dy}b(y) < 0 \quad \text{for every } y > y_1.$$

Finally, since $\lim_{y\to\infty} b(y) = 0$, there exists $\tilde{y}_1 > 0$ such that

$$b(y) > 0 \quad \text{for every } y \geqslant \tilde{y}_1.$$

This completes the proof.

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