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APPROXIMATION OF ELLIPTIC INTEGRALS OF THE THIRD KIND FOR LARGE VALUES OF ARGUMENT AND MODULUS

1. Introductory remarks. Elliptic integrals of the third kind appear in many technical applications, e.g. in problems of carrying capacity of cylinders subject to combined loadings [9], and in problems of optimization of cylindrical shells with respect to their stability [11]. They also appear in such a classical problem as bending of slightly curved rods subject to normal pressure [6].

Approximation and tabulation of these integrals, which can be reduced to the function

$$\pi(\varphi, n, k) = \int_0^{\varphi} \frac{d\psi}{(1+n\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}, \tag{1}$$

is a much more difficult problem than approximation and tabulation of integrals of the first and second kind, since π is a function of three variables. A certain computational algorithm using repeated Landen's transformations was proposed by Fettis [3] and extended by Neuman [7] to the particular case $n = -k^2$. However, function (1) shows singularities along the line $\varphi = \pi/2$, $k = 1$ (similarly to the integral of the first kind), furthermore along the line $\varphi = \pi/2$, $n = -1$ and on the surface $n = -1/\sin^2\varphi$. These singularities bring additional complications both to the analysis of behaviour of the function and to the computational methods.

The purpose of the present paper is to derive approximation formulae for $\pi(\varphi, n, k)$ in domains close to the above-mentioned singularities and to describe analytically the behaviour of that function in the vicinity of the singularities. In particular, we investigate the regions of argument φ close to $\pi/2$, of modulus k close to 1 and, additionally, for the parameter n close to -1. Attempts of derivation of formulae of this type initiated by Hamel [5] were extensively continued by Radon [8]. The authoress [8] has given several expansions into power series with respect to variously chosen arguments. We are interested only in series for the argument φ close to $\pi/2$ and simultaneously for the modulus k close to 1, and hence in series (19), (20), (21) [8] whose arguments are

$$k' = \sqrt{1-k^2}, \quad \xi = k^2+n, \quad \zeta = \sqrt{1-k^2\sin^2\varphi}-\cos\varphi,$$

respectively.

The practical limitations of applicability of series (19) and (20) given by Radon can generally be characterized as follows: any increase of argument must cause a relevant increase of modulus. Additional limitations for negative n cause further decrease of the admissible region. However, the domain of convergence of series (20) does not depend on the value of argument and in the plane (k^2, n) has the shape of a parallelogram.

To illustrate the complicated limitation forms, the domains of convergence for two chosen values of argument $\varphi = 85°$ (Fig. 1) and $\varphi = 80°$ (Fig. 2) are shown in Figs. 1 and 2. In both figures the axes are broken in order to show more precisely the domains we are interested in.

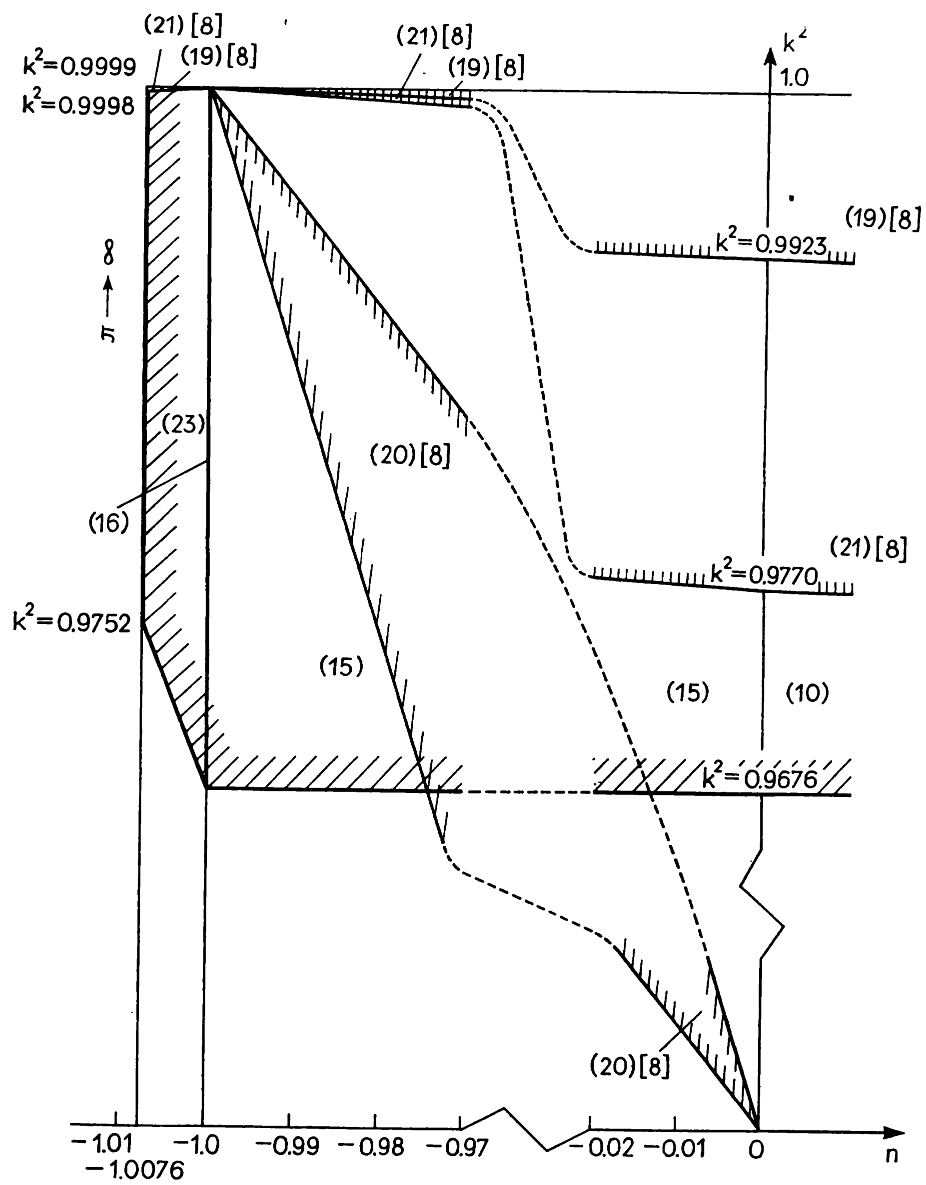

Fig. 1. $\varphi = 85°$

Series (19) and (20) do not describe the singularity $\pi \to \infty$ which occurs on the surface $n = -1/\sin^2\varphi$ and which results in an essential limitation of their applicability. For $n \to -1$ the domains of convergence decrease so that for $n = -1$ the value of the integral can be given only for the modulus $k = 1$.

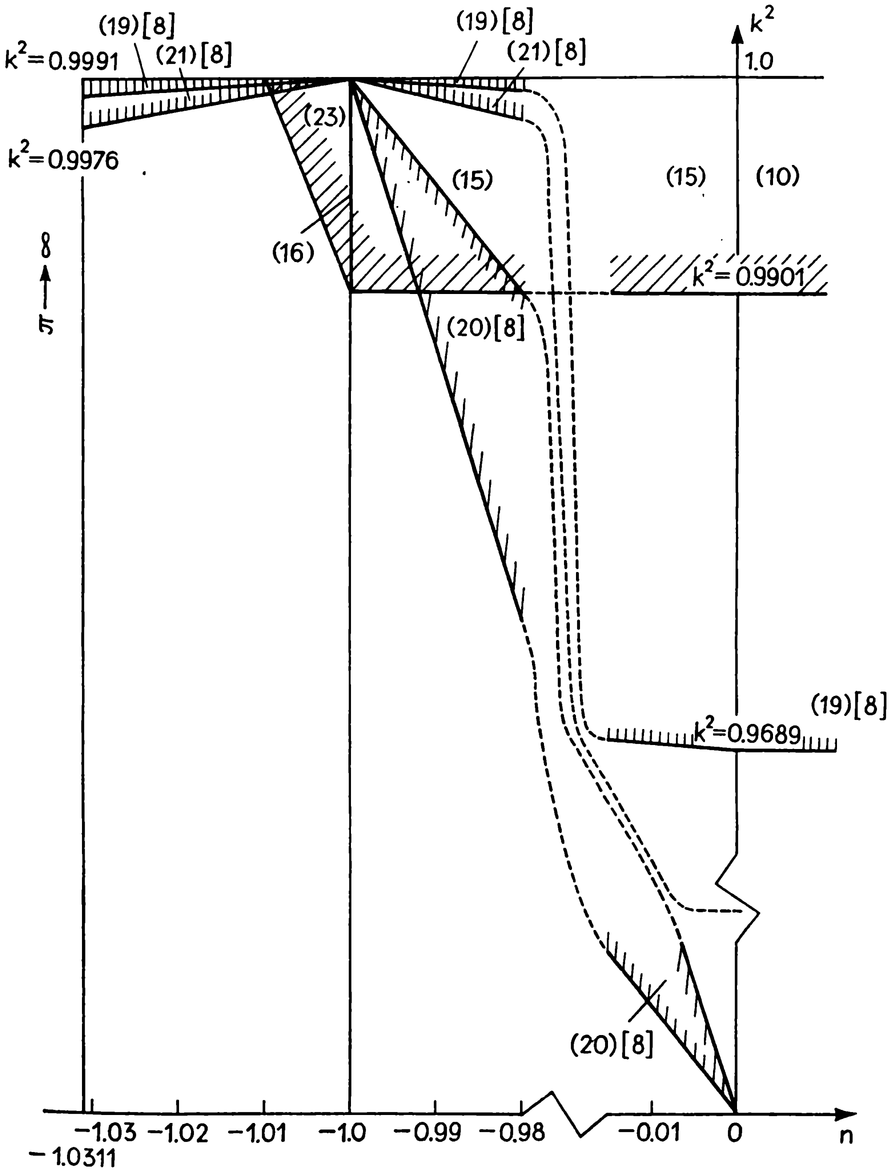

Fig. 2. $\varphi = 80°$

Figs. 1 and 2 show that the neighbourhood of the plane $n = -1$ is practically not described by Radon's series. The purpose of the present paper is to propose approximation formulae valid in this domain.

2. Suggested approximation formulae. The idea of deriving the proposed formulae which approximate elliptic integrals of the third kind is based on introducing such a small parameter ε to the integral that certain elements of the integrand could be neglected. This operation allows us to integrate effectively a simplified function and to obtain, in this way, an approximate solution.

The definition of the elliptic integral in Jacobi's form is applied here, namely

$$\pi(y, n, k) = \int_0^y \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2}$$

(substitution $x = \sin\varphi$ leads to the integral in Legendre's form (1)).

For further considerations it is more convenient to introduce into (2) the new variable $u = \sqrt{1-k^2x^2}$. Then we have

$$\pi = \int_{\sqrt{1-k^2y^2}}^{1} \frac{du}{[1+(n/k^2)(1-u^2)]\sqrt{(k^2-1+u^2)(1-u^2)}}. \tag{3}$$

From now on we use the quantities $k' = \sqrt{1-k^2}$ (complementary modulus) and $y' = \sqrt{1-y^2}$. Within the range we are interested in, they are small and are expressed by a small parameter in which we include furthermore (in one of the cases under consideration) the expression $1+n$. A relatively large range can be described by introducing a small parameter ε defined by the formulae

$$\begin{aligned} k' &= \varepsilon\cos\alpha, \\ y' &= \varepsilon\sin\alpha, \end{aligned} \qquad \varepsilon = \sqrt{k'^2+y'^2}. \tag{4}$$

Substituting (4) into (3) we obtain

$$\pi = \int_{\varepsilon z}^{1} \frac{du}{\left[1+\dfrac{n}{1-\varepsilon^2\cos^2\alpha}(1-u^2)\right]\sqrt{(1-u^2)(u^2-\varepsilon^2\cos^2\alpha)}}, \tag{5}$$

where $z = \sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}$.

Since integral (5) contains a singularity, we use here a method analogical to that applied in [10] for integrals of the first kind, namely, we split integral (5) into a sum of two integrals over appropriate intervals:

$$\begin{aligned} \pi = &\int_{\varepsilon z}^{\sqrt{\varepsilon z}} \frac{du}{\left[1+\dfrac{n}{1-\varepsilon^2\cos^2\alpha}(1-u^2)\right]\sqrt{(1-u^2)(u^2-\varepsilon^2\cos^2\alpha)}} + \\ &+ \int_{\sqrt{\varepsilon z}}^{1} \frac{du}{\left[1+\dfrac{n}{1-\varepsilon^2\cos^2\alpha}(1-u^2)\right]\sqrt{(1-u^2)(u^2-\varepsilon^2\cos^2\alpha)}}. \end{aligned} \tag{6}$$

Substitution $u = \varepsilon w \cos\alpha$ into the first integral in (6) and $u = \sqrt{1-\xi^2}$ into the second one allows us to analyze small quantities (and their appropriate neglecting):

(7)

$$\pi = \int\limits_{z/\cos\alpha}^{\sqrt{z}/\sqrt{\varepsilon}\cos\alpha} \frac{dw}{\left[1+\frac{n}{1-\varepsilon^2\cos^2\alpha}(1-\varepsilon^2\cos^2\alpha w^2)\right]\sqrt{(1-\varepsilon^2\cos^2\alpha w^2)(w^2-1)}} +$$

$$+\int\limits_0^{\sqrt{1-\varepsilon z}} \frac{d\xi}{\left(1+\frac{n}{1-\varepsilon^2\cos^2\alpha}\xi^2\right)\sqrt{(1-\xi^2)(1-\xi^2-\varepsilon^2\cos^2\alpha)}}.$$

From now on we denote the first integral of (7) by I and the second one by II.

In view of the form of the integrals [4] two cases, $n \geqslant 0$ and $n \leqslant 0$, should be considered separately; the second one will be subject to further subdivision.

A. $n \geqslant 0$.

The integral I, after appropriate neglecting small quantities, is of the following form and the result of integration yields the right-hand side of this relation:

(8)
$$I \cong \int\limits_{z/\cos\alpha}^{\sqrt{z}/\sqrt{\varepsilon}\cos\alpha} \frac{dw}{(1+n)\sqrt{w^2-1}}$$

$$= \frac{1}{1+n}\ln\frac{\sqrt[4]{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}+\sqrt{\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}-\varepsilon\cos^2\alpha}}{\sqrt{\varepsilon}\left(\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}+\sin\alpha\sqrt{1-\varepsilon^2\cos^2\alpha}\right)}.$$

The integral II may be approximated by

(9)
$$II \cong \int\limits_0^{\sqrt{1-\varepsilon z}} \frac{d\xi}{(1-\xi^2)(1+n\xi^2)}$$

$$= \frac{1}{2(1+n)}\left[\ln\frac{1+\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}}{1-\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}}+\right.$$

$$\left.+2\sqrt{n}\,\mathrm{arc\,tg}\left(\sqrt{n}\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\right)\right].$$

The sum of (8) and (9), after neglecting small quantities of order ε^2 and after substitution of (4), furnishes an approximate value of the elliptic integral of the third kind:

$$(10)\qquad \pi(y, n, k) \cong \frac{1}{1+n} \times$$
$$\times \left\{ \ln \frac{4-k'^2/\sqrt{y'^2+k'^2}-\frac{1}{2}\sqrt{y'^2+k'^2}}{y'+\sqrt{y'^2+k'^2}} + \sqrt{n}\operatorname{arctg}\left[\sqrt{n}\left(1-\frac{\sqrt{y'^2+k'^2}}{2}\right)\right]\right\}.$$

Substituting $n = 0$ into (10) we obtain an approximate value of the elliptic integral of the first kind for large values of argument and modulus.

B. $-1 \leqslant n \leqslant 0$.

In this case, neglecting the same quantities as for $n > 0$ in the integrand of the integral I leads to results of poor accuracy and for $n \to -1$ the boundary value would be erroneous. Namely, we would have disregarded certain expressions with respect to quantities of the same order ($\varepsilon^2 w \cos^2\alpha$ with respect to $1+n$) and then the integral I for $n = -1$ would be equal to ∞ independently of the values of argument and modulus.

It has been decided here to consider the integrand more precisely. The expression which occurs in the square brackets of formula (7), after transformations and permissible omissions, takes the form

$$1+\frac{n}{1-\varepsilon^2\cos^2\alpha}(1-\varepsilon^2\cos^2\alpha\, w^2) \cong 1+n-n\varepsilon^2\cos^2\alpha(w^2-1),$$

and the integral after neglecting the quantity $\varepsilon^2\cos^2\alpha\, w$ appearing under the root is as follows:

$$(11)\qquad I \cong -\frac{1}{n\varepsilon^2\cos^2\alpha}\int\limits_{z/\cos\alpha}^{\sqrt{z}/\sqrt{\varepsilon}\cos\alpha} \frac{dw}{\left[\dfrac{1+n}{-n\varepsilon^2\cos^2\alpha}+w^2-1\right]\sqrt{w^2-1}}.$$

The form of the solution of integral (11) depends on the value n, therefore different results of integration for two subranges are the following:

for

$$-\frac{1}{1+\varepsilon^2\cos^2\alpha} \leqslant n \leqslant 0$$

we have

$$(12)\qquad I \cong \frac{1}{2\sqrt{(1+n)[n(1+\varepsilon^2\cos^2\alpha)+1]}} \times$$
$$\left(\ln \frac{\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\sin^2\alpha}-\varepsilon\cos^2\alpha}+\sqrt{\dfrac{n+1}{n(1+\varepsilon^2\cos^2\alpha)+1}}\sqrt[4]{1-\varepsilon^2\cos^2\alpha\sin^2\alpha}}{\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\sin^2\alpha}-\varepsilon\cos^2\alpha}-\sqrt{\dfrac{n+1}{n(1+\varepsilon^2\cos^2\alpha)+1}}\sqrt[4]{1-\varepsilon^2\cos^2\alpha\sin^2\alpha}} \right.$$

$$+\ln\frac{\sqrt{1-\varepsilon^2\cos^2\alpha}\sin\alpha-\sqrt{\dfrac{n+1}{n(1+\varepsilon^2\cos^2\alpha)+1}}\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}{\sqrt{1-\varepsilon^2\cos^2\alpha}\sin\alpha+\sqrt{\dfrac{n+1}{n(1+\varepsilon^2\cos^2\alpha)+1}}\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\Bigg);$$

for

$$-1<n\leqslant-\frac{1}{1+\varepsilon^2\cos^2\alpha}$$

we have

$$(13)\qquad I\cong\frac{1}{\sqrt{-(1+n)[n(1+\varepsilon^2\cos^2\alpha)+1]}}\times$$

$$\times\Bigg(\operatorname{arctg}\sqrt{\frac{n+1}{-[n(1+\varepsilon^2\cos^2\alpha)+1]}}\,\frac{\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}{\sin\alpha\sqrt{1-\varepsilon^2\cos^2\alpha}}-$$

$$-\operatorname{arctg}\sqrt{\frac{n+1}{-[n(1+\varepsilon^2\cos^2\alpha)+1]}}\,\frac{\sqrt[4]{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}{\sqrt{\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}-\varepsilon\cos^2\alpha}}\Bigg).$$

In the limit case

$$n=-\frac{1}{1+\varepsilon^2\cos^2\alpha},$$

formulae (12) and (13) lead to the same results. In this case the form of the integral II is the same as for $n>0$ but in view of negative n the solution is different:

$$(14)\qquad II\cong\frac{1}{2(1+n)}\times$$

$$\times\ln\frac{\left(1+\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\right)\left(1-\sqrt{-n}\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\right)^{\sqrt{-n}}}{\left(1-\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\right)\left(1+\sqrt{-n}\sqrt{1-\varepsilon\sqrt{1-\varepsilon^2\sin^2\alpha\cos^2\alpha}}\right)^{\sqrt{-n}}},$$

$$-1<n\leqslant 0.$$

Neglecting, like previously, small quantities of order ε^2 and substituting (4) we obtain the formulae for the integral of the third kind (for $n=-1$ an appropriate limiting procedure should be applied; the result is used in formula (16)). Since the expression $\varepsilon^2\cos^2\alpha$ appears in the upper bound of variability of n, formula (12) after permissible neglections includes the whole range of the considered n and the left boundary of this interval requires a strong inequality. Formula (16) holds only for the limit case $n=-1$.

Thus for $-1 < n \leqslant 0$ we have

$$\pi(y, n, k) \cong \frac{1}{2(1+n)} \ln \frac{16\left(1-k'^2/2\sqrt{y'^2+k'^2}\right)\left(1-\left(\sqrt{y'^2+k'^2}\right)/4\right)}{(y'^2+k'^2)\left(1+y'/\sqrt{y'^2+k'^2}\right)^2} + \frac{\sqrt{-n}}{2(1+n)} \ln \frac{1-\sqrt{-n}+\left(\sqrt{-n}/2\right)\sqrt{y'^2+k'^2}}{1+\sqrt{-n}\left(1-\left(\sqrt{y'^2+k'^2}\right)/2\right)}, \tag{15}$$

and for $n = -1$ we obtain

$$\pi(y, -1, k) \cong \frac{1}{k'^2}\left(\frac{\sqrt{k'^2+y'^2}}{y'}-1\right) - \frac{1}{4}\left(1-\frac{4-\sqrt{y'^2+k'^2}}{\sqrt{y'^2+k'^2}}\right). \tag{16}$$

C. $n < -1$.

The considerations are limited here to the values of n slightly smaller than -1; this limitation is motivated by the fact that for $n = -1/\sin^2\varphi$ the integral increases infinitely.

The definitions of k' and y' are left without change whereas another small quantity $n' = -(1+n)$ is introduced. A new small parameter ε denoted by the same symbol as before (its definition is changed) describes those quantities as follows:

$$\left.\begin{aligned} k' &= \varepsilon\cos\alpha\cos\beta, \\ y' &= \varepsilon\cos\alpha\sin\beta, \\ n' &= \varepsilon^2\sin^2\alpha, \end{aligned}\right\} \qquad \varepsilon = \sqrt{k'^2+y'^2+n'}. \tag{17}$$

Introducing equations (17) into integral (3) and splitting it into a sum of two integrals $(I+II)$ we obtain

(18)

$$\pi = \int_{\varepsilon v\cos\alpha}^{\sqrt{\varepsilon v\cos\alpha}} \frac{du}{\left[1-\dfrac{1+\varepsilon^2\sin^2\alpha}{1-\varepsilon^2\cos^2\alpha\cos^2\beta}(1-u^2)\right]\sqrt{(1-u^2)(u^2-\varepsilon^2\cos^2\alpha\sin^2\beta)}} + \int_{\sqrt{\varepsilon v\cos\alpha}}^{1} \frac{du}{\left[1-\dfrac{1+\varepsilon^2\sin^2\alpha}{1-\varepsilon^2\cos^2\alpha\cos^2\beta}(1-u^2)\right]\sqrt{(1-u^2)(u^2-\varepsilon^2\cos^2\alpha\sin^2\beta)}},$$

where $v = \sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}$.

The substitution of $u = \varepsilon w\cos\alpha\cos\beta$ into the first integral of (18) leads to a more convenient form of the integrand and, furthermore, to an easier analysis of small quantities. The second integral is left without change. Thus we have

(19) $$I = \int_{v/\cos\beta}^{\sqrt{v/\cos\beta}\sqrt{\varepsilon\cos\alpha}} \frac{1}{\left[1-\frac{1+\varepsilon^2\sin^2\alpha}{1-\varepsilon^2\cos^2\alpha\cos^2\beta}(1-\varepsilon^2\cos^2\alpha\cos^2\beta w^2)\right]}\times$$
$$\times\frac{dw}{\sqrt{(w^2-1)(1-\varepsilon^2\cos^2\alpha\cos^2\beta w^2)}}.$$

The expression in the square brackets of integral (19), after transformations and permissible neglections, takes the form

$$1-\frac{1+\varepsilon^2\sin^2\alpha}{1-\varepsilon^2\cos^2\alpha\cos^2\beta}(1-\varepsilon^2\cos^2\alpha\cos^2\beta w^2)$$
$$\cong \varepsilon^2[(w^2-1)\cos^2\alpha\cos^2\beta-\sin^2\alpha],$$

and the integral, after neglecting $\varepsilon^2 w^2\cos^2\alpha\cos^2\beta$, is determined by

(20) $$I\cong\frac{1}{\varepsilon^2\cos^2\alpha\cos^2\beta}\int_{v/\cos\beta}^{\sqrt{v/\cos\beta}\sqrt{\varepsilon\cos\alpha}}\frac{dw}{\left(w^2-1-\frac{\sin^2\alpha}{\cos^2\alpha\cos^2\beta}\right)\sqrt{w^2-1}}.$$

In the integral II there are different permissible neglections transforming it to the form

(21) $$II\cong\int_{\sqrt{\varepsilon v\cos\alpha}}^{1}\frac{du}{u^3\sqrt{1-u^2}}.$$

As a result of integration I and II we obtain an approximate value of the integral of the third kind

(22) $$\pi\cong\frac{1}{2\varepsilon^2\sin^2\alpha A}\times$$
$$\ln\frac{A\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}-\varepsilon\cos^2\beta\cos^2\alpha}-\sqrt[4]{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}\sin\alpha}{A\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}-\varepsilon\cos^2\beta\cos^2\alpha}+\sqrt[4]{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}\sin\alpha}\,\frac{A\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}-\cos^2\beta}-\sqrt[4]{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}\sin\alpha}{A\sqrt{\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}-\cos^2\beta}+\sqrt[4]{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}\sin\alpha}$$
$$+\frac{\sqrt{1-\varepsilon\cos\alpha\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}}}{2\varepsilon\cos\alpha\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}}+$$
$$+\frac{1}{4}\ln\frac{1+\sqrt{1-\varepsilon\cos\alpha\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}}}{1-\sqrt{1-\varepsilon\cos\alpha\sqrt{1-\varepsilon^2\cos^2\alpha\cos^2\beta\sin^2\beta}}},$$

where $A = \sqrt{\cos^2\alpha\cos^2\beta+\sin^2\alpha}$. After permissible neglections of small quantities of order ε^2 and after introduction of (17), expression (22) takes the following form:

$$\pi(y,n,k) \cong \frac{1}{\sqrt{n'(k'^2+n')}} \times \tag{23}$$

$$\times \ln \frac{[y'\sqrt{k'^2+n'}+\sqrt{n'(k'^2+y'^2)}]\sqrt{1-\dfrac{k'^2+n'}{\sqrt{k'^2+y'^2}}\left(1-\dfrac{k'^2}{4\sqrt{k'^2+y'^2}}\right)}}{\left[\sqrt{k'^2+n'}\left(1-\dfrac{k'^2}{2\sqrt{k'^2+y'^2}}\right)+\sqrt{n'}\right]\sqrt{y'^2-n'}}+$$

$$+\frac{1}{2\sqrt{k'^2+y'^2}}-\frac{1}{4}\left(1-\ln\frac{4-\sqrt{k'^2+y'^2}}{\sqrt{k'^2+y'^2}}\right).$$

The expression y'^2-n' which occurs in the denominator of the logarithm in (23) should not be negative, whence $n > -2+y^2$. This limitation can be transformed (for large values of the argument) to the previously discussed one for $n > -1/\sin^2\varphi$.

Formulae (10), (15), (16), and (23) give a complete set describing the singular behaviour of the integral of the third kind within the whole interval $-1/y^2 < n < \infty$. The accuracy of those formulae increases when argument and modulus increase.

Assuming that the maximal value neglected is constant and equals $\varepsilon^2 = V = \text{const}$ for formulae (10), (15), and (16) we obtain a relation between k' and y' simultaneously limiting their values:

$$k'^2+y'^2 < V. \tag{24}$$

For $n < -1$, limitation (24) may be replaced by

$$k'^2+y'^2+n' < V, \tag{25}$$

combining three parameters.

Limitations (24) and (25) show that an increase of the argument y causes an increase of the interval of the admissible modulus k and conversely.

Figs. 1 and 2, for $V = 0.04$, present the domains of applicability of the given formulae. The boundaries of the domains are marked off by diagonal lines.

3. Example. As an illustration of the accuracy of the formulae proposed, a numerical example is given. Values of the argument $y = 0.996190$ ($\varphi = 85°$) and of the modulus $k^2 = 0.99$ are chosen in such a way that the value of the integral can directly be taken from the tables [1].

The computation for $n = -1.001$ was done by using a transformation formula from [2], p. 13, since the tabulated values are given only for $-1 \leqslant n \leqslant +100$.

The results of calculations are presented in Table 1 together with the result computed by using Radon's series (the zeroth, first and second terms were taken into account). For comparison also the tabulated values are given. A much better accuracy of the formulae derived in the present paper is seen.

TABLE 1

$n = 2$	tables [1]	1.415945
	(10)	1.402030
	(19) [8]	1.149219
	(21) [8]	1.395307
$n = -0.5$	tables [1]	4.590004
	(15)	4.628121
	(19) [8]	4.715727
	(21) [8]	4.520406
$n = -1$	tables [1]	53.571850
	(16)	53.050842
	(19) [8]	∞
	(21) [8]	∞
$n = -1.001$	tables [1]	56.931351
	(23)	56.387464
	(19) [8]	63.864221
	(21) [8]	52.053862

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APROKSYMACJA CAŁEK ELIPTYCZNYCH TRZECIEGO RODZAJU DLA DUŻYCH WARTOŚCI ARGUMENTU I MODUŁU

STRESZCZENIE

W pracy podano wzory aproksymujące całki eliptyczne trzeciego rodzaju w pobliżu osobliwości, mianowicie dla wartości argumentu φ bliskich $\pi/2$, modułu k bliskich 1, a ponadto parametru n bliskich -1.

Metoda wyprowadzenia tych wzorów polega na takim wprowadzeniu małego parametru ε do całki, by pewne człony funkcji podcałkowej można było zaniedbać. Operacja ta pozwala na efektywne scałkowanie uproszczonej funkcji i otrzymanie tą drogą rozwiązań przybliżonych.

Podano również przykład liczbowy, a uzyskane wyniki porównano z wynikami otrzymanymi z szeregów Radon [8] i z wartościami tablicowymi [1] całek trzeciego rodzaju; uzyskana dokładność jest wyraźnie większa w całym badanym zakresie.
