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MAXIMIZATION OF A LINEAR FORM OVER A CONVEX HULL OF VERTICES OF A CONVEX POLYHEDRAL SET

1. Introduction. We deal with a solution method for the problem

$$\max\{f(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X_p\}, \tag{1.1}$$

where X_p is the convex hull of all vertices of the convex polyhedral set

$$X = \{\boldsymbol{x} \in R^n \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geqslant \boldsymbol{0}\}, \tag{1.2}$$

$\boldsymbol{c} \in R^n$, $\boldsymbol{b} \in R^m$, and A is an $(m \times n)$-matrix.

We assume that the set X is nonempty and, consequently, problem (1.1) is consistent.

Clearly, if the function $f(\boldsymbol{x})$ is bounded from above on the set X, we obtain an optimal solution of problem (1.1) applying the simplex method to the following problem:

$$\max\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in X\}. \tag{1.3}$$

However, if the function $f(\boldsymbol{x})$ is unbounded from above on X, a solution method of problem (1.1) becomes complicated.

Theoretically, by the Weierstrass theorem, there exists an optimal solution of problem (1.1) since the set X_p is compact and could be described by a system of linear inequalities. However, in practice, the construction of such a system would require finding all vertices of the set X, which is very troublesome or even impossible; effective methods for finding all vertices of X are not known.

In this paper a solution method for problem (1.1) is presented for the case where the function $f(\boldsymbol{x})$ is unbounded from above on X. The method, which was inspired by the algorithms from [1] and [2], is some adaptation of the algorithm described in [3] for ranking the vertices of the set X in the nonincreasing order of the values of the linear form $f(\boldsymbol{x})$, being bounded from above on X. It is based on introducing to the constraints from (1.2) the additional constraint

$$\boldsymbol{t}^{\mathrm{T}}\boldsymbol{x} \leqslant M, \tag{1.4}$$

where t is some vector fixed according to (3.4), and M is a number so great that each vertex of the set X satisfies condition (1.4) as a strict inequality.

It should be stressed that the number M does not need to be specified and the set

$$X(M) = \{x \in X \mid t^T x \leqslant M, \ M \geqslant 0\}$$

does not have to be bounded. But the function $f(x)$ is bounded from above on $X(M)$.

In order to get an optimal solution of problem (1.1) we apply the above-mentioned method [3] for ranking vertices of the set $X(M)$ till we obtain the first vertex which does not satisfy condition (1.4) as an equality. The latest vertex is an optimal solution of (1.1).

Continuing the above procedure we obtain the sequence X^* of all vertices of the set X according to nonincreasing values of $f(x)$. This enables us to propose a general method for ranking the vertices of X even in the case where the function $f(x)$ in unbounded from above on X.

Another method for generating the set X^* under the assumption that $f(x)$ is bounded from above on X is presented in [4]. It consists in a construction of special hyperplanes to cut off successively the ranked vertices. Each hyperplane passes through vertices adjacent to the vertex which is to be cut off and it is parallel to all infinite edges emanating from that vertex. However, in order to apply the latter method to solving problem (1.1), one has to specify the number M, which is troublesome.

The idea of the Murty method [3] is described in Section 2.

The procedure of the method for solving problem (1.1) in the case where the function $f(x)$ is unbounded from above on the set X is given in Section 3.

The algorithm is illustrated by numerical examples in Section 4.

2. An idea of the Murty method. Let us suppose that there exists an optimal solution of problem (1.3) (i.e., the function $f(x)$ is bounded from above on the set X defined by (1.2)) and $\operatorname{rank} A = m \leqslant n$.

Let $\alpha = \{j_1, j_2, \ldots, j_m\}$ and $\xi = \{1, 2, \ldots, n\} - \alpha$ denote sets of indices of basic and nonbasic variables, respectively, corresponding to the basis $A_\alpha = [A_{j_1}, A_{j_2}, \ldots, A_{j_m}]$ constructed from linearly independent columns of the matrix A, spanning R^m. Similarly, x_α and x_ξ denote subvectors of the vector x consisting of coordinates with indices belonging to the sets α and ξ, respectively, and A_ξ denotes a submatrix of A formed from the columns of A with indices belonging to the set ξ.

Denoting by $x^\alpha \in X$ the basic feasible solution relative to the basis A_α, we can write the system of linear equations from (1.2) in the equivalent canonical form

(2.1)
$$x_a + E_\xi x_\xi = x_a^a,$$

where $x_a^a = A_a^{-1} b$ and $E_\xi = A_a^{-1} A_\xi$.

Furthermore, we introduce the following notation for $j \in \xi$:

d_j^a – the relative optimality coefficient of the nonbasic variable x_j (i.e., $d_j^a = c_j - c_a^T E_j$, where E_j is the j-th column of the matrix E_ξ from (2.1));

x^{a^j} – the adjacent basic feasible solution of x^a with x_j as a basic variable (i.e., $a^j - a = \{j\}$);

v_j^a – the value of x_j in x^{a^j}.

Thus, obviously,

$$f(x^{a^j}) = f(x^a) + v_j^a d_j^a \quad \text{for } j \in \xi.$$

Let $F(x^a)$ denote the set of all adjacent vertices of x^a such that $f(x) \leqslant f(x^a)$ for each $x \in F(x^a)$. It can be defined as

(2.2)
$$F(x^a) = \bigcup_{p=1}^{r} \{x^{a_p^j} \mid j \in \xi_p,\ d_j^{a_p} \leqslant 0\},$$

where $a_1 = a, a_2, \ldots, a_r$ are sets of indices of basic variables such that $x^{a_p} = x^a$ for $p \in \{2, \ldots, r\}$.

It should be noticed that $r > 1$ whenever x^a is degenerate, and in the case where x^a is nondegenerate (i.e., $v_j^a > 0$ for all $j \in \xi$) the set (2.2) is of the form

$$F(x^a) = \{x^{a^j} \mid j \in \xi,\ d_j^a \leqslant 0\},$$

so it consists only of the adjacent basic feasible solutions of x^a.

Denoting by X^* the sequence of all vertices of the set X according to nonincreasing values of the function $f(x)$, we can state the following theorem proved in [3]:

THEOREM 1. *If $x^1, x^2, \ldots, x^{k-1} \in X^*$ are known, then the vertex x^k, which follows x^{k-1}, satisfies*

(2.3)
$$f(x^k) = \max\Big\{f(x) \mid x \in \bigcup_{i=1}^{k-1} F(x^i) - \{x^1, \ldots, x^{k-1}\}\Big\},$$

where $F(x^i)$ is defined by (2.2).

This theorem enables us to generate successively the elements of the sequence X^*.

3. A solution method of problem (1.1). We consider the case of problem (1.1) where the function $f(x)$ is unbounded from above on the set X.

Let $x^a \in X$ be the basic feasible solution of problem (1.3) corresponding to the canonical form (2.1).

THEOREM 2. *If $E_\xi \leqslant [0]$ (i.e., all elements of the matrix E_ξ appearing in* (2.1) *are nonpositive), then x^a is an optimal solution of problem* (1.1).

Proof. Suppose $E_\xi \leqslant [0]$. Since $x_a \geqslant 0$, $x_\xi \geqslant 0$, and $x_a^a \geqslant 0$, from (2.1) we obtain $x_a = x_a^a - E_\xi x_\xi \geqslant 0$ for each $x_\xi \geqslant 0$. Hence $(x_a, x_\xi) = (x_a^a, 0)$ is the only vertex of the set X (defined by (1.2)), and it is also an optimal solution of problem (1.1).

Now, let us define the following set:

$$\xi^0 = \{j \in \xi \mid E_j \leqslant 0\}.$$

Applying the simplex method to problem (1.3) we can obtain its equivalent form relative to (2.1), the optimality coefficients of which satisfy

$$\{j \in \xi \mid d_j^a > 0\} = \{j \in \xi^0 \mid d_j^a > 0\} \neq \emptyset. \tag{3.1}$$

It is known that the simplex method terminates whenever ([1])

$$\max_{j \in \xi^0} d_j^a > 0,$$

regardless of the existence of $j \in \xi - \xi^0$ such that $d_j^a > 0$ (that means already the unboundedness of $f(x)$ on X). However, in the latter situation we can select

$$d_l^a = \max_{j \in \xi - \xi^0} d_j^a \tag{3.2}$$

and transform the problem (introducing x_l to the basic variables) till we get the case (3.1).

The above-described modification of the simplex method transforming problem (1.3) to the case (3.1) is the first stage of the solution method for problem (1.1).

The second stage of the method consists in the construction of the set $X(M)$ according to (1.4).

Suppose then that x^a is the current basic feasible solution of problem (1.3) relative to (2.1) and that condition (3.1) is fulfilled ([2]) while

$$d_k^a = \max_{j \in \xi^0} d_j^a. \tag{3.3}$$

In order to determine the vector t appearing in (1.4), we postulate the following:

1° In the next iteration of the simplex method an optimal solution of the problem

$$\text{(i)} \quad \max\{(d_\xi^a)^T x_\xi \mid x_a + E_\xi x_\xi = x_a^a,\ t^T x + x_{n+1} = M,\ x \geqslant 0,\ x_{n+1} \geqslant 0,\ M \gg 0\}$$

([1]) Obviously, it terminates also in the case where $\{j \in \xi \mid d_j^a > 0\} = \emptyset$, but this situation cannot occur for $f(x)$ unbounded from above on X.

([2]) It is obvious that the situation described in Theorem 2 does not occur (i.e., the matrix E_ξ has some positive elements).

should be obtained by introducing x_k as a basic variable (see (3.3)) instead of x_{n+1}.

2° We wish to avoid generating any vertices on infinite edges emanating from $\boldsymbol{x}^a$ for which the function $f(\boldsymbol{x})$ does not increase (e.g., the hyperplane $\boldsymbol{t}^{\mathrm{T}}\boldsymbol{x} = M$ may be parallel to such edges).

Notice that at the next iteration of (i) we have

(ii) $$d_j^{a^k} = d_j^a - \frac{t_j}{t_k} d_k^a \quad \text{for } j \in \xi \cup \{n+1\}$$

and, in particular, $\boldsymbol{d}_a^{a^k} = \boldsymbol{0}$ whenever $\boldsymbol{t}_a = \boldsymbol{0}$.

In order to realize 1° it is required that

(iii) $$d_j^{a^k} \leqslant 0 \quad \text{for } j \in [\xi - \{k\}] \cup \{n+1\}.$$

Obviously, $d_k^{a^k} = 0$ for any $t_k \in R - \{0\}$, but since $d_{n+1}^{a^k} = -d_k^a/t_k$, by (ii) and (3.3) we get

(3.4) $$t_k > 0 \quad \text{and} \quad t_j \geqslant \frac{d_j^a t_k}{d_k^a} \text{ for } j \in \xi - \{k\}.$$

Since $d_j^a \leqslant d_k^a$ for all $j \in \xi - \{k\}$ (see (3.3)), condition (iii) is fulfilled when $t_j = 1$ for all $j \in \xi$. However, if $d_j^a \leqslant 0$ for some $j \in \xi - \{k\}$, we may set (³) $t_j = 0$ satisfying 2°. Thus we have

(3.5) $$t_j = \begin{cases} 0 & \text{for all } j \in a \cup \{j \in \xi \mid d_j^a \leqslant 0\}, \\ 1 & \text{for all } j \in \{j \in \xi^0 \mid d_j^a > 0\} \ (^4). \end{cases}$$

Let us put $\xi_+^0 = \{j \in \xi^0 \mid d_j^a > 0\}$. Now, according to (3.5), we consider the following problem:

(3.6)
$$\max\Big\{(\boldsymbol{d}_\xi^a)^{\mathrm{T}}\boldsymbol{x}_\xi + f(\boldsymbol{x}^a) \mid \boldsymbol{x}_a + \boldsymbol{E}_\xi \boldsymbol{x}_\xi = \boldsymbol{x}_a^a, \sum_{j \in \xi_+^0 \cup \{n+1\}} x_j = M, \boldsymbol{x} \geqslant \boldsymbol{0}, M \geqslant 0\Big\}.$$

THEOREM 3. *An optimal solution of problem* (3.6) *exists and can be obtained in one iteration of the simplex method as follows*:

$$\boldsymbol{x}_a^{a^k} = \boldsymbol{x}_a^a - M\boldsymbol{E}_k, \quad x_k^{a^k} = M, \quad \boldsymbol{x}_{\xi-\{k\}}^{a^k} = \boldsymbol{0}, \quad x_{n+1}^{a^k} = 0.$$

Proof. Notice that the above-defined point is obtained by introducing (according to (3.3)) x_k to the basic variables of problem (3.6)

(³) If $d_j^a \leqslant 0$ for some $j \in \xi^0 - \{k\}$, then $f(x) \leqslant f(x^a)$ for each x belonging to the set $\{x = x^a + \lambda\bar{x} \mid \bar{x}_a = -E_j, \bar{x}_j = 1, \bar{x}_{\xi-\{j\}} = \boldsymbol{0}, \lambda \geqslant 0\}$.

(⁴) According to (3.1), $d_j^a > 0$ for $j \in \xi^0$ only.

instead of x_{n+1}. Its optimality coefficients are

$$d_j^{a^k} = \begin{cases} 0 & \text{for } j \in a \cup \{k\}, \\ d_j^a & \text{for } j \in \xi - \xi_+^0, \\ d_j^a - d_k^a & \text{for } j \in \xi_+^0, \\ -d_k^a & \text{for } j = n+1. \end{cases}$$

Taking into account (3.1) and (3.3) we can easily prove that $d_j^{a^k} \leqslant 0$ for all $j \in [\xi - \{k\}] \cup \{n+1\}$. Hence x^{a^k} is an optimal solution of problem (3.6).

Denote by $\bar{X}(M)$ the set of feasible solutions of problem (3.6) and let $\bar{X}^*(M)$ be the sequence of all vertices of $\bar{X}(M)$ ranked by the Murty method (see [3] or Section 2) according to nonincreasing values of the function $f(x)$, i.e.,

$$(d_\xi^a)^{\mathrm{T}} x_\xi^i \leqslant (d_\xi^a)^{\mathrm{T}} x_\xi^j \quad \text{for } x^i, x^j \in \bar{X}^*(M) \text{ and } i > j$$

(the same holds for the corresponding vertices of $X^*(M)$).

THEOREM 4. *If the sequence* $x^1, x^2, \ldots, x^s \in \bar{X}^*(M)$ *satisfies*

$$\sum_{j \in \xi_+^0} x_j^i = M \quad \textit{for } i = 1, 2, \ldots, s-1 \tag{3.7}$$

and

$$\sum_{j \in \xi_+^0} x_j^s < M, \tag{3.8}$$

then $(x_a^s, x_\xi^s)^{\mathrm{T}}$ *is an optimal solution of problem* (1.1).

Proof. Clearly, $X_p \subset X(M) \subset X$. Besides, from the definition of the number M it follows that $(x_a, x_\xi)^{\mathrm{T}}$ is a vertex of the set X (i.e., $(x_a, x_\xi)^{\mathrm{T}} \in X^* \subset X_p$) if and only if the corresponding basic solution $(x_a, x_\xi, x_{n+1})^{\mathrm{T}} \in \bar{X}^*(M)$ satisfies (3.8). Hence, by (3.7), (3.8), and the definition of $X^*(M)$, the point $(x_a^s, x_\xi^s)^{\mathrm{T}}$ is an optimal solution of problem (1.1).

Theorem 4 indicates that in order to solve problem (1.1) we have to rank vertices of the set $\bar{X}(M)$ till the first vertex satisfying (3.8) (or, equivalently, the inequality $x_{n+1} > 0$) is obtained.

Applying the Murty method to problem (3.6) with the unspecified number M we have to compare the expressions of the form $Ma+b$. In order to do this we assume that $Ma_1+b_1 = Ma_2+b_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$ while $Ma_1+b_1 < Ma_2+b_2$ if and only if the pair (a_1, b_1) is lexicographically less than (a_2, b_2) (i.e., $a_1 < a_2$ or $a_1 = a_2$ and $b_1 < b_2$). Therefore, two right hand-side vectors should be computed.

The results given above enable us to propose the following

ALGORITHM. Step 1. Solve problem (1.3). If an optimal solution of the problem exists, it is an optimal solution of problem (1.1).

Step 2. Check whether $E_\xi \leqslant [\boldsymbol{0}]$ in the latest canonical form (2.1) of problem (1.3) obtained in Step 1. If so, then $\boldsymbol{x}^a$ is an optimal solution of problem (1.1).

Step 3. Check whether condition (3.1) is fulfilled. If so, go to Step 4. Otherwise, applying the modification of the simplex method according to (3.2) transform the problem to the case (3.1) and go to Step 4.

Step 4. Find an optimal solution of problem (3.6).

Step 5. Apply the Murty method [3] to generate the sequence $\bar{X}^*(M)$ and verify conditions (3.7) and (3.8). The first vertex in $\bar{X}^*(M)$ satisfying (3.8) gives an optimal solution of problem (1.1).

Notice that the continuation of Step 5 with deleting all vertices satisfying (3.7) allows us to obtain the sequence X^* of all vertices of the set X defined by (1.2) according to nonincreasing values of the linear form of $f(\boldsymbol{x})$, being unbounded from above on X.

4. Numerical examples. We consider the following two examples to illustrate the algorithm.

Example 1. Solve the problem

$$\max\{3x_1+2x_2 \mid (x_1, x_2) \in V_p\},$$

where V_p is the convex hull of all vertices of the set

$$V = \{(x_1, x_2) \mid -2x_1+x_2 \leqslant 2,\ x_1-2x_2 \leqslant 1,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}.$$

The problem can be solved graphically (see Fig. 1).

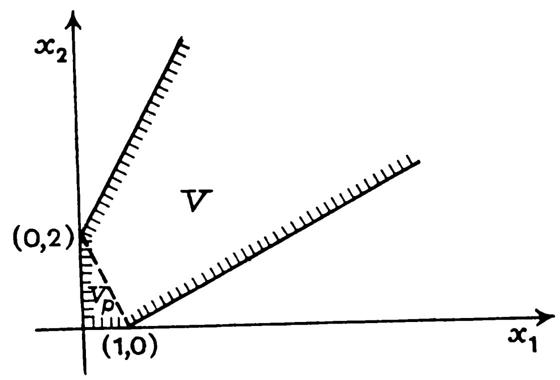

Fig. 1

The set V_p is the triangle with the vertices $\boldsymbol{v}^1 = (0, 0)$, $\boldsymbol{v}^2 = (1, 0)$, and $\boldsymbol{v}^3 = (0, 2)$. Since the function $f(x_1, x_2) = 3x_1+2x_2$ attains its maximum on that triangle at the point $\boldsymbol{v}^3 = (0, 2)$, it is an optimal solution of the problem.

Now, we solve the problem applying the algorithm described in Section 3.

First we consider the following standard form corresponding to (1.3):

Maximize $f(\boldsymbol{x}) = 3x_1+2x_2$ subject to

$$-2x_1+x_2+x_3 = 2, \quad x_1-2x_2+x_4 = 1,$$

where $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$, $x_4 \geqslant 0$.

Thus we have

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{c} = [3 \ 2 \ 0 \ 0]^{\mathrm{T}}.$$

Step 1. The simplex method applied to the above problem terminates generating the basic feasible solution

$$\boldsymbol{x}^{a} = [1 \ 0 \ 4 \ 0]^{\mathrm{T}}$$

(yielding the vertex $\boldsymbol{v}^2 = (1,0)$) corresponding to the following canonical form (see (2.1)):

$$8x_2-3x_4+3 \to \max$$

subject to (5)

$$-3x_2+x_3+2x_4 = 4, \quad x_1-2x_2+x_4 = 1.$$

Then $a = \{1,3\}$, $\xi = \{2,4\}$, $\xi^0 = \{2\}$, and $f(\boldsymbol{x}^a) = 3$. Since $d_2^a = \max\{8, -3\} = 8 > 0$ and $2 \in \xi^0$, the function $f(\boldsymbol{x})$ is unbounded from above on the set X.

Step 2. Notice that $\xi \neq \xi^0$, so $E_\xi \nleq [\boldsymbol{0}]$.

Step 3. Condition (3.1) is fulfilled since $\{j \in \xi \mid d_j^a > 0\} = \xi^0$.

Step 4. According to (3.5) we set $\boldsymbol{t} = [0 \ \ 1 \ \ 0 \ \ 0]^{\mathrm{T}}$ and solve the following problem corresponding to (3.6):

$$8x_2-3x_4+3 \to \max$$

subject to

$$-3x_2+x_3+2x_4 = 4, \quad x_1-2x_2+x_4 = 1, \quad x_2+x_5 = M.$$

An optimal solution of the problem (see Theorem 3) is of the form

$$\boldsymbol{x}^{a^2} = [2M+1 \quad M \quad 3M+4 \quad 0 \quad 0]^{\mathrm{T}}$$

corresponding to the canonical form

$$(4.1) \quad \begin{cases} \qquad -3x_4-8x_5+8M+3 \to \max \\ \text{subject to} \\ \quad x_3+2x_4+3x_5 = 3M+4, \quad x_1+x_4+2x_5 = 2M+1, \\ \qquad x_2+x_5 = M, \end{cases}$$

where $f(\boldsymbol{x}^{a^2}) = 8M+3$.

(5) For convenience we omit the nonnegativity constraints in the rest of the paper.

Step 5. We apply the Murty method to problem (4.1), where $\alpha_1 = \{1, 2, 3\}$ and $\xi_1 = \{4, 5\}$. Taking

$$x^1 = [2M+1 \quad M \quad 3M+4 \quad 0 \quad 0]^T$$

(obtained in Step 4 as x^{α^2}) as the first vertex of the sequence $\bar{X}^*(M)$, we find the set (according to (2.2)) $F(x^1) = \{x^{\alpha_1^4}, x^{\alpha_1^5}\}$ (with x_4 and x_5 as the basic variables, respectively). Since

$$\nu_4^{\alpha_1} = \tfrac{3}{2}M+2, \qquad \nu_5^{\alpha_1} = M, \qquad \text{and} \qquad \nu_4^{\alpha_1} d_4^{\alpha_1} = -\tfrac{9}{2}M-6 > \nu_5^{\alpha_1} d_5^{\alpha_1} = -8M,$$

we infer, according to (2.3), that $x^{\alpha_1^4}$ follows x^1 in the sequence $\bar{X}^*(M)$. Hence we obtain

$$x^2 = [\tfrac{1}{2}M-1 \quad M \quad 0 \quad \tfrac{3}{2}M+2 \quad 0]^T$$

corresponding to the canonical form

$$\tfrac{3}{2}x_3 - \tfrac{7}{2}x_5 + \tfrac{7}{2}M - 3 \to \max$$

subject to

$$\tfrac{1}{2}x_3 + x_4 + \tfrac{3}{2}x_5 = \tfrac{3}{2}M+2, \qquad x_1 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_5 = \tfrac{1}{2}M-1, \qquad x_2 + x_5 = M,$$

where $f(x^2) = \tfrac{7}{2}M-3$, $\alpha_2 = \{1, 2, 4\}$, and $\xi_2 = \{3, 5\}$.

Notice that the vertices x^1 and x^2 are nondegenerate. Then (see Theorem 1) we have

$$\bigcup_{i=1}^{2} F(x^i) - \{x^1, x^2\} = \{x^{\alpha_1^5}, x^{\alpha_2^5}\}.$$

Comparing the values

$$f(x^{\alpha_1^5}) = f(x^1) + \nu_5^{\alpha_1} d_5^{\alpha_1} = 8M+3-8M = 3$$

and

$$f(x^{\alpha_2^5}) = f(x^2) + \nu_5^{\alpha_2} d_5^{\alpha_2} = \tfrac{7}{2}M - 3 - \tfrac{7}{2}M + 7 = 4,$$

we choose $x^{\alpha_2^5}$ as following x^2 in the sequence $\bar{X}^*(M)$. Thus we obtain $x^3 = [0 \quad 2 \quad 0 \quad 5 \quad M-2]^T$ corresponding to the canonical form

$$7x_1 - 2x_3 + 4 \to \max$$

subject to

$$-3x_1 + 2x_3 + x_4 = 5, \qquad 2x_1 - x_3 + x_5 = M-2, \qquad -2x_1 + x_2 + x_3 = 2.$$

Since $x_5^3 = M-2 > 0$, x^3 satisfies condition (3.8). Hence $x_1^3 = 0$ and $x_2^3 = 2$ are the coordinates of the optimal solution of the considered problem.

Example 2. Solve the problem

$$\max\{2x_1+x_2 \mid (x_1, x_2) \in V_p\},$$

where V_p is the convex hull of all vertices of the set

$$V = \{(x_1, x_2) \mid -x_1+x_2 \leqslant 2,\ -2x_1+x_2 \leqslant 1,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}.$$

The vertices of the set V are $v^1 = (0, 0)$, $v^2 = (0, 1)$, and $v^3 = (1, 3)$ (see Fig. 2). Hence $v^3 = (1, 3)$ is an optimal solution of the problem.

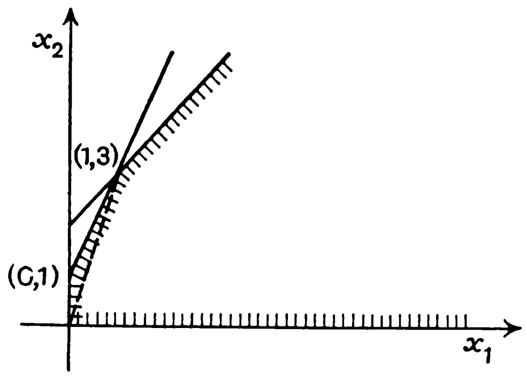

Fig. 2

The initial canonical standard form of the problem is (according to (1.3)) the following:

$$2x_1+x_2 \to \max$$

subject to

$$-x_1+x_2+x_3 = 2, \qquad -2x_1+x_2+x_4 = 1.$$

Hence

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad c = [2 \quad 1 \quad 0 \quad 0]^T,$$

$$a = \{3, 4\}, \quad \xi = \{1, 2\}, \quad \text{and} \quad \xi^0 = \{1\}.$$

Since $\max\{d_1^a, d_2^a\} = \max\{2, 1\} = d_1^a$ and $1 \in \xi^0$ (the first column of the matrix E_ξ is nonpositive), the function $f(x)$ is unbounded from above on X but condition (3.1) is not fulfilled ($d_2^a = 1 > 0$ and $2 \in \xi - \xi^0$). Then in Step 3 we have to use the modification of the simplex method introducing, according to (3.2), x_2 to the basic variables. Thus we obtain successively

$$4x_1 - x_4 + 1 \to \max$$

subject to

$$x_1+x_3-x_4 = 1, \qquad -2x_1+x_2+x_4 = 1,$$

and then (introducing x_1 to the basic variables)

$$-4x_3+3x_4+5 \to \max$$

subject to

$$x_1+x_3-x_4 = 1, \quad x_2+2x_3-x_4 = 3.$$

Now, in Step 4 we introduce the additional constraint $x_4 \leqslant M$ (since $\boldsymbol{t} = [0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}}$ by (3.5)) and solve, according to (3.6), the following problem:

$$(4.2) \quad \begin{cases} -4x_3+3x_4+5 \to \max \\ \text{subject to} \\ x_1+x_3-x_4 = 1, \quad x_2+2x_3-x_4 = 3, \quad x_4+x_5 = M. \end{cases}$$

The optimal solution of problem (4.2), which follows

$$\boldsymbol{x}^1 = [M+1 \quad M+3 \quad 0 \quad M \quad 0]^{\mathrm{T}},$$

is the first vertex of the sequence $\bar{X}^*(M)$; the solution corresponds to the canonical (equivalent to (4.2)) form

$$-4x_3-3x_5+3M+5 \to \max$$

subject to

$$x_1+x_3+x_5 = M+1, \quad x_2+2x_3+x_5 = M+3, \quad x_4+x_5 = M,$$

where $\alpha_1 = \{1, 2, 4\}$, $\xi_1 = \{3, 5\}$ and $F(\boldsymbol{x}^1) = \{\boldsymbol{x}^{\alpha_1^3}, \boldsymbol{x}^{\alpha_1^5}\}$. We select $\boldsymbol{x}^{\alpha_1^3}$ which follows $\boldsymbol{x}^1$ in the sequence $\bar{X}^*(M)$. Thus we obtain

$$\boldsymbol{x}^2 = [\tfrac{1}{2}M-\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2}M+\tfrac{3}{2} \quad M \quad 0]^{\mathrm{T}}$$

corresponding to the canonical form

$$2x_2-x_5+M-1 \to \max$$

subject to

$$x_1-\tfrac{1}{2}x_2+\tfrac{1}{2}x_5 = \tfrac{1}{2}M-\tfrac{1}{2}, \quad \tfrac{1}{2}x_2+x_3+\tfrac{1}{2}x_5 = \tfrac{1}{2}M+\tfrac{3}{2}, \quad x_4+x_5 = M.$$

Now, we have

$$\bigcup_{i=1}^{2} F(\boldsymbol{x}^i) - \{\boldsymbol{x}^1, \boldsymbol{x}^2\} = \{\boldsymbol{x}^{\alpha_1^5}, \boldsymbol{x}^{\alpha_2^5}\},$$

where $\alpha_2 = \{1, 3, 4\}$ and $\xi_2 = \{2, 5\}$. Since the value of $f(\boldsymbol{x})$ is greater for $\boldsymbol{x}^{\alpha_1^5}$ than for $\boldsymbol{x}^{\alpha_2^5}$, we take $\boldsymbol{x}^3 = \boldsymbol{x}^{\alpha_1^5}$. Thus we obtain

$$\boldsymbol{x}^3 = [1 \quad 3 \quad 0 \quad 0 \quad M]^{\mathrm{T}}$$

which corresponds to the canonical form (4.2). Since $x_5^3 = M > 0$, $\boldsymbol{x}^3$ satisfies (3.8). Hence $x_1^3 = 1$ and $x_2^3 = 3$ are the coordinates of the optimal solution of the considered problem.

Finally, it should be noticed that in Example 1 the simplex method terminates at the point $\boldsymbol{v}^2 = (1, 0)$ while the Murty method allows us to state that the optimal solution of the problem is $\boldsymbol{v}^3 = (0, 2)$.

In Example 2 the simplex method terminates at $v^1 = (0, 0)$ but due to its modification we obtained $v^3 = (1, 3)$ which turned out to be also (after applying the Murty method) the optimal solution of the considered problem.

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