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$M/G/1$ QUEUEING SYSTEM WITH "FAGGING" SERVICE CHANNEL

1. Problem formulation. This paper is devoted to the problem of rational work of a machine operator. The psychological point of view and an attempt of mathematical formulation of this problem is given in [1]. In many practically interesting systems, the efficiency of the operator is not constant in time and generally decreases as time elapses.

Thus, in such systems "fagging" of the operator should be taken into account; it can be interpreted in many ways, dependent upon the character of the operator's work. For instance, if the operator is tooling some products, one can treat the tooling times as independent random variables with different distributions. These distributions can reflect the fagging of the operator.

PROBLEM A. Here we deal with service systems in which service times need not be identically distributed. Consider a queueing system with Poisson input having a constant intensity λ, in which customers are serviced by one operator. Let us number the customers according to their arrival and let the i-th customer have his service time distributed as $B_i(t)$. An interesting characteristics of the system is the number of customers in the system at the moment t which is denoted by $\eta(t)$. To investigate this characteristics let us introduce the vector process $\zeta(t) = \{\eta(t), \gamma(t)\}$ in which the component $\gamma(t)$ is the number of the service time distribution of the customer being serviced at the moment t. For the steady-state distribution of this process to exist, it is necessary that the renewal moments of this process exist. Define them as those moments in which leaving customers make the system empty. Assume that the service channel resumes its initial efficiency, i.e. the efficiency at the moment $t = 0$, at any renewal moment. That means that after renewals the process is independent of its past and also that the first customer arriving after a renewal has the service distribution $B_1(t)$, the second one $B_2(t)$, etc. Such a system will be considered in chapter 2.

PROBLEM B. In many practical situations one must assume that the renewal of the service channel takes some time. As it has been observed in [1], there can not exist a finite time interval needed for a complete renewal of the service channel. If, in addition, its expected value is infinite, the process $\zeta(t)$ has no steady-state distribution. Since we are interested in the steady-state distribution of $\zeta(t)$, we shall not deal with this case. We assume thus that the service channel has a finite expected renewal time. This problem is considered in chapter 3, in which we show that it does not lead to an essentially more general system than that considered in problem A. Recall our assumption that the renewal of the service channel takes place when the system is empty. Of course, this assumption is far from what we observe in many real situations. Note, however, that taking into account breakdowns of the service channel during its working time (see [2] and [3]) does not lead to a more general system. In fact, it is sufficient to assume that the customer service time is equal to the time interval from the beginning of the service to the moment of the customer leaving the system. Different breakdown streams and various renewal times of the service channel can be covered by non-identical service time distributions of the customers.

PROBLEM C. Another modification of problem A is the assumption that the service channel is renewed after having served k customers and customers which have not been serviced or which arrived during the renewal time are lost. We show in chapter 4 that this model is a particular case of problem A.

2. Solution of problem A.

2.1. Derivation of the fundamental equations. The analysis of the process $\zeta(t) = \{\eta(t), \gamma(t)\}$ in continuous time without having it extended to a Markov process is rather difficult. There are theorems relating the steady-state distribution of the process $\zeta(t)$ in continuous time with the steady-state distribution of some chain (see [4]). It suffices thus to investigate the steady-state distribution of the states of the process $\zeta(t)$ in the discrete time.

Let t_n be the moment of the n-th customer leaving the system, counting from the moment $t = 0$. We write sequences of random variables $\eta_n = \eta(t_n+0)$ and $\gamma_n = \gamma(t_n-0)$. Here η_n is the number of customers in the system directly after the n-th customer was leaving the system and takes on the values $0, 1, 2, \ldots$, and γ_n is the number, counting from the last renewal, of the customer which leaves the system at the moment t_n. The variable γ_n takes on the values $1, 2, 3, \ldots$

Assume that the system is empty at the moment $t = 0$ and that the first customer arrives exactly at that moment. Notice that the sequence of random variables $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ is a homogeneous, irreducible and aperiodic

Markov chain. This follows directly from the obvious formulae for the one-step-transition probabilities of this chain which are equal to

$$(2.1)\quad \begin{aligned} \mathrm{P}\{(i,j)\to(k,l)\} &= \int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{k+1-i}}{(k+1-i)!}\,dB_{j+1}(u) \\ &\qquad \text{for } i,j=1,2,\ldots,\ k=i-1,i,\ldots,\ l=j+1, \\ \mathrm{P}\{(0,j)\to(k,1)\} &= \int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{k}}{k!}\,dB_{1}(u) \\ &\qquad \text{for } k=0,1,2,\ldots,\ j=1,2,\ldots \end{aligned}$$

The probability that the n-th customer, counting from the moment $t=0$, is the l-th customer, counting from the last renewal, and after his exit there remain in the system k customers, is shortly denoted by

$$P_n(k,l) = \mathrm{P}\{\eta_n = k, \gamma_n = 1\} \quad \text{for } k=0,1,2,\ldots,\ l=1,2,\ldots,$$

whereas

$$P_n(0) = \mathrm{P}(\eta_n = 0) = \sum_{l=1}^{\infty} P_n(0,l)$$

denotes the probability that the n-th customer leaving the system remains it empty.

The limits of these probabilities are denoted by

$$P(k,l) = \lim_{n\to\infty} P_n(k,l) \quad \text{and} \quad P(0) = \lim_{n\to\infty} P_n(0).$$

They exist since it is easily seen that the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ is homogeneous, irreducible and aperiodic.

Let

$$\psi_n(z,l) = \sum_{k=0}^{\infty} P_n(k,l)z^k \quad \text{and} \quad \psi(z,l) = \sum_{k=0}^{\infty} P(k,l)z^k$$

be the probability generating functions of $P_n(k,l)$ and $P(k,l)$, respectively, and let $K_i(z) = \beta_l(\lambda(1-z))$, where $\beta_l(z)$ is the Laplace-Stieltjes transform of the service distribution of the l-th customer, counting from the last renewal. We call this time interval *l-interval.*

THEOREM 2.1. *For the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$, the following recurrence relations are satisfied:*

$$(2.2)\quad z\psi(z,l) = K_l(z)\,[\psi(z,l-1) - P(0,l-1)] \quad \textit{for } l=2,3,\ldots$$

and

$$(2.3)\quad \psi(z,l) = P(0)K_l(z).$$

Proof. The following equalities are easily verified:

$$(2.4)\qquad P_{n+1}(k,l) = \sum_{i=1}^{k+1} P_n(i,l-1)\int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{k+1-i}}{(k+1-i)!}\,dB_l(u)$$

for $k = 0, 1, \ldots,\ l = 2, 3, \ldots,$

$$(2.5)\qquad P_{n+1}(k,1) = P_n(0)\int_0^\infty e^{-\lambda u}\frac{(\lambda u)^k}{k!}\,dB_1(u) \quad \text{for } k = 0, 1, 2, \ldots$$

Multiplying both sides of (2.4) by z^k and summing it after k leads to

$$\sum_{k=0}^\infty P_{n+1}(k,l)z^k = \sum_{k=0}^\infty\sum_{i=1}^{k+1} P_n(i,l-1)z^k\int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{k+1-i}}{(k+1-i)!}\,dB_l(u).$$

After simple derivations one obtains

$$\psi_{n+1}(z,l) = \beta_l\big(\lambda(1-z)\big)\sum_{i=1}^\infty P_n(i,l-1)z^{i-1}.$$

By the introduced notation, this can be expressed as

$$z\psi_{n+1}(z,l) = K_1(z)\,[\psi_n(z,l-1)-P_n(0,l-1)] \quad \text{for } l = 2, 3, \ldots$$

Analogically, from (2.3) we obtain $\psi_{n+1}(z,1) = P_n(0)K_1(z)$. Since, for every k and l, the limits

$$\lim_{n\to\infty} P_n(k,l) = P(k,l) \quad \text{and} \quad \lim_{n\to\infty} P_n(0) = P(0)$$

exist, the statement of theorem 2.1 follows.

One can easily show by induction that the function $\psi(z,l)$ can be expressed in the form

$$(2.6)\qquad \begin{aligned} \psi(z,l) &= P(0)z^{1-l}\prod_{j=1}^{l} k_j(z) - \sum_{i=1}^{l-1} z^{i-l}\prod_{j=i+1}^{l} k_j(z)P(0,i) \quad \text{for } l = 2, 3, \ldots, \\ \psi(z,1) &= P(0)K_1(z). \end{aligned}$$

While stating problem A we have said that we are mainly interested in the distribution of the number of customers in the system. If this distribution exists, it can be obtained from $P(k) = \sum_{l=1}^\infty P(k,l)$. Denote by $\psi(z)$ the generating function of the sequence $\{P(k)\}_{k=0}^\infty$. Hence

$$\psi(z) = \sum_{k=0}^\infty P(k)z^k = \sum_{l=1}^\infty \psi(z,l).$$

If the sequence $\{P(k)\}_{k=0}^{\infty}$ is a probability distribution, $\psi(z)$ is the generating function of the number of customers in the system.

2.2. Distribution of the number of services in the busy time interval. Let us find now the probabilities $P(0, l)$ for $l = 1, 2, \ldots$ Introduce the following notation and definitions:

Let X_i be a function defined on the sequence of non-negative integers $\{j_n\}_{n=1}^{\infty}$ as follows:

$$x_i = x(j_1, j_2, \ldots, j_i) = \sum_{n=1}^{i} j_n.$$

Let $f(\cdot)$ be the function defined on the set $\{(i, s)\colon i = 0, 1, 2, \ldots, s-1;\ s = 2, 3, \ldots\}$ such that

$$f(i, s) = \begin{cases} 0 & \text{for } i < s-1, \\ 1 & \text{for } i = s-1, \end{cases}$$

and let

$$b_j = \int_0^{\infty} x dB_j(x).$$

Also, let $K_l^i(0)$ denote the i-th derivative of $K_l(z)$ at the point $z = 0$. Note that the number $K_l^j(0)$ equals the probability that in the l-interval j customers arrive to the system.

We prove now the following

LEMMA 2.1. *If there exists the steady-state distribution of the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$, the following relations are satisfied*:

$$P(0, l) = P(0)\tilde{P}(0, l),$$

(i) $$\tilde{P}(0, l) = K_l(0) \sum_{j_1=1}^{l-1} \sum_{j_2=f(l-1-x_1, l-1)}^{l-1-x_1} \cdots \sum_{j_{l-1}=f(l-1-x_{l-2}, 2)}^{l-1-x_{l-2}} \prod_{m=1}^{l-1} \frac{1}{j_m!} k_m^{j_m}(0)$$

for $l = 2, 3, \ldots,$

$$\tilde{P}(0, 1) = K_1(0),$$

(ii) $$P(0) = \Big[\sum_{l=1}^{\infty} \tilde{P}(0, l) l\Big]^{-1}.$$

If, in addition, we assume that there exists $\lim_{n\to\infty} b_n = b$, *the following holds*:

(iii) $$P(0) = (i - \lambda b)\Big[i - \lambda \sum_{i=1}^{\infty} \tilde{P}(0, i)\Big(ib - \sum_{j=1}^{i} b_j\Big)\Big]^{-1}.$$

Formula (iii) is given here because in many situations it is easily applied. This is so, for instance, when the mean service times satisfy the condition $b_s = b_{s+k}$ for $k \geqslant 0$, where s is some fixed non-negative integer.

The proof of lemma 2.1 is begun from (i). We now have from equalities (2.4) and (2.5)

$$(2.7)\qquad \begin{aligned} P(0,l) &= P(1,l-1)K_l(0) \quad \text{for } l = 2,3,\ldots, \\ P(0,1) &= P(0)K_1(0). \end{aligned}$$

The probabilities $P(1,l)$ for $l = 1,2,\ldots$ are now to be determined. Introduce the following notation:

$A_{l,s}$ is the event that at the end of the l-interval there remain s customers in the system;

$B_{l,s}$ is the event that during an l-interval s customers arrive to the system;

$C_{l+1,s,i}$ is the event that in the $l+1, l+2, \ldots, (l+s)$-intervals i customers arrive to the system and that at the end of the $(l+s)$-interval there remains only one customer in the system.

Remark 2.1. The event $C_{l+1,s,i}$ indicates how customers should arrive in order to have no renewal, i.e. no empty system, in the time interval between the $(l+1)$-interval and the $(l+s)$-interval.

Because the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ has a steady-state distribution, $\mathrm{P}(A_{l,r}) = P(r,l) > 0$ holds. The probabilities $\mathrm{P}(C_{l+1,s-1,i}/A_{l,s-i})$ are understood as conditional probabilities in the usual sense. From the formula for the total probability we have

$$(2.8)\qquad P(1,s) = \sum_{i=1}^{s} P(i,1)\mathrm{P}(C_{2,s-1,s-i}/A_{1,i}),$$

where $\mathrm{P}(A_{1,i}) = P(i,1) = K_1^i(0)/i!$. The probability $P(i,1)$ is determined from $\psi(z,1)$ given by (2.3).

The probabilities $\mathrm{P}(C_{l+1,s-1,i}/A_{l,s-i})$ satisfy the following recurrence relations:

$$(2.9)\quad \mathrm{P}(C_{l+1,s,i}/A_{l,s+1-i}) = \sum_{j=f(i,s+1)}^{i} \frac{1}{j!} k_{l+1}^{j}(0)\mathrm{P}(C_{l+2,s-1,i-j}/A_{l+1,s+j-i})$$

$$\text{for } l = 0,1,\ldots,\ s = 2,3,\ldots,\ i = 0,1,\ldots,s-1,$$

$$(2.10)\qquad \mathrm{P}(C_{l,1,0}/A_{l-1,2}) = K_l(0) \quad \text{for } l = 1,2,\ldots,$$

$$(2.11)\qquad \mathrm{P}(C_{l,1,1}/A_{l-1,1}) = K_l^1(0) \quad \text{for } l = 1,2,\ldots$$

These formulae follow directly from the formula for the total probability. They can be proved formally by using the following formulae:

$$B_{l,j}\cap A_{l-1,s+1-i} = A_{l,s-i+j},$$

$$\{C_{l+1,s,i}/A_{l,s+1-i}\} = \bigcup_{j=f(i,s+1)}^{i} \{(B_{l+1,j}\cap C_{l+1,s-1,i-j})/A_{l,s+1-i}\}$$

for $s = 2, 3, \ldots$ and $i = 0, 1, \ldots, s-1$.

For fixed l, s and i, the sets of the family described by the right-hand side of the last equality are pairwise disjoint. Hence

$$\mathrm{P}(C_{l,s,i}/A_{l,s+1-i}) = \sum_{j=f(i,s+1)}^{i} \mathrm{P}\big((B_{l+1,j}\cap C_{l+2,s-1,i-j})/A_{l,s+1-i}\big)$$

$$= \sum_{j=f(i,s+1)}^{i} \mathrm{P}(B_{l+1,j}/A_{l,s+1-i})\mathrm{P}(C_{l+2,s-1,i-j}/B_{l+1,j}\cap A_{l,s+1-i}).$$

Since the input stream is Poissonian, we have

$$\mathrm{P}(B_{l+1,j}/A_{l,s+1-i}) = \mathrm{P}(B_{l+1,j}) = K_{l+1}^{j}(0)/j!$$

and

$$\mathrm{P}(C_{l+2,s-1,i-j}/B_{l+1,j}\cap A_{l,s+1-i}) = \mathrm{P}(C_{l+2,s-1,i-j}/A_{l+1,s+j-i}).$$

This completes the proof of (2.9). Formulae (2.10) and (2.11) follow directly from the interpretation of the events $C_{1,1,0}/A_{l-1,2}$ and $C_{1,1,1}/A_{l-1,1}$. The use of formulae (2.7) and (2.8) and an $(l-2)$-fold usage of (2.9) leads to the formula for $\tilde{P}(0, l)$ given in (i).

Remark 2.2. From the equality $P(0, l) = P(0)\tilde{P}(0, l)$ for $l = 1, 2, \ldots$ and from the fact that $P(0) > 0$ and

$$P(0) = \sum_{l=1}^{\infty} P(0, l) = P(0)\sum_{l=1}^{\infty}\tilde{P}(0, l)$$

we have

$$\sum_{l=1}^{\infty}\tilde{P}(0, l) = 1,$$

i.e. $\tilde{P}(0, l)$ form a probability distribution. From the interpretation of the probabilities $P(0, l)$ it follows that $\tilde{P}(0, l)$ form the distribution of the number of serviced customers during the busy period. The Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ is ergodic, therefore, the above-mentioned distribution has a finite expected value.

We prove now equality (ii). From (2.2) and (2.3) and from the fact that $K_l(1) = 1$ for every l we have

$$\psi(1, l) = \psi(1, l-1) - P(0, l-1) \quad \text{for } l = 2, 3, \ldots,$$
$$\psi(1, 1) = P(0).$$

Finally, we obtain from this

$$\psi(1, l) = P(0) \sum_{s=l}^{\infty} \tilde{P}(0, s). \tag{2.12}$$

Since $\psi(1) = 1$, we have

$$\sum_{l=1}^{\infty} \psi(1, l) = P(0) \sum_{l=1}^{\infty} \sum_{s=l}^{\infty} \tilde{P}(0, s) = 1. \tag{2.13}$$

In the last expression the order of summation can be changed because $\tilde{P}(0, s) > 0$ and $\sum_{s=1}^{\infty} \tilde{P}(0, s)s < \infty$. Therefore,

$$1 = P(0) \sum_{s=1}^{\infty} \tilde{P}(0, s)s.$$

From the above-mentioned we obtain at last formula (ii).

It remains still to prove formula (iii). Consider such a case of problem A in which, for any fixed s, $B_{s+1} \equiv B_{s+i}$ holds for every $i \geqslant 1$. From (2.2) and (2.3) we have

$$z \sum_{l=2}^{\infty} \psi(z, l) = \sum_{l=2}^{s} k_l(z) \psi(z, l-1) - \sum_{l=2}^{s} k_l(z) P(0, l-1) + \tag{2.14}$$
$$+ \sum_{l=s+1}^{\infty} k_{s+1}(z) \psi(z, l-1) - \sum_{l=s+1}^{\infty}{}' k_{s+1}(z) P(0, l-1).$$

To underline the role of s we write

$$\sum_{l=1}^{\infty} \psi(z, l) = \psi_s(z).$$

Subtracting from both sides of (2.14) the function $K_{s+1}(z)\psi_s(z)$, we obtain

$$z\psi_s(z) - z\psi(z, 1) - k_{s+1}(z)\psi_s(z) = \sum_{l=1}^{s-1} k_{l+1}(z)\psi(z, l) - \sum_{l=1}^{s-1} k_{l+1}(z) P(0, l) -$$
$$- k_{s+1}(z) \sum_{l=1}^{s-1} \psi(z, l) - k_{s+1}(z) \sum_{l=s}^{\infty} P(0, l).$$

Hence

(2.15) $$\psi_s(z) = \big(z-K_{s+1}(z)\big)^{-1}\varphi_s(z),$$

where

$$\varphi_s(z) = zP(0)K_1(z)+\sum_{l=1}^{s-1}k_{l+1}(z)\psi(z,l)-\sum_{l=1}^{s-1}k_{l+1}(z)P(0,l)-$$
$$-k_{s+1}(z)\sum_{l=1}^{s-1}\psi(z,l)-k_{s+1}(z)\sum_{l=s}^{\infty}{}' P(0,l).$$

At the point $z = 1$ the left-hand side of (2.15) equals 1 and the right-hand side is a symbol of type $0/0$. Take the derivative of $\varphi_s(z)$. Its value at the point $z = 1$ exists and is equal to

$$\varphi_s'(z) = P(0)\,(1+\lambda b_1)+\sum_{l=1}^{s-1}\lambda b_{l+1}\psi(1,l)-\sum_{l=1}^{s-1}\lambda b_{l+1}P(0,l)-$$
$$-\lambda b_{s+1}\sum_{l=1}^{s-1}\psi(1,l)-\lambda b_{s+1}\sum_{l=s}^{\infty}{}' P(0,l).$$

We have used here the relations $K_l(1) = 1$ and $K_l^1(1) = \lambda b_l$, where

$$b_l = \int_0^\infty x dB_l(x).$$

From formulae (2.12) and (2.13) and from the fact that

$$\lim_{s\to\infty}\sum_{l=s}^{\infty}\tilde P(0,l) = 0$$

it follows the existence of $\lim\limits_{s\to\infty}\varphi_s'(1)$ and the equality

$$\lim_{s\to\infty}\varphi_s'(1)$$
$$= P(0)\Big[1+\lambda b_1+\sum_{l=1}^{\infty}\lambda b_{l+1}\sum_{i=l}^{\infty}\tilde P(0,i)-\sum_{l=1}^{\infty}\lambda b_{l+1}\tilde P(0,l)-\lambda b\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\tilde P(0,\ i)\Big],$$

where $b = \lim\limits_{n\to\infty} b_n$.

Changing the summation order in the second and last expressions, we obtain

$$\lim_{s\to\infty}\varphi_s'(1)$$
$$= P(0)\Big[1+\lambda b_1+\sum_{i=1}^{\infty}\tilde P(0,i)\sum_{l=1}^{i}\lambda b_{l+1}-\sum_{l=1}^{\infty}\lambda b_{l+1}\tilde P(0,l)-\lambda b\sum_{i=1}^{\infty}i\tilde P(0,i)\Big].$$

Hence

$$\lim_{s\to\infty} \varphi_s'(1) = P(0)\Big[1+\sum_{i=1}^{\infty}\tilde{P}(0,i)\Big(\sum_{l=1}^{i}\lambda b_l - \lambda i b\Big)\Big].$$

The above-mentioned and formula (2.15) lead to formula (iii).

2.3. Distribution of the busy time. Let the random variable T_i be an i-interval and let N be the number of services in the busy period. The distribution of the random variable N has been found in lemma 2.1. Of course, the busy time Y of the system equals

$$Y = \sum_{i=1}^{N} T_i.$$

The knowledge of the distributions of N and of T_i, $i = 1, 2, \ldots$, allows to find easily the Laplace-Stieltjes transform of the distribution of Y. It is given by

$$\sum_{n=1}^{\infty} \mathrm{P}(N=n)\prod_{i=1}^{n}\beta_i(z) = \sum_{n=1}^{\infty}\tilde{P}(0,n)\prod_{i=1}^{n}\beta_i(z).$$

Hence, the expected value of the length of the busy time has the form

$$\sum_{n=1}^{\infty}\tilde{P}(0,n)\sum_{i=1}^{n} b_i.$$

2.4. Existence of the steady-state distribution of the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$. The analogue of the system described in chapter 1 as problem A, which additionally assumes that the input stream is Erlangian, will be called A-*system*. The renewal moments are formed here by the moments in which a customer leaves the system remaining it empty and directly thereafter a new arrival takes place.

A sufficient condition for the existence of the steady-state distribution of the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$, being, however, a strong assumption posed on the mean service times, is the following one: $\lambda \sup b_n < 1$, where b_n is the mean service time of the n-th customer, counting from the last renewal. A weaker, but sufficient, condition for the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ to be steady-state is $\lambda \lim\limits_{n\to\infty} b_n < 1$. These statements are immediate corollaries from a theorem which is presented in this section.

Remark 2.3. If $\lim\limits_{n\to\infty} b_n$ exists, the condition

$$\lambda \overline{\lim_{n\to\infty}}\, b_n < 1$$

is necessary for the existence of the steady-state distribution of the considered Markov chain.

The assumption that the chain has also a steady-state distribution in the case $\lambda \lim_{n\to\infty} b_n \geq 1$ leads to contradiction. In particular, consider the case $\lambda \lim_{n\to\infty} b_n = 1$. Then, it follows from lemma 2.1, formula (iii), that $P(0) = 0$; thus $P(0, l) = 0$ and $P(k, l) = 0$ which contradicts the assumption about the existence of a steady-state distribution of the considered Markov chain. As it will be shown later, the condition $\lambda \overline{\lim_{n\to\infty}} b_n < 1$, in the case where the sequence $\{b_n\}_{n=1}^{\infty}$ is not converging, is not necessary for the existence of a steady-state distribution of the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$.

LEMMA 2.2. *If the input stream to an A-system has the m-th Erlang distribution with parameter λ, and if b_n is the mean service time of the n-th customer, counting from the last renewal, then a necessary condition for the existence of a steady-state distribution of the number of customers in the system at the moments if a customer exits is given by*

$$\frac{\lambda}{m}\sup_n b_n < 1.$$

Proof. Consider the sequence of random variables $\{\bar{\eta}_n, \bar{\gamma}_n, \nu_n\}_{n=1}^{\infty}$, where $\bar{\eta}_n$ is the number of customers in the system directly after the exit of the n-th customer, counting from the moment $t = 0$, $\bar{\gamma}_n$ is the number of the service time distribution of the n-th customer, and ν_n is the phase number (at the moment of the exit of the n-th customer) of the customer which will arrive to the system after the exit of the n-th customer. In other words, ν_n denotes the number of customers arriving in a Poisson stream with parameter λ from the moment of the last arrival to the moment of the n-th customer leaving the system.

Note that the sequence of random variables $\{\bar{\eta}_n, \bar{\gamma}_n, \nu_n\}_{n=1}^{\infty}$ forms a homogeneous, irreducible and aperiodic Markov chain. This is evident from the following one-step-transition probabilities in the chain considered:

$$\mathrm{P}\{(i,j,k)\to(l,j+1,r)\} = \int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{(l-i+1)m+r-k}}{\big((l-i+1)m+r-k\big)!}\,dB_{j+1}(u)$$

for $i, j = 1, 2, \ldots, l = i-1, i, \ldots,\ 0 \leqslant k, r \leqslant m-1$; $r \geqslant k$ for $l = i-1$;

$$\mathrm{P}\{(0,j,k)\to(l,1,r)\} = \int_0^\infty e^{-\lambda u}\frac{(\lambda u)^{lm+r}}{(lm+r)!}\,dB_1(u) \quad \text{for } 0 \leqslant k,\ r \leqslant m-1.$$

B_j denotes here the cumulative distribution function of the service time of the customer j, counting from the last renewal. The probabilities of all other one-step-transitions are equal to zero.

To prove this lemma the following theorem from [3] is used:

THEOREM 2.2. *For a homogeneous, irreducible and aperiodic Markov chain with one-step-transition probability matrix* $P = (p_{ij})$ *to have a steady-state distribution it suffices that there exist the number* $\varepsilon > 0$, *the non-negative integer* i_0 *and non-negative numbers* $x_0, x_1, \ldots$ *such that*

$$\text{(i)} \qquad \sum_{j=0}^{\infty} p_{ij} x_j \leqslant x_i - \varepsilon \qquad \textit{for } i > i_0,$$

$$\text{(ii)} \qquad \sum_{j=0}^{\infty} p_{ij} x_j < \infty \qquad \textit{for } i \leqslant i_0.$$

Now, let us define the numbers i_0, $\varepsilon > 0$ and the number $x_{l,j,r}$ associated with the state (l, j, r). Let us number the states of the Markov chain $\{\bar{\eta}_n, \bar{\gamma}_n, \nu_n\}_{n=1}^{\infty}$ in the following way:

For $r = 0, 1, \ldots, m-1$, the states $(0, 1, r)$ have numbers $0, 1, \ldots, m-1$, respectively, the states $(1, 1, r)$ have numbers $m, m+1, \ldots, 2m-1$, respectively, and all other states have numbers greater than $2m-1$.

Let $i_0 = 2m-1$. Write $b = \sup\limits_n b_n$, $d = b - b_1/2$, $\varepsilon_1 = 1 - \lambda b/m$ and $\varepsilon = \varepsilon_1 \min\{b_1, d\}$.

The assumption of lemma 2.2 states that $b\lambda/m < 1$; this also implies that $b_j \leqslant b$. Therefore, $\varepsilon > 0$.

With every state (l, j, r) let us associate the number $x_{l,j,r}$ as follows:

$$\begin{aligned}
x_{0,1,r} &= 0 && \text{for } 0 \leqslant r \leqslant m-1,\\
x_{0,j,r} &= b_1 && \text{for } 0 \leqslant r \leqslant m-1,\ j \geqslant 2,\\
x_{l,1,r} &= b(l + r/m) && \text{for } 0 \leqslant r \leqslant m-1,\ l \geqslant 1,\\
x_{l,j,r} &= d(l + r/m) + b_1 && \text{for } 0 \leqslant r \leqslant m-1,\ l \geqslant 1,\ j \geqslant 2.
\end{aligned}$$

It is seen that, for every (l, j, r), $x_{l,j,r} \geqslant 0$ holds. Notice also that, for $l \geqslant 2$, $x_{l,1,r} \geqslant x_{l,j,r}$ holds. We shall show that, for the defined i_0, ε and $x_{l,j,r}$, inequalities (i) and (ii) of theorem 2.2 are satisfied. First, let us show that, for the states (i, j, k) whose numbers exceed $i_0 = 2m-1$, inequality (i) is satisfied. Consider the case $i \geqslant 2$. The left-hand side of (i) equals then

$$(2.16)\qquad \sum_{l,r} \mathrm{P}\{(i,j,k)\to(l,j+1,r)\}x_{l,j+1,r}$$

$$= \int_0^\infty \sum_{l=i-1}^{\infty} \sum_{r=(0\vee k)(l-i+1)}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{(l-i+1)m+r-k}}{\big((l-i+1)m+r-k\big)!}x_{l,j+1,r}dB_{j+1}(u)$$

$$= \int_0^\infty \sum_{l=0}^{\infty} \sum_{r=(0\vee k)l}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm+r-k}}{(lm+r-k)!}x_{l+i-1,j+1,r}dB_{j+1}(u),$$

where the symbol $(0\vee k)l$ denotes zero if $l>0$, and the number k if $l=0$. In this case $l+i-1\geqslant 1$ and $j+1\geqslant 2$; thus $x_{l+i-1,j+1,r} = d(l+i-1+r/m)+b_1$. The right-hand side of (2.16) can be transformed as follows:

$$\int_0^\infty \sum_{l=0}^{\infty} \sum_{r=(0\vee k)l}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm-k+r}}{(lm-k+r)!}\left(d\left(l+i-1+\frac{r}{m}\right)+b_1\right)dB_{j+1}(u)$$

$$=b_1+d(i-1)+\frac{kd}{m}+\frac{d}{m}\int_0^\infty \sum_{l=0}^{\infty} \sum_{r=(0\vee k)l}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm-k+r}}{(lm-k+r)!}(lm-k+r)dB_{j+1}(u)$$

$$= x_{i,j,k}-d+\frac{d}{m}\int_0^\infty \lambda u\,dB_{j+1}(u) = x_{i,j,k}-d\left(1-\frac{\lambda b_{j+1}}{m}\right)\leqslant x_{i,j,k}-\varepsilon \leqslant x_{i,1,k}-\varepsilon.$$

Next, consider the case $i=1$, $j\geqslant 2$. Now, the left-hand side of inequality (i) equals

$$(2.17)\qquad \sum_{l=0}^{\infty} \sum_{r=(0\vee k)l}^{m-1} \mathrm{P}\{(1,j,k)\to(l,j+1,r)\}x_{l,j+1,r}$$

$$= \sum_{r=k}^{m-1} \mathrm{P}\{(1,j,k)\to(0,j+1,r)\}x_{0,j+1,r}+\sum_{l=1}^{\infty}\sum_{r=0}^{m-1} \mathrm{P}\{(1,j,k)\to(l,j+1,r)\}x_{l,j+1,r}$$

$$= b_1\sum_{r=k}^{m-1} \mathrm{P}\{(1,j,k)\to(0,j+1,r)\}+b_1\sum_{l=1}^{\infty}\sum_{r=0}^{m-1} \mathrm{P}\{(1,j,k)\to(l,j+1,r)\}+$$

$$+d\sum_{l=1}^{\infty}\sum_{r=0}^{m-1} \mathrm{P}\{(1,j,k)\to(l,j+1,r)\}(l+r/m).$$

After simple derivations the right-hand side of (2.17) assumes the form

$$b_1+\frac{d}{m}\int\limits_0^\infty \sum_{l=1}^{\infty}\sum_{r=0}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm+r-k}}{(lm+r-k)!}(lm+r)\,dB_{j+1}(u)$$

$$\leqslant b_1+\frac{kd}{m}+\frac{d}{m}\int\limits_0^\infty \sum_{l=1}^{\infty}\sum_{r=0}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm-k+r}}{(lm-k+r)!}(lm-k+r)\,dB_{j+1}(u)$$

$$\leqslant b_1+\frac{kd}{m}+\frac{d}{m}\lambda b_{j+1} = b_1+d\left(\frac{m+k}{m}\right)-d\left(1-\frac{\lambda b_{j+1}}{m}\right)\leqslant x_{1,j,k}-\varepsilon.$$

It remains to consider the case $i = 0$ and $j > 1$. The states $(0, j, r)$ have then numbers greater than i_0. Since $x_{0,1,r} = 0$, the left-hand side of inequality (i) is now equal to

$$\sum_{l=1}^{\infty}\sum_{r=0}^{m-1} \mathrm{P}\{(0, j, k)\to(l, 1, r)\}x_{l,1,r}$$

$$= b\int\limits_0^\infty \sum_{l=1}^{\infty}\sum_{r=0}^{m-1} e^{-\lambda u}\frac{(\lambda u)^{lm+r}}{(lm+r)!}\left(l+\frac{r}{m}\right)dB_1(u) = \frac{1}{m}b\lambda b_1+b_1-b_1\leqslant x_{0,j,r}-\varepsilon.$$

We must still verify that, for the states (i, j, k) having numbers not greater than $i_0 = 2m-1$, inequality (ii) holds, i.e. the expression

$$\sum_{l,r,s}\mathrm{P}\{(i, j, k)\to(l, s, r)\}x_{l,s,r} \tag{2.18}$$

is finite. First, consider the states whose numbers are less than m. They are represented by $(0, 1, k)$, where $k = 0, 1, \ldots, m-1$. For those states the sum in (2.18) is extended from $l = 0$ to infinity and $s = 1$. The expression is bounded by the number $\lambda bb_1/m$ which is, of course, finite. For the states whose numbers are greater than $m-1$ and less than $2m$, i.e. for $(1, 1, k)$, where $k = 0, 1, \ldots, m-1$, the sum in (2.18) is extended from $l = 0$ to infinity and $s = 2$. The expression is bounded by the number $b_1+d+db_2\lambda/m$, which also is finite. This completes the proof of lemma 2.2.

A sufficient condition for the existence of a steady-state distribution of the number of customers in an A-system whose input is Poissonian with parameter λ will be given now.

THEOREM 2.3. *If there exists a natural number m such that, for*

$$\bar{b} = \sup_{k\geqslant 1}\sum_{i=m(k-1)+1}^{mk} b_i, \tag{2.19}$$

$\lambda\bar{b}/m < 1$ holds, the Markov chain $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$ has a steady-state distribution, where b_i is the mean service time of the i-th customer, counting from the last renewal.

Proof. The A-system whose input stream is Poissonian with parameter λ will be called the first system now. Consider also the following system with Poissonian input having the parameter λ:

Only customers having numbers $m(j-1)+1$ for $j = 1, 2, \ldots$, counting from the last renewal, where m is given by theorem 2.3, remain in the system. Assume also that customer numbered $m(j-1)+1$ for $j = 1, 2, \ldots$ has a service time equal to

$$Y_j = \sum_{i=m(j-1)+1}^{mj} X_i,$$

where X_i is the service time of the customer i, counting from the last renewal in the first system. Write $\bar{b}_j = \mathrm{E}\,Y_j$. The cumulative distribution function of Y_j is denoted by $\bar{B}_j$. It has the form

$$\bar{B}_j = B_{m(j-1)+1} * B_{m(j-1)+2} * \ldots * B_{mj},$$

where the asterisk denotes a composition, and B_i is the cumulative distribution function of X_i. Such a system will be called the second system.

Since the rejected customers have here no influence on the distribution of the number of customers in the system, one can assume that the input stream of the second system has the m-th Erlang distribution with parameter λ.

Denote by $P_i(0)$, $i = 1, 2$, the probability that a customer leaving system i remains it empty. From the construction of the second system it follows that $P_1(0) \geqslant P_2(0)$, because the busy time interval in the second system is stochastically not smaller than that in the first system. From lemma 2.2 we have $P_2(0) > 0$ since

$$\frac{\lambda}{m} \sup_j \bar{b}_j = \frac{\lambda}{m} \bar{b} < 1.$$

This completes the proof of theorem 2.3.

The remarks which were formulated at the begin of this section follow from theorem 2.3. In fact, if $\lim_{n\to\infty} b_n = c < 1$, then, for $\varepsilon > 0$ such that $c+\varepsilon < 1$, there exists an n_0 such that, for $n > n_0$, $b_n\lambda < c+\varepsilon/2 < 1$ holds. Take now m_1 such that

$$\frac{\lambda}{m_1} \sum_{i=1}^{n_0} b_i < c+\varepsilon.$$

If we take $m = n_0$ for $m_1 < n_0$, we obtain

$$\frac{\lambda}{m}\sum_{i=1}^{m} b_i < 1.$$

It follows from this that the conditions of theorem 2.3 are satisfied for $m = n_0$; thus the statement of theorem 2.3 is true. If $m_1 > n_0$, write $C_n = \sum_{i=1}^{n} b_i$. Then

$$\frac{\lambda}{n} C_n = \frac{\lambda}{n} C_{n_0} + \frac{\lambda}{n}(C_n - C_{n_0}). \tag{2.20}$$

Notice that the following holds:

$$\frac{\lambda}{n}(C_n - C_{n_0}) \leqslant \frac{\lambda}{n}(n-n_0)\sup_i b_i \leqslant \frac{n-n_0}{n}\left(c+\frac{\varepsilon}{2}\right) < c+\frac{\varepsilon}{2} \qquad \text{for } n > n_0.$$

If we take m_2 such that, for $n > m_2$, there is

$$\frac{\lambda}{n} C_{n_0} < \frac{\varepsilon}{2},$$

then we obtain from (2.20) for $n \geqslant m$, where $m = \max\{n_0, m_2\}$, the relation

$$\frac{\lambda}{n} C_n < \frac{\varepsilon}{2+c} + \frac{\varepsilon}{2} < 1.$$

We have thus found such $m \geqslant n_0$ that

$$\frac{\lambda}{m}\sum_{i=1}^{m} b_i < 1.$$

The inequality

$$\sup_k \frac{\lambda}{m} \sum_{i=m(k-1)+1}^{mk} b_i < 1$$

is also satisfied, because

$$\frac{\lambda}{m} \sum_{i=m(k-1)+1}^{mk} b_i \leqslant c+\varepsilon < 1.$$

This, in turn, proves that the condition $\lambda \varlimsup_{n\to\infty} b_n < 1$ is stronger than the condition $\lambda\bar{b}/m < 1$ in theorem 2.3, where $\bar{b}$ is given by (2.19).

In lemma 2.2 it was given a sufficient condition for an A-system with Erlangian input stream to have a steady-state distribution of the number of customers in the system. This condition can be weakened.

THEOREM 2.4. *If the input stream of an A-system has the s-th Erlang distribution with parameter λ and if b_n is the mean service time of the n-th customer, counting from the last renewal, then a sufficient condition for the existence of the steady-state distribution of the number of customers in the system is the existence of such a natural number m that if*

$$\bar{b} = \sup_{k\geqslant 1} \sum_{i=m(k-1)+1}^{mk} b_i,$$

then $\lambda\bar{b}/ms < 1$.

Proof. Let the system described in theorem 2.4 be called the first system. Similarly as in the proof of theorem 2.3, we construct the second system. In that system there remain only customers numbered $m(j-1)+1$, $j = 1, 2, \ldots$, counting from the last renewal. All other customers are rejected. It follows from this that the input stream has the ms-th Erlang distribution with parameter λ. The j-th customer, counting from the last renewal, which remains in the system, has a service time equal to

$$Y_j = \sum_{i=m(j-1)+1}^{mj} X_i \quad \text{for } j = 1, 2, \ldots$$

If $P_i(0)$ denotes the probability that a customer leaving system i remains it empty, then $P_1(0) \geqslant P_2(0)$. This follows from the fact that the busy time interval of the second system is stochastically not smaller than that of the first system. Notice that

$$\mathrm{E}Y_j = \sum_{i=m(j-1)+1}^{mj} b_i.$$

From the assumptions of the theorem it follows that

$$\frac{\lambda}{ms} \sup_j \mathrm{E}Y_j = \frac{\lambda b}{ms} < 1.$$

Using lemma 2.2, we obtain the thesis of theorem 2.4.

2.5. Example of problem A. Now, we shall show how in practice the generating function of the distribution of the number of customers in an A-system can be found.

Assume that

$$B_i(t) = \begin{cases} 0 & \text{for } t \leqslant 0, \\ 1-\exp(-\mu_i t) & \text{for } t > 0, \end{cases}$$

and also that $B_{3+j} \equiv B_3$ for $j \geqslant 0$.

With these assumptions the generating function $\psi(z)$ can be found from formula (2.15) in which $s = 2$. Formula (2.15) gives the general form of the generating function of the distribution of the number of customers in the system considered as problem A in which, for fixed s, there holds $B_{s+j} \equiv B_{s+1}$ for $j \geqslant 1$. From (2.15) we have

$$(2.21)\quad \psi(z) = P(0)\left[K_1(z)+\left(z-K_3(z)\right)^{-1}K_2(z)\left(K_1(z)-P(0,1)\right)-\right.$$
$$\left.-\left(z-K_3(z)\right)^{-1}K_3(z)\left(1-P(0,1)\right)\right].$$

Substituting in (2.21) $K_i(z) = \mu_i/(\lambda+\mu_i-\lambda z)$ for $i = 1, 2, 3$ and $\tilde{P}(0,1) = \mu_1/(\lambda+\mu_1)$ (which has been calculated from (i) in lemma 2.1), we obtain, finally,

$$\psi(z) = P(0)\left[\frac{\mu_1}{\lambda+\mu_1-\lambda z}-\right.$$
$$-\frac{\mu_2\mu_1\lambda z(\lambda+\mu_3-\lambda z)}{(\lambda+\mu_2-\lambda z)(\lambda+\mu_1-\lambda z)\left(\lambda z^2-(\lambda+\mu_3)z+\mu_3\right)(\lambda+\mu_1)}-$$
$$\left.-\frac{(\lambda+\mu_3-\lambda z)\lambda}{\left(\lambda z^2-(\lambda+\mu_3)z+\mu_3\right)(\lambda+\mu_1)}\right].$$

The function $\psi(z)$ can easily be expressed as a power series from which $P(k, l)$ can be calculated. $P(0)$ is to be calculated from (iii) as

$$P(0) = \left(1-\frac{\lambda}{\mu_3}\right)\left[1+\lambda\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)-2\frac{\lambda}{\mu_3}+\frac{\lambda\mu_1}{\lambda+\mu_1}\left(\frac{1}{\mu_3}-\frac{1}{\mu_2}\right)\right]^{-1}.$$

3. Solution of problem B. Problem B differs from problem A in taking into account the renewal of the service mechanism, i.e. one assumes that the renewal time is a random variable with given distribution. Let τ_i be the i-th renewal moment of the process $\zeta(t)$, i.e. the i-th moment of the system being empty, and let Y_i be the renewal time of the service mechanism, counting from the moment τ_i+0. Assume also that the random variables of the sequence $\{Y_n\}$ are independent and have an identical cumulative distribution function $F(t)$. Denote by Z_i the random variable expressing the time interval from moment τ_i+0 to the moment of the next arrival. The sequence of random variables Z_i, $i = 1, 2, \ldots$, is stochastically independent. From the assumption about the input stream it follows that the random variables $Z_1, Z_2, \ldots$ have identical distributions, i.e. are exponential with parameter λ.

Consider a system in which the service of the first customer begins at the moment of its arrival and is equal to $\tilde{X}_1 = (Y_1-Z_1)^+ + X_1$, where

$$(Y_1-Z_1)^+ = \begin{cases} 0 & \text{if } Y_1-Z_1 \leqslant 0, \\ Y_1-Z_1 & \text{if } Y_1-Z_1 > 0. \end{cases}$$

Notice that the number of customers in the system at the moment of the first customer leaving the system is equal to the number of customers in the system at the moment of the first customer leaving a system in which the first customer waits for the end of the renewal of the service mechanism and has a service time of length X_1.

To find the distribution of the number of customers in the system in the moments of customer exits, the results obtained for the A-system can be used. One has only to substitute $B_1 := \tilde{B}_1$, where $\tilde{B}_1$ is the composition of the cumulative distribution functions of $(Y_1 - Z_1)^+$ and X_1, i.e. $\tilde{B}_1 = B_1 * C$, with $C(t)$ being the cumulative distribution function of $(Y_1 - Z_1)^+$ of the form

$$C(t) = \begin{cases} 0 & \text{for } t < 0, \\ \lambda \int_0^\infty e^{-\lambda x} F(t+x)\,dx & \text{for } t \geqslant 0. \end{cases}$$

4. Example of problem C. Assume that

(1) the service mechanism works without breakdown in a time interval equal to the sum of the service times of the first r customers, and

(2) the service time of the i-th customer ($i \leqslant r$) has the cumulative distribution function

$$B_i(t) = \begin{cases} 0 & \text{for } t \leqslant 0, \\ 1 - \exp(-\mu_i t) & \text{for } t > 0. \end{cases}$$

Having the generating function of the number of customers in the system at the moments of the exit of the first r customers, it is easy to obtain that function in the case where r is a random variable.

Let us find now this generating function for r fixed. Write, for $l = 1, 2, \ldots$, $\nu_l = \mu_l/\lambda$ and $\varrho_l = \lambda/(\lambda + \mu_l)$. Using the notation from chapter 2 and our assumptions, we obtain

$$K_l(z) = \frac{\nu_l \varrho_l}{1 - z\varrho_l} \quad \text{and} \quad K_l^i(0) = i!\,\nu_l \varrho_l^{i+1}. \tag{4.1}$$

From formula (i) of lemma 2.1 we obtain

$$\tilde{P}(0, n) = \prod_{i=1}^{n} \nu_i \varrho_i Q(n-1), \tag{4.2}$$

where $Q(0) = 1$, and

$$Q(n-1) = \sum_{j_1=1}^{n-1} \sum_{j_2=f(l-1-x_1,\, l-1)}^{n-1-x_1} \cdots \sum_{j_{l-1}=f(l-1-x_{l-2},\, 2)}^{l-1-x_{l-2}} \prod_{m=1}^{n-1} \varrho_m^{j_m} \quad \text{for } n = 2, 3, \ldots$$

Let $\sum_{V_k^s}$ denote summation extended over all sequences $(i_1, i_2, \ldots, i_s)$ which are s-element variations of the set $\{1, 2, \ldots, k\}$. Assume also $\sum_{i=m}^{n} \ldots$ equal to zero for $m > n$ (the set of summation is empty).

The following theorem states the form of the generating function of the distribution of the number of customers in the system at the moment of the exit of the l-th customer, counting from the last renewal. From the assumption that r is finite it follows $P(0) > 0$, i.e. the system has a steady-state distribution of the number of customers in the queue.

THEOREM 4.1. *If the cumulative distribution functions $B_i(t)$ satisfy condition* (2) *and if $0 < \mu_i < \infty$ for $0 \leqslant i \leqslant l$, $l = 2, 3, \ldots$, then there hold the relations*

$$\text{(i)}\qquad \begin{aligned} \psi(z, l) &= P(0)\tilde{\psi}(z, l), \\ \tilde{\psi}(z, l) &= \prod_{i=1}^{l} \nu_i \varrho_i \Big(\prod_{i=1}^{l} (1 - z\varrho_i)\Big)^{-1} \times \\ &\quad \times \Big\{ \sum_{k=0}^{[l/2]-1} Q(l-2-k) \sum_{s=k+1}^{l-1-k} \frac{(-1)^{s-1}}{s!} z^{s-k-1} \sum_{V_{l-1-k}^s} \prod_{j=1}^{s} \varrho_{i_j} \Big\}, \end{aligned}$$

$$\text{(ii)}\qquad \tilde{P}(0, l) = \sum_{k=0}^{[l/2]-1} \tilde{P}(0, l-1-k) \prod_{i=l-k}^{l} \nu_i \varrho_i \frac{(-1)^k}{(k+1)!} \sum_{V_{l-1-k}^{k+1}} \prod_{j=1}^{k+1} \varrho_{i_j},$$

where $[\cdot]$ is the entier function.

The proof of theorem 4.1 is based on induction with respect to l. For $l = 2$, formula (i) has the form

$$\psi(z, 2) = \prod_{i=1}^{2} \nu_i \varrho_i \Big(\prod_{i=1}^{2} (1 - z\varrho_i)\Big)^{-1} \varrho_1. \tag{4.3}$$

On the other hand, calculation of $\psi(z, 2)$ from (2.2) gives

$$\psi(z, 2) = P(0) K_2(z)\big(K_1(z) - \tilde{P}(0, 1)\big)/z = P(0)\tilde{\psi}(z, 2),$$

where $\tilde{\psi}(z, 2) = \psi(z, 2)/P(0)$.

Substituting into $\psi(z, 2)$ formulae (4.1) and (4.2) with appropriate l and n, we obtain (4.3). Assume now that formula (i) holds for some fixed l. We shall prove it for $l+1$. Denote by $R_l(z)$ the expression

$$\sum_{k=0}^{[l/2]-1} Q(l-2-k) \sum_{s=1+k}^{l-1-k} \frac{(-1)^{s-1}}{s!} z^{s-k-1} \sum_{V_{l-1-k}^s} \prod_{j=1}^{s} \varrho_{i_j}.$$

We have from theorem 2.1

$$z\tilde{\psi}(z, l+1) = K_{l+1}(z)\,[\tilde{\psi}(z, l) - \tilde{P}(0, l)].$$

Substitution of $\tilde{\psi}(z, l)$ from theorem 4.1 and of $P(0, l)$ and $K_{l+1}(z)$ from (4.1) and (4.2) leads to

$$(4.4)\quad z\tilde{\psi}(z, l+1) = \prod_{i=1}^{l+1} \nu_i \varrho_i \Big(\prod_{i=1}^{l+1}(1-z\varrho_i)\Big)^{-1}\Big\{R_l(z) - Q(l-1)\prod_{i=1}^{l}(1-z\varrho_i)\Big\}.$$

Denote the function $R_l(z) - Q(l-1)\prod_{i=1}^{l}(1-z\varrho_i)$ by $W_l(z)$. It is easily verified that

$$(4.5)\qquad \prod_{i=1}^{l}(1-z\varrho_i) = 1 + \sum_{s=1}^{l}\frac{(-1)^s}{s!}z^s \sum_{V_l^s}\prod_{j=1}^{s}\varrho_{i_j}.$$

Substitution of (4.5) into (4.4) gives

$$W_l(z) = R_l(z) + Q(l-1)\sum_{s=1}^{l}\frac{(-1)^{s-1}}{s!}z^s\sum_{V_l^s}\prod_{j=1}^{s}\varrho_{i_j} - Q(l-1).$$

Transform now $W_l(z)$ to the form

$$(4.6)\quad W_l(z) = \Big\{\sum_{k=0}^{[l/2]-1} Q(l-2-k)\sum_{s=k+2}^{l-1-k}\frac{(-1)^{s-1}}{s!}z^{s-k-1}\sum_{V_{l-1-k}^s}\prod_{j=1}^{s}\varrho_{i_j}\Big\} +$$

$$+\Big\{Q(l-1)\sum_{s=1}^{l}\frac{(-1)^{s-1}}{s!}z^s\sum_{V_l^s}\prod_{i=1}^{s}\varrho_{i_j}\Big\} +$$

$$+\sum_{k=0}^{[l/2]-1} Q(l-2-k)\frac{(-1)^k}{(k+1)!}\sum_{V_{l-1-k}^{k+1}}\prod_{j=1}^{k+1}\varrho_{i_j} - Q(l-1).$$

From (4.4) it follows

$$(4.7)\qquad \tilde{\psi}(z, l+1) = \prod_{i=1}^{l+1}\nu_i\varrho_i\Big(\prod_{i=1}^{l+1}(1-z\varrho_i)\Big)^{-1}\frac{W_l(z)}{z}.$$

Since $\tilde{\psi}(z, l+1)$ is an analytic function in the region $|z| \leqslant 1$, $W_l(z)$ is a polynomial with respect to z without the free term (this follows from the definition of $W_l(z)$). Therefore,

$$(4.8)\qquad Q(l-1) = \sum_{k=0}^{[l/2]-1} Q(l-2-k)\frac{(-1)^k}{(k+1)!}\sum_{V_{l-1-k}^{k+1}}\prod_{j=1}^{k+1}\varrho_{i_j}.$$

One can show that

$$(4.9)\qquad W_l(z) = \Big(\sum_{k=0}^{[l/2]-1} Q(l-1-k) \sum_{s=k+1}^{l-k} \frac{(-1)^{s-1}}{s!} z^{s-k-1} \sum_{V_{l-k}^{s}} \prod_{j=1}^{s} \varrho_{i_j}\Big) z.$$

Formula (4.9) is based on the following observation:

Firstly, consider (4.6) for $l = 2n$, i.e. such that $[l/2] = [(l+1)/2]$. For $k = [l/2]-1 = n-1$, the expression

$$\sum_{s=k+2}^{l-1-k} \frac{(-1)^{s-1}}{s!} z^{s-k-1} \sum_{V_{l-1-k}^{s}} \prod_{j=1}^{s} \varrho_{i_j}$$

is then equal to zero, since $k+2 > l-1-k$. Therefore, the sum in the first term of (4.5) is extended from $k = 0$ to $[l/2]-2 = n-2$. Substitution of $k: = k+1$ into the first term of (4.6) gives

$$\sum_{k=1}^{n-1} Q(l-1-k) \sum_{s=k+1}^{l-k} \frac{(-1)^{s-1}}{s!} z^{s-k} \sum_{V_{l-k}^{s}} \prod_{j=1}^{s} \varrho_{i_j}$$

and summation with the second term of (4.6) leads to (4.9).

Similarly, consider the case $l = 2n+1$. Substitute $k: = k+1$ into the first term of (4.6). One obtains then

$$\sum_{k=1}^{n} Q(l-1-k) \sum_{s=k+1}^{l-k} \frac{(-1)^{s-1}}{s!} z^{s-k} \sum_{V_{l-k}^{s}} \prod_{j=1}^{s} \varrho_{i_j}.$$

Addition of the second term of (4.6) leads to (4.9). Substitution of (4.9) into (4.7) leads to the conclusion that (i) holds for $l+1$, thus also for every $l \geqslant 2$.

From formula (4.8) one obtains easily (ii). It suffices to multiply both sides of (4.8) by $\prod_{i=1}^{l} \nu_i \varrho_i$.

Remark. In the case of exponential service times formula (ii) of theorem 4.1 enables the calculation of $\tilde{P}(0, l+1)$ when all $\tilde{P}(0, i)$ for $i = 1, 2, \ldots, l$ are known. This formula is easier to handle than formula (i) of lemma 2.1.

COROLLARY 4.1. *If $\varrho_i \neq \varrho_j$ for $i \neq j$, then, for $l = 2, 3, \ldots,$ there holds*

$$\tilde{\psi}(z, l) = \left\{ \sum_{i=1}^{[l/2]-1} \tilde{P}(0, l-1-k) \prod_{i=l-k}^{l} \nu_i \varrho_i \sum_{s=k+1}^{l-1-k} \frac{(-1)^s}{s!} z^{s-k-1} \sum_{V_{l-1-k}^s} \prod_{j=1}^{s} \varrho_{i_j} \right\} \times$$

$$\times \left\{ \sum_{n=0}^{\infty} \sum_{i=1}^{r} \varrho^{n+1-l} \Big(\prod_{\substack{j=1 \\ j \neq i}}^{l} (\varrho_i - \varrho_j) \Big)^{-1} z^n \right\}.$$

Proof. Using formula (i) of theorem 4.1, formula (4.1) and the equality

$$\Big(\prod_{i=1}^{l} (1 - z\varrho_i) \Big)^{-1} = \sum_{i=1}^{l} \varrho_i^{l-1} \Big(\prod_{\substack{j=1 \\ j \neq i}}^{l} (\varrho_i - \varrho_j) \Big)^{-1} \frac{1}{1 - z\varrho_i}$$

which follows from the equality

$$\Big(\sum_{i=1}^{l} (z + a_i) \Big)^{-1} = \sum_{i=1}^{l} \frac{1}{(z + a_i)\varphi'(-a_i)},$$

where $\varphi(z) = \prod_{i=1}^{l} (z + a_i)$, one obtains corollary 4.1.

Formula (i) of theorem 4.1 gives the generating function of the distribution of the number of customers in the system at the moment of the exit of the l-th customer under the assumption that $0 < \mu_i < \infty$ for $i \leqslant l$. The distribution of the number of customers in the system during the time when the service mechanism is able to serve customers, which is of interest to us, can be obtained from the following generating function of this distribution:

$$\psi(z)_r = \Big(\sum_{l=1}^{r} \psi(1, l) \Big)^{-1} \sum_{l=1}^{r} \psi(z, l), \tag{4.10}$$

where $\psi(z, l)$ is given by (i) in theorem 4.1. The distribution of the number of services in the busy period is given by

$$\tilde{P}_r^*(0, l) = \tilde{P}(0, l) \Big(\sum_{i=1}^{r} \psi(1, i) \Big)^{-1} \quad \text{for } l \leqslant r.$$

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„MĘCZĄCY SIĘ” SYSTEM MASOWEJ OBSŁUGI TYPU $M/G/1$

STRESZCZENIE

Praca poświęcona jest badaniu systemów masowej obsługi, dla których zakładamy, że czasy obsługi jednostek są niezależnymi zmiennymi losowymi, niekoniecznie o tym samym rozkładzie. Kolejność obsługi jednostek jest zgodna z kolejnością ich zgłoszeń do systemu, natomiast czas obsługi i-tej jednostki — obliczając od ostatniego momentu odnowy systemu — ma dystrybuantę B_i. Przyjmujemy, że momentami odnowy systemu są te, w których system zostaje pusty.

Rozpatrujemy dwa typy systemów. W systemach typu A urządzenie obsługujące nie ulega awarii ani w czasie obsługi jednostki, ani też gdy jest wolne od obsługi jednostek. W systemach typu B uwzględniamy możliwość awarii i odnowy urządzenia obsługującego. Podziału dokonaliśmy tylko dla wygody badań, bowiem — po odpowiedniej interpretacji — systemy typu B sprowadzają się do systemów typu A.

Badaniem systemów typu A, przy założeniu poissonowskiego strumienia zgłoszeń jednostek do systemu, zajęliśmy się w rozdziale 2. Do badania procesu liczby jednostek w systemie zastosowaliśmy metodę włożonych łańcuchów Markowa. Badaliśmy więc charakterystyki łańcucha Markowa $\{\eta_n, \gamma_n\}_{n=1}^{\infty}$, gdzie η_n jest liczbą jednostek w systemie w momencie wyjścia n-tej jednostki z systemu, obliczając od chwili $t = 0$, natomiast γ_n jest numerem tej jednostki, obliczonym od ostatniej odnowy. W rozdziale tym udowodniliśmy twierdzenie 2.1, podające związki rekurencyjne między funkcjami tworzącymi $\psi(z, l)$ prawdopodobieństw liczby jednostek w systemie typu A w momencie wyjścia l-tej jednostki z systemu, obliczając od ostatniej odnowy systemu. Znalezienie jawnej postaci funkcji tworzącej $\psi(z)$ rozkładu liczby jednostek w systemie typu A jest dosyć trudne. W praktyce jednak często można założyć, że dla pewnego ustalonego s zachodzi $B_{s+i} \equiv B_{s+1}$ dla każdego $i \geqslant 1$. Dla tego założenia funkcja tworząca $\psi(z)$ określona jest wzorem (2.15). Prosty przykład znajdowania funkcji $\psi(z)$ przy tym założeniu i przy założeniu wykładniczości obsług podany jest w punkcie 2.5.

Interesującymi charakterystykami systemów typu A są również rozkład przedziału zajętości systemu oraz prawdopodobieństwo, że system jest pusty. Otóż, jeżeli istnieje stacjonarny rozkład liczby jednostek w systemie typu A, to wspomniane charakterystyki dane są wzorami (i), (ii) oraz (iii) w lemacie 2.1.

W rozdziale 1 wspomnieliśmy o pewnym zagadnieniu C, które polega na tym, że urządzenie obsługujące ma z góry dany czas bezawaryjnej pracy. W tym przypadku ograniczyliśmy się jedynie do pewnego przykładu, tzn. przyjęliśmy, że po zakończeniu r obsług — obliczając od ostatniej odnowy — urządzenie obsługujące jest odnawiane, a jednostki, które nie zostały obsłużone lub te, które zgłoszą się do systemu w czasie odnowy urządzenia obsługującego, są stracone. Przy założeniu, że czasy obsług są wykładnicze, przykład ten podaliśmy w rozdziale 4. Wspomnieliśmy tam również, że r może być dowolną zmienną losową. Przy tych założeniach wzór (4.10) daje funkcję tworzącą liczby jednostek w systemie w momentach wyjść r pierwszych jednostek z systemu, obliczając od ostatniej odnowy. Funkcje $\psi(z, l)$ określone są wzorem (i) w twierdzeniu 4.1. W twierdzeniu 4.1 mamy również wzór (ii), który dla założeń z rozdziału 4 pozwala na łatwiejsze niż wzór (i) w lemacie 2.1 obliczenie prawdopodobieństw $\tilde{P}(0, i)$.

Przedmiotem naszych badań w tej pracy są również warunki, dla których istnieje stacjonarny rozkład liczby jednostek w systemie typu A. Zagadnieniem tym zajęliśmy się w punkcie 2.4. Najmocniejsze w tym punkcie jest twierdzenie 2.4. Mówi ono, że jeżeli strumień zgłoszeń jednostek do systemu typu A jest erlangowski rzędu s z parametrem λ, natomiast b_n jest średnią czasu obsługi n-tej jednostki, obliczając od ostatniej odnowy, to warunkiem dostatecznym na to, aby istniał rozkład stacjonarny liczby jednostek w systemie, jest istnienie liczby naturalnej m, takiej że $\lambda\bar{b}/ms < 1$, gdzie $\bar{b}$ określone jest wzorem (2.19).