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ON MINIMAX SEQUENTIAL ESTIMATION OF THE MEAN VALUE OF A STATIONARY GAUSSIAN MARKOV PROCESS

Introduction. In this paper we consider the problem of minimax sequential estimation of the mean value of the Ornstein-Uhlenbeck process. The loss incurred by the statistician is due not only to the error of estimation but also to the cost of observation. We prove that for a quadratic loss function connected with the error of estimation the fixed-time plan is minimax.

1. Preliminaries. Let us introduce the following notation:

$C[0, \infty)$ denotes the space of real continuous functions x: $[0, \infty) \to R$.

$\mathscr{F}$ is the smallest σ-algebra of subsets of the set $C[0, \infty)$ with respect to which the functions $x(t)$: $C[0, \infty) \to R$, $t \in [0, \infty)$, are measurable.

$\mathscr{F}_t$ is the smallest σ-algebra of the subsets of the set $C[0, \infty)$ with respect to which the functions $x(s)$: $C[0, \infty) \to R$, $s \in [0, t]$, are measurable.

μ_θ is a probability measure on $(C[0, \infty), \mathscr{F})$ dependent on the real parameter $\theta \in A \subset R$.

$\mu_{\theta,t}$ is the restriction of μ_θ to the σ-algebra $\mathscr{F}_t$.

τ is a stopping time, i.e.

$$\tau : C[0, \infty) \to [0, \infty],$$

$$\{x(\cdot) \in C[0, \infty) : \tau(x(\cdot)) \leqslant t\} \in \mathscr{F}_t, \qquad t \in [0, \infty),$$

$$\mu_\theta(\{x(\cdot) : \tau(x(\cdot)) < \infty\}) = 1 \quad \text{for all } \theta \in A.$$

Let us suppose that the measure $\mu_{\theta,t}$ is absolutely continuous with respect to the measure $\mu_{\theta_0,t}$ and that the density function takes the form

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = g(t, S_t(x(\cdot)), \theta, \theta_0),$$

where g is a continuous function and S_t is a mapping from $C([0, \infty))$ to R, $\mathscr{F}_t$-measurable and right continuous μ_θ-almost surely with respect to t.

Let $U = [0, \infty)\times R = T\times R$, $u = (t(u), y(u)) \in U$, $t(u)\in [0, \infty)$, $y(u)\in R$. Let $\mathscr{B}_U$ be the σ-algebra of Borel subsets of the set U. On $(U, \mathscr{B}_U)$ we define the measure m_θ in the following way:

$$m_\theta(B) = \mu_\theta\Big(\big\{x(\cdot): \big(\tau(x(\cdot)), S_{\tau(x(\cdot))}(x(\cdot))\big)\in B\big\}\Big) \quad \text{for all } B\in\mathscr{B}_U.$$

Under these assumptions we can formulate the following lemma proved in [4]:

LEMMA 1. *The measure m_θ is absolutely continuous with respect to the measure m_{θ_0} and*

$$\frac{dm_\theta}{dm_{\theta_0}} = g(u, \theta, \theta_0) = g\big(t(u), y(u), \theta, \theta_0\big).$$

Definition 1. The $\mathscr{B}_U$-measurable mapping $f\colon U\to R$ is called an *estimator of the parameter* θ.

Definition 2. By a *sequential plan* we mean any pair $\delta = (\tau, f)$ consisting of a stopping time τ and an estimator f of the parameter θ.

By $L(\theta, f)$ we denote the loss function which determines the loss incurred by the statistician if θ is the true value of the parameter and f is an estimator chosen by the statistician. Let $c(t)$, $t\in[0, \infty)$, be the cost function which represents the cost to the statistician of observing the sample path of the process up to time t. We assume that $c(t)$ is a non-negative continuous function and $\lim\limits_{t\to\infty} c(t) = \infty$. Let

$$R(\theta, \delta) = \int\limits_U \big[L\big(\theta, f(u)\big) + c\big(t(u)\big)\big]\, dm_\theta(u) = \mathrm{E}_\theta\big[L\big(\theta, f(\tau, S_\tau)\big) + c(\tau)\big].$$

The function $R(\theta, \delta)$ is called the *risk function*. In the sequel we assume that $R(\theta, \delta) < \infty$.

Definition 3. A sequential plan $\delta^* = (\tau^*, f^*)$ is *minimax* if

$$\sup_\theta R(\theta, \delta^*) = \inf_\delta \sup_\theta R(\theta, \delta).$$

Let $\mathscr{B}_A$ be the σ-algebra of Borel subsets of the set A. Let us suppose that on $(A, \mathscr{B}_A)$ the prior probability distribution of the parameter θ is defined by the distribution function $\Phi(\theta)$.

Let $R(\theta, \delta)$ be a $\mathscr{B}_A$-measurable function of the variable θ. Then for a given sequential plan δ the expected risk with respect to the prior distribution Φ is defined by

$$r(\Phi, \delta) = \int\limits_A R(\theta, \delta)\, d\Phi(\theta).$$

Definition 4. Let Φ be the prior distribution function of the parameter θ and let f be an estimator of this parameter. Then the *posterior* *risk* corresponding to Φ and f is given by the formula

$$r_1\big(\Phi(\theta \mid t(u) = t, y(u) = y), f\big)$$
$$= \int_A L\big(\theta, f(t(u), y(u))\big) d\Phi(\theta \mid t(u) = t, y(u) = y).$$

Definition 5. A sequential plan $\hat{\delta} = (\hat{\tau}, \hat{f})$ is called the *Bayes plan* for Φ if

$$r(\Phi, \hat{\delta}) = \inf_{\delta}\big(r(\Phi, \delta)\big).$$

Definition 6. A sequential plan $\delta = (\tau, f)$ is the *fixed-time plan* if τ is equal with probability 1 to a constant t.

Definition 7. Let Φ be the prior distribution function of the parameter θ. An estimator f^{*t} is said to be the *t-Bayes estimator* of the parameter θ with respect to Φ if

$$\int_A L\big(\theta, f^{*t}(u)\big) d\Phi(\theta \mid t(u) = t, y(u) = y)$$
$$= \inf_f \int_A L\big(\theta, f(u)\big) d\Phi(\theta \mid t(u) = t, y(u) = y) \quad \text{for all } (t, y) \in T \times R.$$

In our further considerations the following theorem is used:

THEOREM 1 ([1]). *Suppose that for every $t \geqslant 0$ there exists a sequence of distribution functions Φ_n $(n = 1, 2, \ldots)$ for which there are corresponding t-Bayes estimators f_n^{*t} such that the posterior risk*

$$r_1\big(\Phi_n(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\big)$$

corresponding to Φ_n and f_n^{*t} is independent of the value y. Moreover, suppose that there exists δ_0^t for which

$$\sup_{\theta} R_1(\theta, \delta_0^t) = \lim_{n\to\infty} r_1\big(\Phi_n(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\big)$$

and there exists t_0 for which

$$c(t_0) + \sup_{\theta} R_1(\theta, \delta_0^{t_0}) = \min_{t\geqslant 0}[c(t) + \sup_{\theta} R_1(\theta, \delta_0^t)].$$

Then the fixed-time plan $\delta_0^{t_0} = (t_0, f^{t_0})$ is minimax among the plans $\delta = \big(\tau, f(\tau, S_\tau)\big)$.

Remark. $R_1(\theta, \delta^t) = \mathrm{E}_\theta L(\theta, \delta^t)$.

In the proof of Theorem 1 we need the following

LEMMA 2. *Let us assume that for every $t \geqslant 0$ there exists a t-Bayes estimator f^{*t} of the parameter θ with respect to some prior distribution Φ for which the posterior risk $r_1\big(\Phi(\theta \mid t(u) = t, y(u) = y), f^{*t}\big)$ corresponding to Φ and f^{*t} is independent of the value y. Then for each sequential plan $\delta = (\tau, f)$*

the inequality

$$r(\Phi, \delta) \geqslant \int_T \Big[c(t) + r_1\big(\Phi(\theta \mid t(u) = t, y(u) = y), f^{*t}\big)\Big] d\nu(t)$$

holds, where the measure ν is defined in the proof.

Proof. We have

$$r(\Phi, \delta) = \int_\Theta d\Phi(\theta) \Big[\int_U \Big(L\big(\theta, f(t(u), y(u))\big) + c\big(t(u)\big)\Big) m_\theta\big(dt(u), dy(u)\big)\Big]$$

$$= \int_U m(dt, dy) \int_\Theta \big[L\big(\theta, f(t(u), y(u))\big) + c\big(t(u)\big)\big] d\Phi\big(\theta \mid t(u) = t, y(u) = y\big).$$

The measure m is defined by

$$m(B) = \int_A m_\theta(B) d\Phi(\theta) \quad \text{for all } B \in \mathscr{B}_U.$$

By Definition 7 we can write

$$\int_A L\big(\theta, f(t(u), y(u))\big) d\Phi(\theta \mid t(u) = t, y(u) = y)$$

$$\geqslant \int_A L\big(\theta, f^{*t}(t(u), y(u))\big) d\Phi\big(\theta \mid t(u) = t, y(u) = y\big)$$

and by assumption the integral

$$\int_A L\big(\theta, f^{*t}(t(u), y(u))\big) d\Phi\big(\theta \mid t(u) = t, y(u) = y\big)$$

is independent of the value y. Thus

$$r(\Phi, \delta) = \int_U m(dt, dy) \int_A L(\theta, f^{*t}) d\Phi\big(\theta \mid t(u) = t, y(u) = y\big)$$

$$= \int_T \nu(dt) \int_R \Big[c(t) + \int_A (\theta, f^{*t}) d\Phi\big(\theta \mid t(u) = t, y(u) = y\big)\Big] m(dy \mid t)$$

$$= \int_T \Big[c(t) + r_1\big(\Phi(\theta \mid t(u) = t, y(u) = y), f^{*t}\big)\Big] \nu(dt),$$

where

$$\nu(dt) = \int_R m(dt, dy)$$

or, more strictly,

$$\nu\big([a, b)\big) = m\big([a, b) \times R\big) \quad \text{for } [a, b) \subset [0, \infty),$$

which completes the proof.

Proof of Theorem 1. Let us suppose that the plan $\delta_0^{t_0} = (t_0, f_0^{t_0})$ is not minimax. Then there exists a plan $\tilde{\delta} = (\tilde{\tau}, \tilde{f})$ for which

$$\sup_\theta R(\theta, \tilde{\delta}) < \sup_\theta R(\theta, \delta_0^{t_0}).$$

But, by Fatou's lemma, Lemma 2 and by the assumptions of Theorem 1, we have

$$\sup_{\theta} R(\theta, \tilde{\delta}) = \sup_{\Phi} r(\Phi, \tilde{\delta}) \geqslant r(\Phi_n, \tilde{\delta})$$
$$\geqslant \int_T d\nu(t)\left[c(t) + r_1\left(\Phi_n(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\right)\right]$$

and, consequently,

$$\sup_{\theta} R(\theta, \tilde{\delta}) \geqslant \varliminf_{n\to\infty} \int_T d\nu(t)\left[c(t) + r_1\left(\Phi_n(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\right)\right]$$
$$\geqslant \int_T d\nu(t)[c(t) + \sup_{\theta} R_1(\theta, f_0^{t_0})].$$

Further,

$$\int_T d\nu(t)[c(t) + \sup_{\theta} R_1(\theta, f_0^{t_0})]$$
$$\geqslant \int_T d\nu(t)[c(t_0) + \sup_{\theta} R_1(\theta, f_0^{t_0})] = \sup_{\theta} R(\theta, \delta_0^{t_0}).$$

Thus we obtain

$$\sup_{\theta} R(\theta, \delta_0^{t_0}) \leqslant \sup_{\theta} R(\theta, \tilde{\delta}) < \sup_{\theta} R(\theta, \delta_0^{t_0}),$$

which contradicts the assumption.

2. Minimax sequential estimation of the mean value of a stationary Gaussian Markov process. Let us consider a stationary Gaussian Markov process (Ornstein-Uhlenbeck process) with unknown mean value θ and correlation function

$$B(t, s) = \exp\left[-[\beta|t-s|]\right], \quad \beta > 0.$$

This process generates the measure $\mu_{\theta,t}$ ([2], p. 63-69). In [3] it is proved that the measure $\mu_{\theta,t}$ is absolutely continuous with respect to the measure $\mu_{\theta_0,t}$ for $\theta_0 = 0$ and

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = \exp\left[-\frac{\theta^2}{2}\left(1 + \frac{\beta t}{2}\right) + \frac{\theta}{2}\left(x(0) + x(t) + \beta\int_0^t x(s)\,ds\right)\right],$$

where $x(s)$ is a sample path of this process, belonging to $C([0, \infty))$. Here we have

$$S_t(x(\cdot)) = x(0) + x(t) + \beta\int_0^t x(s)\,ds.$$

By Lemma 1, the measure m_θ is absolutely continuous with respect to the measure m_{θ_0} and

$$\frac{dm_\theta}{dm_{\theta_0}}(u)\exp\left[-\frac{\theta^2}{2}\left(1+\frac{\beta}{2}t(u)\right)+\frac{\theta}{2}y(u)\right] = g(u,\theta,\theta_0) = g(u\,|\,\theta),$$
$$\theta\in A = R.$$

We prove the following

THEOREM 2. *Let* $L(\theta,f) = (f-\theta)^2$ *and let*

$$R(\theta,\delta) = \mathrm{E}_\theta\left[L\left(\theta, f(\tau, S_\tau)\right)+c(\tau)\right].$$

Then there exists a fixed-time plan $\delta_0^{t_0} = (t_0, f^{t_0})$ *which is minimax for the estimation of the mean value of the Ornstein-Uhlenbeck process.*

Proof. Assume that the sequence of prior distributions is given by the sequence of density functions

$$\varphi_n(\theta) = \frac{\sqrt{\beta t_n}}{2\sqrt{\pi}}\exp\left[-\frac{\beta}{2}t_n\frac{\theta^2}{2}\right].$$

Then the density of the posterior probability distribution of the parameter θ takes the form

$$\varphi_n\left(\theta\mid t(u)=t, y(u)=y\right) = \frac{g\left(t(u),y(u),\theta,\theta_0\right)\varphi_n(\theta)}{\int_{-\infty}^{+\infty} g\left(t(u),y(u),\theta,\theta_0\right)\varphi_n(\theta)d\theta}$$
$$= \frac{\exp\left[-\frac{\theta^2}{2}\left(1+\frac{\beta}{2}(t+t_n)\right)+\frac{\theta}{2}y\right]}{\int_{-\infty}^{+\infty}\exp\left[-\frac{\theta^2}{2}\left(1+\frac{\beta}{2}(t+t_n)\right)+\frac{\theta}{2}y\right]d\theta}.$$

Further we obtain

$$\varphi_n\left(\theta\mid t(u)=t, y(u)=y\right) = \frac{\sqrt{\beta(t+t_n)+2}}{2\sqrt{\pi}}\times$$
$$\times\exp\left[-\frac{y^2}{4[\beta(t+t_n)+2]}\right]\exp\left[-\frac{\theta^2}{2}\left(1+\frac{\beta}{2}(t+t_n)\right)+\frac{\theta}{2}y\right].$$

Let us consider the estimator f_n^{*t} of the form

$$f_n^{*t}(t,y) = \int_{-\infty}^{+\infty}\theta\varphi_n\left(\theta\mid t(u)=t, y(u)=y\right).$$

We obtain

$$f_n^{*t}(t, y) = \frac{y}{\beta(t+t_n)+2}.$$

The posterior risk for this estimator takes the form

$$r_1\big(\varphi(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\big)$$

$$= \int_{-\infty}^{+\infty} \left[\frac{y}{\beta(t+t_n)+2} - \theta\right]^2 \varphi_n\big(\theta \mid t(u) = t, y(u) = y\big) d\theta$$

$$= \frac{1}{[\beta(t+t_n)+2]^2} \exp\left[-\frac{y^2}{4[\beta(t+t_n)+2]}\right] \frac{\sqrt{\beta(t+t_n)+2}}{2\sqrt{\pi}} \times$$

$$\times \int_{-\infty}^{+\infty} \big[y - \theta\big(\beta(t+t_n)+2\big)\big]^2 \exp\left[-\frac{\theta^2}{2}\left(1+\frac{\beta}{2}(t+t_n)\right) + \frac{\theta}{2} y\right] d\theta$$

$$= \frac{2}{\beta(t+t_n)+2}.$$

Thus the posterior risk corresponding to Φ_n and to the t-Bayes estimators $f_n^{*t}(u)$ is independent of y. Let us consider the estimator

$$f_0^t(t, S_t) = \frac{S_t}{\beta t+2}$$

with the risk

$$R_1(\theta, f_0^t) = \mathrm{E}_\theta\left[\frac{S_t}{\beta t+2} - \theta\right]^2 = \frac{2}{\beta t+2}.$$

Taking $t_n = 1/n$, we obtain

$$f_n^{*t}(t, y) = \frac{y}{\beta(t+1/n)+2}$$

with

$$r_1\big(\varphi_n(\theta \mid t(u) = t, y(u) = y), f_n^{*t}\big)$$

$$= \frac{2}{\beta(t+1/n)+2} \to \frac{2}{\beta t+2} = \sup_\theta R_1(\theta, f_0^t) \quad \text{as } n \to \infty.$$

We can choose some t_0 for which

$$\min_t \left[c(t) + \frac{2}{\beta t+2}\right] = c(t_0) + \frac{2}{\beta t_0+2}.$$

Therefore, all the assumptions of Theorem 1 are fulfilled and we conclude that the plan

$$\delta_0^{t_0} = (t_0, f_0^{t_0}), \qquad f_0^{t_0}(t_0, S_{t_0}) = \frac{S_{t_0}}{\beta t_0 + 2},$$

is minimax for the parameter θ in the class of sequential plans $\delta = \big(\tau, f(\tau, S_\tau)\big)$.

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MINIMAKSOWA ESTYMACJA SEKWENCYJNA WARTOŚCI ŚREDNIEJ STACJONARNEGO GAUSSOWSKIEGO PROCESU MARKOWA

STRESZCZENIE

W pracy rozważa się zagadnienie minimaksowej estymacji sekwencyjnej wartości średniej stacjonarnego gaussowskiego procesu Markowa, gdy strata poniesiona przez statystyka związana jest nie tylko z błędem estymacji, ale także z kosztami obserwacji procesu. Udowodniono, że dla kwadratowej funkcji strat, związanej z błędem estymacji, plan o stałym czasie obserwacji jest minimaksowy.
