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ON A FUNCTIONAL EQUATION FOR CHARACTERISTIC FUNCTIONS

1. Introduction. An infinitely divisible characteristic function $\varphi(u)$ is said to belong to the *class* U if its Lévy spectral function $M(x)$ is convex on $(-\infty, 0)$ and concave on $(0, \infty)$. The class U of infinitely divisible characteristic functions has attracted a great deal of attention during the last fifteen years. In 1967 Medgyessy [4] established the unimodality for symmetric distribution functions whose characteristic functions belong to the class U. He also established the unimodality for symmetric distribution functions whose characteristic function is a power mixture of a real characteristic function of the class U. In 1977 Alf and O'Connor [1] proved that $M(x)$ is convex on $(-\infty, 0)$ and concave on $(0, \infty)$ if and only if the characteristic function $\varphi(u)$ can be written in the form

$$\varphi(u) = \exp\left\{\int_0^1 \log f(uy)\,dy\right\},$$

where $f(u)$ is an infinitely divisible characteristic function.

In 1978 Wolfe [6] proved that a necessary condition for a distribution function to be imbedded in a unimodal semigroup is that its characteristic function belongs to the class U. Finally, in 1979 O'Connor [5] proved that the membership in this class is related to the solution of the equations

$$\varphi(u) = \varphi^{\tau}(\tau u)\varphi_{\tau}(u),$$

where $\tau \in (0,1)$ and $\varphi_{\tau}(u)$ is a characteristic function.

Let $f(u)$ be an infinitely divisible characteristic function and let p $(p > 1)$ be a real number. Then

$$\varphi(u) = \exp\left\{p\int_0^1 \log f(uy)\,y^{p-1}\,dy\right\} \tag{1}$$

is also an infinitely divisible characteristic function [2]. The characteristic function $\varphi(u)$ is said to belong to the *class* U_p $(U_1 \equiv U)$.

In this paper, certain properties of U_p are established.

2. Results. We begin by extending Theorem 1 of O'Connor [5].

THEOREM 1. *Let $\varphi(u)$ be a characteristic function without real zeros. Then $\varphi(u)$ belongs to the class U_p if and only if the following conditions are satisfied:*

(a) *$\varphi'(u)$ exists on $R\setminus\{0\}$ and*

$$\lim_{u\to 0} u\varphi'(u) = 0;$$

(b) *for each τ, $0<\tau<1$, and $p>1$, there is a characteristic function $\varphi_{\tau,p}(u)$ such that*

$$\varphi(u) = \varphi^{\tau^p}(\tau u)\varphi_{\tau,p}(u).$$

In this case the characteristic function $\varphi_{\tau,p}(u)$ is infinitely divisible.

Proof. We first assume that $\varphi(u)$ belongs to the class U_p. Then we can write

$$\log\varphi(u) = p\int_0^1 \log f(uy)y^{p-1}dy = pu^{-p}\int_0^u \log f(y)y^{p-1}dy.$$

Hence condition (a) is satisfied. Furthermore, for $0<\tau<1$ and $p>1$, $\exp\{p\int_\tau^1 \log f(uy)y^{p-1}dy\}$ is an infinitely divisible characteristic function [2]. So, by letting

$$\varphi_{\tau,p}(u) = \exp\left\{p\int_\tau^1 \log f(uy)y^{p-1}dy\right\},$$

we infer that conditions (a) and (b) are satisfied with $\varphi_{\tau,p}(u)$ infinitely divisible.

Conversely, suppose that (a) and (b) are valid. We define the characteristic function

$$\varphi_{k,p}(u) = \varphi\left(\frac{ku}{k-1}\right)/\varphi^{((k-1)/k)^p}(u),$$

where $p>1$, $k=1,2,\ldots$ Then $\varphi_{k,p}(u)\neq 0$ for all $u\in R$, k and p. For $u\neq 0$ we can write

$$\log\varphi_{k,p}^k(u) = \frac{k^p-(k-1)^p}{k^{p-1}}\log\varphi\left(\frac{k}{k-1}u\right)+ \\ +\left(\frac{k-1}{k}\right)^{p-1} u\,\frac{\log\varphi(ku/(k-1))-\log\varphi(u)}{u/(k-1)}.$$

As $k\to\infty$, the right-hand side of the above equation converges to $p\log\varphi(u)+u\varphi'(u)/\varphi(u)$, and hence

$$\lim_{k\to\infty}\varphi_{k,p}^k(u) = (\varphi(u))^p\exp\{u\varphi'(u)/\varphi(u)\}. \tag{2}$$

Set $f(0) = 1$ and, for $u \neq 0$, put $f(u) = \varphi(u)\exp\{u\varphi'(u)/p\varphi(u)\}$. It follows from (2) and the continuity theorem that $f(u)$ is an infinitely divisible characteristic function. Since

$$\varphi(u) = \exp\left\{p\, u^{-p} \int_0^u \log f(y) y^{p-1} dy\right\},$$

we conclude that $\varphi(u)$ belongs to the class U_p. The infinite divisibility of $\varphi_{\tau,p}(u)$ follows from the necessity part of the theorem.

An infinitely divisible characteristic function $\varphi(u)$ is said to be *self-decomposable* (or of *class* L) if

$$\varphi(u) = \varphi(\tau u)\varphi_\tau(u),$$

where $\tau \in (0,1)$ and $\varphi_\tau(u)$ is a characteristic function. In Theorem 2 we examine the relations of L and U with the class U_p.

THEOREM 2. *Let $\varphi(u)$ be a characteristic function of the class U_p. Then*

(i) $\varphi(u)\exp\{p \int_0^u y^{-1}\log\varphi(y)\,dy\}$ *belongs to the class L, where the function* $\int_0^u y^{-1}\log\varphi(y)\,dy$ *is a continuous function of u;*

(ii) $\varphi(u)\exp\{pu^{-1} \int_0^u \log\varphi(y)\,dy\}$ *belongs to the class U.*

Proof. (i) The characteristic function $\varphi(u)$ can be written in the form (1) with $f(u)$ being an infinitely divisible characteristic function. Since $\int_0^u y^{-1}\log\varphi(y)\,dy$ is a continuous function of u and

$$\exp\left\{p \int_0^u y^{-1}\log f(y)\,dy\right\} = \varphi(u)\exp\left\{p \int_0^u y^{-1}\log\varphi(y)\,dy\right\},$$

it follows easily that $\gamma(u) = \exp\{p \int_0^u y^{-1}\log f(y)\,dy\}$ is an infinitely divisible characteristic function [2]. Furthermore, it is easy to see that $\gamma(u) = \gamma(\tau u)\gamma_\tau(u)$ for all $\tau \in (0,1)$. Hence $\gamma(u)$ is a self-decomposable characteristic function.

(ii) Suppose that $\varphi(u) \in U_p$. Then

$$\varphi_1(u) = \exp\left\{u^{-1} \int_0^u \log f^p(y)\,dy\right\}$$

and

$$\varphi_2(u) = \exp\left\{u^{-1} \int^u \left[py^{-p} \int_0^y \log f(x) x^{p-1} dx\right] dy\right\}$$

are characteristic functions of the class U. It follows easily that

$$\varphi_1(u)\varphi_2(u) = \varphi(u)\exp\left\{pu^{-1}\int_0^u \log\varphi(y)\,dy\right\}$$

belongs to the class U.

Furthermore, if $\varphi(u)$ is real, then $\varphi(u)\exp\{pu^{-1}\int_0^u \log\varphi(y)\,dy\}$ is the characteristic function of a distribution function having the unique mode at zero [4].

We close this section with a decomposition of characteristic functions of the class U.

THEOREM 3. *Suppose that $\varphi(u)$ is a characteristic function of the class U. Then there exist characteristic functions $\varphi_1(u)$, $\varphi_2(u)$ of the class U_p such that $\varphi(u) = \varphi_1(u)\varphi_2(u)$.*

Proof. Since $\varphi(u) \in U$, we have

$$\varphi(u) = \exp\left\{u^{-1}\int_0^u \log f(y)\,dy\right\},$$

where $f(u)$ is an infinitely divisible characteristic function.

Set

$$\varphi_1(u) = \exp\left\{pu^{-p}\int_0^u \left[y^{-1}\int_0^y \log f^{(p-1)/p}(x)\,dx\right] y^{p-1}\,dy\right\}$$

and

$$\varphi_2(u) = \exp\left\{pu^{-p}\int_0^u \log f^{1/p}(y)\,y^{p-1}\,dy\right\}.$$

The characteristic functions $\varphi_1(u)$ and $\varphi_2(u)$ belong to the class U_p and satisfy $\varphi(u) = \varphi_1(u)\varphi_2(u)$.

3. Remarks. Let $X(t)$ be a stochastic process for $t \geqslant 0$; the random variable $X(t_2)-X(t_1)$ is called the *increment* of the process $X(t)$ over the interval $[t_1, t_2]$. A process $X(t)$ is said to be *homogeneous* if the distribution function of the increment $X(t+v)-X(t)$ depends only on the length v of the interval but is independent of t. A process $X(t)$ is called a *process with independent increments* if the increments over non-overlapping intervals are independent. A process is said to be *continuous in an interval* $[a, b]$ if, for any $t \in [a, b]$ and for any $\varepsilon > 0$,

$$\lim_{v\to 0} P\{|X(t+v)-X(t)| > \varepsilon\} = 0.$$

Let $X(t)$ be a homogeneous and continuous process with independent increments and denote the characteristic function of the increment

$X(t+v)-X(t)$ by $f(u,v)$. It is known that $f(u,v)$ is infinitely divisible and that $f(u,v)=[f(u,1)]^v$. For the sake of simplicity we write $f(u)$ for $f(u,1)$. The stochastic integral $\int_0^1 t^{1/p}dX(t)$ $(p\geqslant 1)$ exists in the sense of convergence in probability and its characteristic function is given by

$$\varphi(u) = \exp\left\{pu^{-p}\int_0^u \log f(y)y^{p-1}dy\right\} \tag{3}$$

(see [3]). Hence (3) can be considered as a transformation which converts an infinitely divisible characteristic function $f(u)$ into an infinitely divisible characteristic function of the class U_p. Naturally, one will be interested in classes of infinitely divisible characteristic functions which are preserved under the transformation (3). Suppose that $f(u)=f(\tau u)f_\tau(u)$, $\tau\in(0,1)$. Then $\varphi(u)=\varphi(\tau u)\varphi_\tau(u)$, which means that the class L is preserved under the transformation (3). In a similar way we can see that the class U is preserved under the transformation (3).

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