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BEST QUADRATURE FORMULA FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

1. Definitions. Denote by $\mathscr{F}$ the class of all real analytic functions in $[-1, 1]$ which have a bounded by 1 analytic continuation in the unit circle G. Let $\{x_k\}_1^n$ satisfy $-1 < x_1 < \ldots < x_n < 1$.

We shall study the methods of approximation of the integral

$$I(f) = \int_{-1}^{1} f(x)\,dx, \quad f \in \mathscr{F},$$

using, as information, only the values $f(x_k)$ and $f'(x_k)$ $(k = 1, 2, \ldots, n)$. An arbitrary method of such a type can be defined by a function S of $2n$ variables in the following way:

$$I(f) \approx S\big(f(x_1), \ldots, f(x_n), f'(x_1), \ldots, f'(x_n)\big). \tag{1}$$

The quantity

$$R(x_1, \ldots, x_n;\ S) = \sup_{f\in\mathscr{F}} \big|I(f) - S\big(f(x_1), \ldots, f'(x_n)\big)\big|$$

is said to be the *error* of the method S in the class $\mathscr{F}$. The purpose of this paper is to construct such a method S_0 for which

$$R(x_1, \ldots, x_n, S_0) = \inf_S R(x_1, \ldots, x_n;\ S) = R(\boldsymbol{x}),$$

where $\inf\limits_S$ is extended over all admissible methods of type (1). The method S_0 will be called *best*.

2. Preliminary results. The following is a consequence of a general result due to Smoljak [5] (see also [1]).

LEMMA 1. *There exist numbers C_k and D_k $(k = 1, 2, \ldots, n)$ such that*

$$\sup_{f\in\mathscr{F}} \Big|I(f) - \sum_{k=1}^{n} \big(C_k f(x_k) + D_k f'(x_k)\big)\Big| = R(\boldsymbol{x}).$$

That means there exists a linear best method of approximation of the integral $I(f)$.

The proof of the lemma produces the next two corollaries.

COROLLARY 1. *There exists a function $f(x) \in \mathcal{F}$ such that $f(x_k) = f'(x_k) = 0$ $(k = 1, 2, \ldots, n)$ for which the best method attains its maximal error in $\mathcal{F}$.*

Let us be given a number ε. Write

$$\mathcal{F}_1^k \equiv \{f \in \mathcal{F}\colon f(x_k) = \varepsilon,\ f(x_i) = 0,\ i \neq k,\ f'(x_i) = 0\ (i = 1, 2, \ldots, n)\},$$

$$\mathcal{F}_2^k \equiv \{f \in \mathcal{F}\colon f(x_i) = 0\ (i = 1, 2, \ldots, n),\ f'(x_k) = \varepsilon,\ f'(x_i) = 0,\ i \neq k\},$$

$$\psi_k(\varepsilon) = \sup_{f \in \mathcal{F}_1^k} \int_{-1}^{1} f(x)\,dx, \qquad \theta_k(\varepsilon) = \sup_{f \in \mathcal{F}_2^k} \int_{-1}^{1} f(x)\,dx. \tag{2}$$

COROLLARY 2. *If $\psi_k'(0)$ and $\theta_k'(0)$ exist, then $C_k = \psi_k'(0)$ and $D_k = \theta_k'(0)$.*

Thus, in order to construct the best quadrature, it is necessary to solve the variational problems (2).

3. Main result. Write, for simplicity,

$$W_k(x) = \frac{x - x_k}{1 - x x_k}, \qquad \omega_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{x - x_i}{1 - x x_i}.$$

LEMMA 2. *If $\varphi(x) \in \mathcal{F}_1^k$, then there exists a function $\tilde{\varphi}(x) \in \mathcal{F}$ such that*

$$\varphi(x) = \omega_k^2(x) \frac{A + CW_k(x) + \big(ACW_k(x) + W_k^2(x)\big)\tilde{\varphi}(x)}{1 + ACW_k(x) + \big(CW_k(x) + AW_k^2(x)\big)\tilde{\varphi}(x)}, \tag{3}$$

where

$$A = \frac{\varepsilon}{\omega_k^2(x_k)}, \qquad C = -\frac{2\varepsilon}{1 - A^2} \frac{\omega_k'(x_k)}{\omega_k^3(x_k) W_k'(x_k)}.$$

On the other hand, every function $\tilde{\varphi}(x)$ from $\mathcal{F}$ produces $\varphi(x) \in \mathcal{F}_1^k$.

Proof. Let $\varphi \in \mathcal{F}_1^k$. Then it can be expressed by $\varphi(x) = \omega_k^2(x)\varphi_0(x)$. Since $|\omega_k^2(x)| = 1$ for $|x| = 1$, it follows, by the principle of maximum, that $\varphi_0(x) \in \mathcal{F}$. From the conditions $\varphi(x_k) = \varepsilon$ and $\varphi'(x_k) = 0$ we get

$$\varphi_0(x_k) = \varepsilon / \omega_k^2(x_k) = A, \qquad 2\omega_k(x_k)\omega_k'(x_k)\varphi_0(x_k) + \omega_k^2(x_k)\varphi_0'(x_k) = 0.$$

The last equality gives

$$\varphi_0'(x_k) = -2\varepsilon \frac{\omega_k'(x_k)}{\omega_k^3(x_k)} = B.$$

It is seen that the function $\varphi_1(x)$, determined by

$$\frac{\varphi_0(x) - A}{1 - A\varphi_0(x)} = W_k(x)\varphi_1(x), \tag{4}$$

belongs to the class $\mathcal{F}$. Differentiating both sides of (4) and puting $x = x_k$, we find

$$\frac{\varphi_0'(x_k)\big(1-A\varphi_0(x_k)\big)}{\big(1-A\varphi_0(x_k)\big)^2} = W_k'(x_k)\varphi_1(x_k)$$

which gives

$$\varphi_1(x_k) = \frac{B}{1-A^2}\,\frac{1}{W_k'(x_k)} = C.$$

From (4) we have

$$\varphi_0(x) = \frac{A+W_k(x)\varphi_1(x)}{1+AW_k(x)\varphi_1(x)}. \tag{5}$$

By analogous calculations we get

$$\varphi_1(x) = \frac{C+W_k(x)\tilde{\varphi}(x)}{1+CW_k(x)\tilde{\varphi}(x)}, \quad \text{where } \tilde{\varphi}(x)\in\mathcal{F}. \tag{6}$$

The presentation (3) follows from (5) and (6).

The reverse statement is obvious.

LEMMA 3. *If $g(x)\in\mathcal{F}_2^k$, then there exists a function $\tilde{g}(x)\in\mathcal{F}$ such that*

$$g(x) = \omega_k^2(x)W_k(x)\frac{E+W_k(x)\tilde{g}(x)}{1+EW_k(x)\tilde{g}(x)}, \quad \textit{where } E = \frac{\varepsilon}{\omega_k^2(x_k)W_k'(x_k)}. \tag{7}$$

On the other hand, every function $\tilde{g}(x)$ from $\mathcal{F}$ produces $g(x)\in\mathcal{F}_2^k$.

The proof is analogous to that of lemma 2.

THEOREM 1. *Let the knots $\{x_k\}_1^n$ be fixed in $(-1,1)$. The quadrature formula*

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{n}\{C_kf(x_k)+D_kf'(x_k)\}, \tag{8}$$

where

$$D_k = \int_{-1}^{1}\frac{\omega_k^2(x)}{\omega_k^2(x_k)}\,\frac{W_k(x)}{W_k'(x)}\big(1-W_k^2(x)\big)\,dx,$$

$$C_k = \int_{-1}^{1}\frac{\omega_k^2(x)}{\omega_k^2(x_k)}\left\{1-W_k^4(x)-\frac{2\omega_k'(x_k)}{\omega_k(x_k)W_k'(x_k)}W_k(x)\big(1-W_k^2(x)\big)\right\}dx,$$

is best in the class $\mathcal{F}$. The error has the value

$$R(x_1,\ldots,x_n) = \int_{-1}^{1}\left(\prod_{k=1}^{n}\frac{x-x_k}{1-xx_k}\right)^2 dx.$$

Proof. From the definition of the function $\psi_k(\varepsilon)$ and (3) we have

$$\psi_k(\varepsilon) = \sup_{\tilde{\varphi}\in\mathscr{F}} \int_{-1}^{1} \omega_k^2(x) \frac{A+CW_k(x)+\{ACW_k(x)+W_k^2(x)\}\tilde{\varphi}(x)}{1+ACW_k(x)+\{CW_k(x)+AW_k^2(x)\}\tilde{\varphi}(x)}\,dx.$$

Define the function $h(x)$ by

$$h(x) = \max_{-1\leqslant t\leqslant 1} p(x,t),$$

where

$$p(x,t) = \frac{A+CW_k(x)+\big(ACW_k(x)+W_k^2(x)\big)t}{1+ACW_k(x)+\big(CW_k(x)+AW_k^2(x)\big)t}.$$

It is clear that

$$\psi_k(\varepsilon) \leqslant \int_{-1}^{1} \omega_k^2(x)h(x)\,dx \quad \text{as } \omega_k^2(x) \geqslant 0 \text{ in } [-1,1].$$

We show that $\omega_k^2(x)h(x)\in\mathscr{F}_1^k$. Let x be fixed in $[-1,1]$. Since

$$\frac{dp(x,t)}{dt} = \frac{(1-A^2)(1-C^2)W_k^2(x)}{1+ACW_k(x)+\big(CW_k(x)+AW_k^2(x)\big)t} \geqslant 0,$$

for small ε we conclude that

$$h(x) = \frac{A+CW_k(x)+ACW_k(x)+W_k^2(x)}{1+ACW_k(x)+CW_k(x)+AW_k^2(x)}.$$

Hence the function $\omega_k^2(x)h(x)$ is of form (3) with $\tilde{\varphi}(x)\equiv 1$. By lemma 2 it follows that $\omega_k^2(x)h(x)\in\mathscr{F}_1^k$. Consequently,

$$\psi_k(\varepsilon) = \int_{-1}^{1} \omega_k^2(x)h(x)\,dx.$$

The function $\psi_k(\varepsilon)$ is differentiable for $\varepsilon = 0$. We have

$$\psi_k'(0) = \int_{-1}^{1} \omega_k^2(x)\big\{A'(0)+C'(0)W_k(x)-W_k^2(x)\big(C'(0)W_k(x)+A'(0)W_k^2(x)\big)\big\}\,dx,$$

where

$$A'(0) = \frac{1}{\omega_k^2(x_k)}, \qquad C'(0) = -\frac{2\omega_k'(x_k)}{\omega_k^3(x_k)W_k'(x_k)}.$$

By using corollary 2 we get C_k.

It remains to determine coefficients D_k $(k = 1, 2, \ldots, n)$. From (1) and (7) we have

$$\theta_k(\varepsilon) = \sup_{\tilde{g}\in\mathscr{F}} \int_{-1}^{1} \omega_k^2(x)W_k(x)\frac{E+W_k(x)\tilde{g}(x)}{1+EW_k(x)\tilde{g}(x)}\,dx.$$

Define the function $v(x)$ by

$$v(x)=\begin{cases}\sup\limits_{-1\leqslant t\leqslant 1}\dfrac{E+W_k(x)t}{1+EW_k(x)t} & \text{for } x_k\leqslant x\leqslant 1,\\[2ex] \inf\limits_{-1\leqslant t\leqslant 1}\dfrac{E+W_k(x)t}{1+EW_k(x)t} & \text{for } -1\leqslant x\leqslant x_k.\end{cases}$$

As $\omega_k^2(x)W_k(x)$ changes its sign in x_k,

$$\theta_k(\varepsilon)\leqslant\int_{-1}^{1}\omega_k^2(x)W_k(x)v(x)\,dx. \tag{9}$$

It can be proved, as in the first part of the theorem, that

$$v(x)=\frac{E+W_k(x)}{1+EW_k(x)}$$

and the function $\omega_k^2(x)W_k(x)v(x)$ is of type (7) with $\tilde{g}\equiv 1$. Differentiating $\theta_k(\varepsilon)$ and putting $\varepsilon=0$ we find coefficients D_k.

In order to evaluate the error $R(x_1,\dots,x_n)$, we use corollary 1. Thus we obtain

$$R(x_1,\dots,x_n)=\sup_{\substack{f\in\mathscr{F}\\ f(x_k)=f'(x_k)=0\\ k=1,2,\dots,n}}\int_{-1}^{1}f(x)\,dx.$$

Let $f\in\mathscr{F}$ vanish with its first derivative at the points x_k ($k=1,2,\dots,n$). The function

$$f^*(x)=\frac{f(x)}{\left(\prod\limits_{k=1}^{n}(x-x_k)/(1-xx_k)\right)^2}$$

is analytic in the circle $|x|<1$. By the principle of maximum for analytic functions, we obtain

$$\sup_{|x|\leqslant 1}\left|\frac{f(x)}{\prod\limits_{k=1}^{n}\big((x-x_k)/(1-xx_k)\big)^2}\right|\leqslant\sup_{|x|=1}|f(x)|=1.$$

Hence, for every $|x|\leqslant 1$, we have

$$|f(x)|\leqslant\left|\prod_{k=1}^{n}\left(\frac{x-x_k}{1-xx_k}\right)^2\right|.$$

It implies

$$R(x_1, \ldots, x_n) = \int_{-1}^{1} \prod_{k=1}^{n} \left(\frac{x-x_k}{1-xx_k}\right)^2 dx. \tag{10}$$

The proof of the theorem is complete.

It is natural to try to solve the following problem: Find these knots $\{x_k^*\}_1^n$ for which error (10) is minimal. It is interesting to note that the coefficients D_k in the best quadrature for these extremal knots vanish. Indeed, the conditions

$$\frac{\partial}{\partial x_k^*} R(x_1^*, \ldots, x_n^*) = 0$$

coincide with $D_k(x_1^*, \ldots, x_n^*) = 0$.

The following estimates for the convergence of the optimal quadrature are obtained in [2]:

$$\exp\left[-\left(2\sqrt{2}+\frac{1}{\sqrt{2}}\right)\pi\sqrt{n}\right] \leqslant R(x_1^*, \ldots, x_n^*) \leqslant \exp\left[-\frac{\pi}{\sqrt{2}}\sqrt{n}\right].$$

The extremal knots $\{x_k^*\}_1^n$ are studied in [3] and [4].

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NAJLEPSZA KWADRATURA DLA PEWNEJ KLASY FUNKCJI ANALITYCZNYCH

STRESZCZENIE

Oznaczmy przez $\mathcal{F}$ klasę wszystkich analitycznych funkcji rzeczywistych na odcinku $[-1, 1]$, dla których istnieje ograniczone przez 1 przedłużenie analityczne w kole jednostkowym. Niech węzły $\{x_k\}_1^n$ spełniają warunek

$$-1 < x_1 < x_2 < \ldots < x_n < 1.$$

W klasie $\mathcal{F}$ buduje się kwadraturę, najlepszą spośród wszystkich metod całkowania przybliżonego wykorzystujących wartości $f(x_k)$ i $f'(x_k)$, $k = 1, 2, \ldots, n$.