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THE INTERVALS OF STABILITY OF RUNGE-KUTTA METHODS AFTER RICHARDSON EXTRAPOLATION

The intervals of stability of Runge–Kutta methods of order $p = 1, 2, 3$ and 4, after Richardson extrapolation applied to the solutions obtained with steps h and h/k ($k = 2, 3, \ldots, 6$), are presented.

1. Introduction. The general s-stage explicit Runge–Kutta method (RK) is defined by

$$y_{n+1} - y_n = h\varphi(x_n, y_n, h), \tag{1}$$

$$\varphi(x, y, h) = \sum_{r=1}^{s} c_r k_r, \tag{2}$$

where

$$\begin{aligned} k_1 &= f(x, y), \\ k_r &= f\Big(x + ha_r,\ y + h\sum_{i=1}^{r-1} b_{ri} k_i\Big) \quad (r = 2, 3, \ldots, s), \\ a_r &= \sum_{i=1}^{r-1} b_{ri} \quad (r = 2, 3, \ldots, s). \end{aligned}$$

Let $p^*(s)$ be the highest order that can be attained by an s-stage method. Then

$$\begin{aligned} &p^*(s) = s, \quad s = 1, 2, 3, 4, \\ &p^*(5) = 4, \\ &p^*(6) = 5, \\ &p^*(7) = 6, \\ &p^*(8) = 6, \\ &p^*(9) = 7, \\ &p^*(s) \leqslant s-2, \quad s = 10, 11, \ldots \end{aligned} \tag{3}$$

When we use the s-stage p-order RK method with the step h to the test problem of the form

$$y' = \lambda y, \quad y(0) = 1, \quad \lambda < 0, \tag{4}$$

from the formulae (1) and (2) the following equality holds

$$y_{n+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2!} + \ldots + \frac{(h\lambda)^p}{p!} + \sum_{i=p+1}^{s} d_i \frac{(h\lambda)^i}{i!}\right) y_n. \tag{5}$$

In (5) the parameters d_i $(i = p+1, p+2, \ldots, s)$ are dependent on the parameters of the RK method.

DEFINITION. The *interval of stability* associated with the RK formula is defined as the set

$P = \{h\lambda$: the formula applied to (4), with constant step size $h > 0$, produces a sequence $\{y_n\}$ satisfying $y_n \to 0$ as $n \to \infty\}$.

Let $z = h\lambda$ and let the polynomial $w(z)$ of the variable z be defined in the form

$$w(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^p}{p!} + \sum_{i=p+1}^{s} d_i \frac{z^i}{i!}. \tag{6}$$

Then for (5) the following equality holds

$$y_{n+1} = w(z)\, y_n. \tag{7}$$

On the basis of the above definition z belongs to the interval of stability P, when $|w(z)| < 1$.

For the p-stage p-order RK methods ($p = 1$, 2, 3 and 4) we have

order of method	polynomial $w(z)$	interval of stability
$p = 1$	$1+z$	$(-2, 0)$
$p = 2$	$1+z+\frac{z^2}{2}$	$(-2, 0)$
$p = 3$	$1+z+\frac{z^2}{2}+\frac{z^3}{3!}$	$(-2.51, 0)$
$p = 4$	$1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\frac{z^4}{4!}$	$(-2.78, 0)$

From (3) and (5) and (6) we obtain that for $p > 4$ the interval of stability depends on the parameters of the RK method.

In this paper we investigate only the RK methods with $p = 1$, 2, 3 and

4. In the following this restriction is assumed with respect to p. In many computational situations we have two solutions:

y_{n+1} – the solution obtained with the step h,

$\bar{y}_{n+1}$ – the solution obtained with the step $h/2$ by twofold application of the RK method.

From (7), for the solution $\bar{y}_{n+1}$ holds

$$\bar{y}_{n+1} = w^2\left(\frac{z}{2}\right)y_n, \tag{8}$$

where the first step gives

$$y_{n+1/2} = w\left(\frac{z}{2}\right)y_n$$

and the second step gives

$$\bar{y}_{n+1} = w\left(\frac{z}{2}\right)y_{n+1/2} = w^2\left(\frac{z}{2}\right)y_n.$$

Usually Richardson extrapolation is applied to the solutions y_{n+1} and $\bar{y}_{n+1}$ and a new solution y^*_{n+1} is formed as follows

$$y^*_{n+1} = \bar{y}_{n+1} + \frac{\bar{y}_{n+1} - y_{n+1}}{2^p - 1}. \tag{9}$$

This solution may be treated as the solution obtained by a new numerical method, i.e. by the Runge–Kutta–Richardson method (RKR). The properties of stability of the RKR method are rather different than the properties of the RK method.

For the solution y^*_{n+1} we have the equality

$$y^*_{n+1} = w^*(z)\,y_n, \tag{10}$$

where $w^*(z)$ is defined as follows

$$w^*(z) = w^2\left(\frac{z}{2}\right) + \frac{w^2(z/2) - w(z)}{2^p - 1}. \tag{11}$$

Now, the variable z belongs to the interval of stability P of the RKR method, if for the polynomial $w^*(z)$ for the value z holds $|w^*(z)| < 1$. It is an interesting question how great is the interval of stability after the use of Richardson extrapolation, i.e. the interval of stability of the RKR method.

In this paper we try to give an answer to the question: is the interval of stability of the RKR method increased or decreased in comparison to the interval of stability of the RK method? In other words, is it better with respect to the properties of stability to use the solution $\bar{y}_{n+1}$ or the solution y^*_{n+1}?

2. The intervals of stability of the RKR methods. Now let $\bar{y}_{n+1}$ be the solution obtained with the step size h/k, where k may have one of the values 2, 3, ..., 6. By analogy to the formulae (8)–(11) we have the relations

$$\bar{y}_{n+1} = w^k\left(\frac{z}{k}\right)y_n,$$

$$y^*_{n+1} = \bar{y}_{n+1} + \frac{\bar{y}_{n+1} - y_{n+1}}{k^p - 1},$$

$$y^*_{n+1} = w^*(z)\, y_n,$$

where

$$w^*(z) = w^k\left(\frac{z}{k}\right) + \frac{w^k(z/k) - w(z)}{k^p - 1} \tag{12}$$

and

$$w^*(z) = 1 + z + \ldots + \frac{z^{p+1}}{(p+1)!} + \sum_{i=p+2}^{ks} d_i^* \frac{z^i}{i!}.$$

The coefficients d_i^* ($i = p+2, p+3, \ldots, ks$) depend on the d_i in (5). The order of the RKR method is $p+1$.

For example for the Euler method ($p = 1$) the formula (12) gives

$$w^*(z) = \frac{z^2}{2} + z + 1,$$

$$w^*(z) = \frac{z^3}{18} + \frac{z^2}{2} + z + 1,$$

$$w^*(z) = \frac{z^4}{192} + \frac{z^3}{12} + \frac{z^2}{2} + z + 1,$$

$$w^*(z) = \frac{z^5}{2500} + \frac{z^4}{100} + \frac{z^3}{10} + \frac{z^2}{2} + z + 1,$$

$$w^*(z) = \frac{z^6}{38880} + \frac{z^5}{1080} + \frac{z^4}{72} + \frac{z^3}{9} + \frac{z^2}{2} + z + 1$$

for $k = 2, 3, \ldots, 6$ respectively.

For finding the polynomials $w^*(z)$ for other values of p and k a special computer language is used [1]. The intervals of stability of the RKR methods are also obtained by computer and they are presented in Table 1 in the form $(a, 0)$, where the value of a is given. The polynomial $w^*(z)$ was tabulated and was found the longest interval $(a, 0)$ such that for $z \in (a, 0)$ holds $|w^*(z)| < 1$.

TABLE 1. The intervals of stability after Richardson extrapolation applied to the solution obtained with the step size h and h/k

p \ k	2	3	4	5	6
1	-2.00	-3.00	-4.00	-5.00	-6.00
2	-5.15	-7.13	-9.12	-11.11	-9.80
3	-4.06	-6.44	-8.07	-9.97	-11.80
4	-6.46	-9.21	-10.76	-12.12	-14.26

From this table we conclude:

1° for $p = 1, 3$ ($k = 2, 3, \ldots, 6$) and $p = 4$ ($k = 4, 5, 6$) the intervals of stability of the RKR methods are shorter than the intervals of stability of the RK methods, with the step size h/k used,

2° for $p = 2$ ($k = 2, 3, \ldots, 5$) and $p = 4$ ($k = 2, 3$) the intervals of stability of the RKR methods are longer than the intervals of stability of the RK methods with the step size h/k used.

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References

[1] W. Świątek, T. Świątek, *Automatyzacja obliczeń analitycznych na wielomianach wielu zmiennych*, Instytut Informatyki Uniwersytetu Wrocławskiego, diploma work, Wrocław, 1981.

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