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ON FREDHOLM INTEGRAL EQUATIONS WITH RANDOM DEGENERATE KERNELS (1)

1. In this paper we consider the random operator equation

$$(T(\omega)-\lambda I)f = g \tag{1}$$

when the random operator $T(\omega)$ is a Fredholm integral operator with random kernel of degenerate type. The results presented compliment and extend those announced in [4]. In Sec. 2 we show that a Fredholm integral equation with random degenerate kernel can be reduced to an algebraic system of random linear equations; and consider the question of the existence, uniqueness and measurability of the solution of the algebraic system of random linear equations. In Sec. 3 we investigate the asymptotic distribution of the eigenvalues of the random Fredholm operator.

2. Consider the Fredholm integral equation of second kind

$$\int_0^1 K(x, y)f(y)dy - \lambda f(x) = g(x). \tag{2}$$

It is well-known (cf. Pogorzelski [6], Tricomi [7]) that if the kernel $K(x, y)$ is *degenerate* (2), that is $K(x, y)$ is of the form

$$K(x, y) = \sum_{i=1}^{n} a_i(x)\beta_i(y) \tag{3}$$

where $\{a_i(x)\}_{i=1}^{n}$ and $\{\beta_i(y)\}_{i=1}^{n}$ are two sets of linearly independent L_2 $(0, 1)$-functions, then the integral equation can be reduced to an algebraic system of n linear equations in n unknowns. If we put

$$\int_0^1 \beta_j(x)f(x)dx = \xi_j, \quad j = 1, 2, \ldots, n, \tag{4}$$

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(2) Degenerate kernels are also referred to as *separable*, kernels of *finite-rank*, or *Pincherle-Goursat* kernels.

eq. (2) with kernel (3) becomes

$$\sum_{j=1}^{n} \xi_j a_j(x) - \lambda f(x) = g(x). \tag{5}$$

If we now multiply (5) by $\beta_i(x)$ ($i = 1, 2, \ldots, n$) and then integrate, we obtain

$$\sum_{j=1}^{n} \xi_j \int_0^1 a_j(x)\beta_i(x)\,dx - \lambda\xi_i = \int_0^1 \beta_i(x)g(x)\,dx;$$

that is

$$\sum_{j=1}^{n} a_{ij}\xi_j - \lambda\xi_i = b_i, \qquad i = 1, 2, \ldots, n, \tag{6}$$

where

$$a_{ij} = \int_0^1 a_j(x)\beta_i(x)\,dx, \tag{7}$$

$$b_i = \int_0^1 \beta_i(x)g(x)\,dx. \tag{8}$$

Rewriting (6) in matrix form, we have

$$(A - \lambda I)\xi = b, \tag{9}$$

where $A = (a_{ij})$ is an $n \times n$ matrix, and ξ and b are n-vectors. Hence we see that a Fredholm integral equation with degenerate kernel reduces to an algebraic system of linear equations; and, moreover, the eigenvalues of a degenerate kernel are the roots of the algebraic equation $|A - \lambda I| = 0$.

Let $(\Omega, \mathscr{A}, \mu)$ be a probability measure space; and consider the *random kernel*

$$K(\omega, x, y) = \sum_{i=1}^{n} a_i(\omega, x)\beta_i(y). \tag{10}$$

In (10) $\{a_i(\omega, x)\}_{i=1}^{n}$ is a family of (almost surely) independent $L_2(0, 1)$--random functions and $\{\beta_i(y)\}_{i=1}^{n}$ is a set of independent $L_2(0, 1)$-determinate functions. Clearly, for every fixed $x, y \in (0, 1)$ the kernel K is a measurable function of ω. Put

$$\xi_i = \int_0^1 \beta_i(x)f(x)\,dx, \qquad i = 1, 2, \ldots, n. \tag{11}$$

Then, proceeding as in the deterministic case, we obtain the algebraic system of random linear equations

$$\sum_{j=1}^{n} a_{ij}(\omega)\xi_j - \lambda\xi_i = b_i, \qquad i = 1, 2, \ldots, n, \tag{12}$$

where

$$a_{ij}(\omega) = \int_0^1 a_j(\omega, x)\beta_i(x)dx, \qquad i, j = 1, 2, \ldots, n, \tag{13}$$

$$b_i = \int_0^1 \beta_i(x)g(x)dx, \qquad i = 1, 2, \ldots, n. \tag{14}$$

The integrals in (11) and (14) are well-defined; and the Riemann-integrability of

$$\int_0^1 \int_0^1 \beta_i(x_1)\beta_i(x_2)R_j(x_1, x_2)dx_1dx_2, \tag{15}$$

where $R_j(x_1, x_2) = \mathscr{E}\{a_j(\omega, x_1)a_j(\omega, x_2)\}$ is the covariance kernel associated with the $a_j(\omega, x)$-process, is sufficient to insure that the integral in (13) exists in mean square and defines, for every pair (i, j), a real-valued random variable.

Equation (12) can be rewritten as the random operator equation

$$(A(\omega) - \lambda I)\xi = b, \tag{16}$$

where $A(\omega)$ is a $n \times n$ random matrix with elements $a_{ij}(\omega)$ defined by (13), and ξ and b are n-vectors. We remark that (16) can be interpreted as a random operator equation in the Euclidean space R_n or the Hilbert space $l_2(n)$.

In [4] the existence, uniqueness and measurability of the solution $\xi(\omega)$ of (16) was established using the Špaček-Hanš probabilistic analogue of the Banach contraction mapping theorem. We now consider the solution of (16) using the following result of Bharucha-Reid [2], [3] and Hanš [5] on the invertibility of linear random operators of the form $T(\omega) - \lambda I$: Let $T(\omega)$ be a random transformation on a separable Banach space $\mathscr{X}$ which is for every $\omega \in \Omega$ linear and bounded. Then, for every real number $\lambda \neq 0$ the set $\Omega(\lambda) = \{\omega: \|T(\omega)\| < |\lambda|\} \in \mathscr{A}$, and the random transformation $T(\omega) - \lambda I$ is invertible for every $\omega \in \Omega(\lambda)$. Furthermore, for every $\omega \in \Omega(\lambda)$ the solution of the random operator equation $(T(\omega) - \lambda I)f = g$ is for every random variable $g(\omega)$ with values in $\mathscr{X}$ given by $f(\omega) = (T(\omega) - \lambda I)^{-1}g(\omega)$, and $f(\omega)$ is measurable with respect to the σ-algebra $\Omega(\lambda) \cap \mathscr{A}$.

A straightforward application of the above theorem enables us to state the following result for the solution of (16): Let $\lambda \neq 0$ be a real number such that

$$\mu\big(\Omega(\lambda)\big) = \mu\Big(\Big\{\omega: \Big(\sum_{i,j=1}^{n} a_{ij}^2(\omega)\Big)^{1/2} < |\lambda|\Big\}\Big) = 1.$$

Then the random matrix $\big(A(\omega) - \lambda I\big)$ is invertible; and the solution $\xi(\omega) = \big(A(\omega) - \lambda I\big)^{-1}b$ of (16) is $\big(\Omega(\lambda) \cap \mathscr{A}\big)$-measurable.

3. In this section we utilize the fact that for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of any $n \times n$ matrix $A = (a_{ij})$

$$\sum_{i=1}^{n} \lambda_i^k = \mathrm{Trace}(A^k) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$$

for $k = 1, 2, \ldots$

Consider the random kernel

$$K(\omega, x, y) = \sum_{i=1}^{n} a_i(\omega, x) a_i(\omega, y), \tag{17}$$

where the a_i's are independent random functions having the same finite-dimensional probability distributions. For the sake of simplicity, we assume

(i) $\quad m_k(x) = \mathscr{E}\{|a_i(\omega, x)|^k\} < \infty \quad$ for all $\quad x \in [0, 1]$,

and

(ii) $\quad \int_0^1 m_k(x)\,dx < \infty \quad$ for every $\quad k = 1, 2, \ldots$

(Stronger results under weaker conditions can be obtained by the technique used in [1].) Put

$$R(x, y) = \mathscr{E}\{a_i(\omega, x) a_i(\omega, y)\}, \tag{18}$$

and assume

$$\mathscr{E}\{a_i(\omega, x)\} \equiv 0. \tag{19}$$

In the case we are considering, the kernel (17) is almost surely symmetric and positive-definite, since

$$\int_0^1 \int_0^1 K(\omega, x, y) h(x) \overline{h(y)}\,dx\,dy = \sum_{i=1}^{n} \left| \int_0^1 a_i(\omega, x) h(x)\,dx \right|^2$$

for any continuous function $h(x)$. Therefore, the eigenvalues of (17) are all real and, moreover, non-negative.

The expectation of K,

$$\mathscr{E}\{K(\omega, x, y)\} = nR(x, y), \tag{20}$$

is also a symmetric and positive-definite kernel, but in general *not* degenerate.

Consider the corresponding random matrix $A(\omega)$ with elements

$$a_{ij}(\omega) = \int_0^1 a_i(\omega, x) a_j(\omega, x)\,dx. \tag{21}$$

Since $a_{ij}(\omega) = a_{ji}(\omega)$ almost surely, $A(\omega)$ is symmetric. The diagonal elements $a_{ii}(\omega)$ are independent and have the same distribution with

$$\mathscr{E}\{a_{ii}(\omega)\} = \int_0^1 R(x, x)\,dx = \mathrm{Trace}(R). \tag{22}$$

The off-diagonal elements also have the same distribution with

$$\mathscr{E}\{a_{ij}(\omega)\} = 0 \tag{23}$$

and

$$\mathscr{E}\{a_{ij}^2(\omega)\} = \int_0^1\int_0^1 R^2(x, y)\,dx\,dy. \tag{24}$$

The only difference between the problem investigated in [1], [8] and [9] and the problem here considered is that in addition to the symmetry condition there are other relations among the matrix elements; hence they are not independent. This difference completely changes the result. However, the elements $a_{ij}(\omega)$ are independent *except* if they are in the same row or column of $A(\omega)$.

Define $R_1 = R$, and the (k-1)-fold iterated kernel R_k by

$$R_k(x, y) = \int_0^1 R_{k-1}(x, s) R(s, y)\,ds; \tag{25}$$

and

$$\mathrm{Trace}(R_k) = \int_0^1 R_k(x, x)\,dx. \tag{26}$$

Then, if the indicies $i_1, i_2, \ldots, i_k$ are all different,

$$\begin{aligned} &\mathscr{E}\{a_{i_1 i_2}(\omega) a_{i_2 i_3}(\omega) \ldots a_{i_k i_1}(\omega)\} \\ &= \underbrace{\int_0^1 \ldots \int_0^1}_{k\text{-fold}} R(x_1, x_2) R(x_2, x_3) \ldots R(x_k, x_1)\,dx_1\,dx_2 \ldots dx_k = \mathrm{Trace}(R_k). \end{aligned} \tag{27}$$

Consequently, for all $k = 1, 2, \ldots,$

$$\lim_{n\to\infty} n^{-k} \sum_{i=1}^{n} \mathscr{E}\{\lambda_i^k(\omega)\} = \mathrm{Trace}(R_k). \tag{28}$$

Let $N_n(\omega, x)$ denote the number of eigenvalues of the random matrix $A(\omega)$ which are less than x. Since $A(\omega)$ is positive, we have $N_n(\omega, 0) = 0$ and $N_n(\omega, \infty) = n$ almost surely. Furthermore,

$$\int_0^1 x^k\,dN_n(\omega, x) = \sum_{i=1}^{n} \lambda_i^k(\omega). \tag{29}$$

Relation (28) can now be rewritten as

$$\lim_{n\to\infty}\int_0^1 x^k d\mathscr{E}\{N_n(\omega, nx)\} = \mathrm{Trace}(R_k). \tag{30}$$

The convergence must be due to the fact that there are, on the average, very many small eigenvalues — but only a few large eigenvalues.

As an example, put $a_i(\omega, x) = X_i(\omega)$. Then $R(x, y) = \mathscr{E}\{X_i^2(\omega)\} = \sigma^2$, and the eigenvalues of (17) are $\lambda_1(\omega) = \lambda_2(\omega) = \ldots = \lambda_{n-1}(\omega) = 0$ and $\lambda_n(\omega) = \sum_{i=1}^{n} X_i^2(\omega)$. From the strong law of large numbers, it follows that

$$\int_0^1 x^k dN_n(\omega, nx) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2(\omega)\right)^k \to \mathrm{Trace}(R_k)$$

almost surely, for every $k = 1, 2, \ldots$

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