ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIV, 4 (1975)

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# ESTIMATION OF RELIABILITY IN THE EXPONENTIAL CASE (II)

**1. Introduction and summary.** In the first part of this paper (see [1]) we have considered the problem of unbiased estimation of the exponential reliability function $R(t) = e^{-\lambda t}$, $t \geqslant 0$, for the life test without replacement of the failed items and with duration of observations until the fixed moment $T$.

In this part of the paper we consider the case where the tested elements have the two-parameter exponential distribution with probability density function (pdf)

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-a)} & \text{for } x \geqslant a, \\ 0 & \text{for } x < a. \end{cases} \tag{1}$$

The reliability function is of the form

$$R_0(t) = e^{-\lambda(t-a)} \quad \text{for } t \geqslant a. \tag{2}$$

The problem concerns the estimation of the function (2) for this same life test.

Let $N$ identical items having the life-time distribution (1) be placed simultaneously on the above-mentioned test. Let $D(t)$ denote the number of failures until the moment $t$ and let $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_{D(T)}$ be the moments of failures until the moment $T$. The joint pdf of the vector $(X_1, X_2, \ldots, X_{D(T)}, D(T))$ is of the form

(3) $$p(x_1, \ldots, x_d, d) = \frac{N!}{(N-d)!} \lambda^d \exp\left\{-\lambda\left[\sum_{i=1}^{d} x_i + (N-d)T - Na\right]\right\} \chi_{(a,\infty)}(x_1)$$

$$\text{for } d = 1, 2, \ldots, N \text{ and } x_1 \leqslant x_2 \leqslant \ldots \leqslant x_d \leqslant T,$$

$$p(\cdot, d) = e^{-\lambda N(T-a)} \quad \text{for } d = 0$$

($\chi_A(x_1)$ denotes the indicator of the set $A$).

The factorization theorem implies that the sufficient statistic for the parameter $\lambda$, when $a$ is known, is the vector $(D(T), S)$, where

$$S = X_1 + X_2 + \ldots + X_{D(T)} + (N - D(T))T$$

is the accumulated life-time of all tested items. In the case where the parameter $a$ is unknown, the sufficient statistic for the parameter vector $(\lambda, a)$ is the vector $(X_1, D(T), S)$.

The statistic

$$\bar{R}(t) = 1-\frac{D(t)}{N} \quad \text{for } 0 \leqslant t \leqslant T \tag{4}$$

is the unbiased estimator of $R_0(t)$ for $t \leqslant T$. In [1] the unbiased estimator of $R(t) = e^{-\lambda t}$ has been given even for $t > T$. It is of the form

(5)

$$\bar{R}_0(t) = \begin{cases} 1-\dfrac{D(t)}{N} & \text{for } 0 \leqslant t \leqslant T, \\ \left(1-\dfrac{D(T)}{N}\right)\left(1-\dfrac{D(T)}{N-1}\right)\cdots\left(1-\dfrac{D(T)}{N-w+1}\right)\left(1-\dfrac{D(t-wT)}{N-w}\right) & \\ \qquad \text{for } wT < t \leqslant (w+1)T,\ w = 1, 2, \ldots, N-1. & \end{cases}$$

If $F$ is the distribution function of any life-time $X$, $\bar{F}$ is its reliability function, and $D(t)$ denotes the number of failures until $t$, then it is easy to verify that

$$\mathrm{E}\,\bar{R}_0(t) = [\bar{F}(T)]^w \bar{F}(t-wT)$$
$$\text{for } wT < t \leqslant (w+1)T,\ w = 0, 1, 2, \ldots, N-1.$$

The condition of unbiasedness of $\bar{R}_0(t)$ is the following:

$$[\bar{F}(T)]^w \bar{F}(t-wT) = \bar{F}(t) = \bar{F}(wT+t-wT) \tag{6}$$
$$\text{for } wT < t \leqslant (w+1)T,\ w = 0, 1, 2, \ldots, N-1.$$

It is well known that the exponential reliability function $\bar{F}(t) = e^{-Ct}$ satisfies (6) only, assuming absolutely continuous distributions of $X$.

J. Łukaszewicz has remarked that if the parameter $a$ is known, the simple modification of the estimator (5) gives the unbiased estimator also for the function $R_0(t)$. This modified estimate is of the form

(7)

$$\hat{R}_0(t) = \begin{cases} 1 & \text{for } t \leqslant a, \\ 1-\dfrac{D(t)}{N} & \text{for } a < t \leqslant T, \\ \left(1-\dfrac{D(T)}{N}\right)\left(1-\dfrac{D(T)}{N-1}\right)\cdots\left(1-\dfrac{D(T)}{N-w+1}\right)\left(1-\dfrac{D\big(t-w(T-a)\big)}{N-w}\right) & \\ \qquad \text{for } a+w(T-a) \leqslant t \leqslant a+(w+1)(T-a),\ w = 1, 2, \ldots, N-1. & \end{cases}$$

This modification is of course the translation $t' = t-a$ and $T' = T-a$, and hence the unbiasedness of $\hat{R}_0(t)$ is obvious.

In Section 2 we give the unbiased estimator

$$\tilde{R}_0(t) = \mathrm{E}[\hat{R}_0(t) \mid D(T), S] \quad \text{for } 0 \leqslant t < a + N(t-a)$$

which, by virtue of the Rao-Blackwell theorem, has a uniformly smaller variance than $R_0(t)$. As it has been proved in [1], the sufficient statistic $(D(T), S)$ is not complete except for the trivial case $N = 1$, and hence one cannot state that $\tilde{R}_0(t)$ has a minimum variance in the class of all unbiased estimators of the function (2).

If the parameter $a$ is unknown, we cannot give the unbiased estimator of $R_0(t)$ for $t > T$ based only on the observations until $T$. The truncation of the sample at the fixed moment $T$ causes that the distribution of $X_1$, which is sufficient for $a$, loses the property of lack of memory, and hence it is impossible to use the previous methods to construct the unbiased estimator of $R_0(t)$.

In Section 3 we derive the unbiased estimator of $R_0(t)$. This estimator is of the form

$$\breve{R}(t) = \mathrm{E}[\bar{R}(t) \mid X_1, D(T), S] \quad \text{for } 0 \leqslant t \leqslant T,$$

where $\bar{R}(t)$ is given by (4). $\breve{R}(t)$ has a smaller variance than $\bar{R}(t)$ but, unfortunately, one cannot also state that it has a minimum variance among the unbiased estimators, since — as it will be proved — the sufficient statistic $(X_1, D(T), S)$ is not complete in general.

## 2. The parameter $a$ known.

THEOREM 1. *The estimator* $\tilde{R}_0(t) = \mathrm{E}[\hat{R}_0(t) \mid D(T), S]$ *is of the form*

$$\tilde{R}_0(t) = \begin{cases} 1 & \text{for } t \leqslant a, \\ 1 - \dfrac{R_0(D(T), S; t)}{N} & \text{for } a < t \leqslant T, \\ \left(1 - \dfrac{D(T)}{N}\right)\left(1 - \dfrac{D(T)}{N-1}\right) \cdots \left(1 - \dfrac{D(T)}{N-w+1}\right)\left(1 - \dfrac{R_0(D(T), S; t - w(T-a))}{N-w}\right) & \\ \quad \text{for } a + w(T-a) \leqslant t \leqslant a + (w+1)(T-a),\ w = 1, 2, \ldots, N-1, & \end{cases}$$

*where*

$$R_0(d, s; t) = \mathrm{E}[D(t) \mid D(T) = d,\ S = s]$$

$$= \begin{cases} 0 & \text{for } d = 0, \\ \displaystyle\sum_{k=0}^{d} k\binom{d}{k} \frac{\sum_{i=0}^{d-k}\sum_{j=0}^{k} (-1)^{i+j}\binom{k}{j}\binom{d-k}{i}[s-(k-j)a-(d-k+j-i)t-(N-d+i)T]_+^{d-1}}{\sum_{j=0}^{d}(-1)^j\binom{d}{j}[s-(d-j)a-(N-d+j)T]_+^{d-1}} & \text{for } d > 0, \end{cases}$$

*and*

$$a_+^m = [\max(0, a)]^m \text{ for } m > 0 \quad \text{and} \quad a_+^0 = \begin{cases} 0 & \text{for } a \leqslant 0, \\ 1 & \text{for } a > 0. \end{cases}$$

The proof of this theorem is quite similar to the proof of Theorem 2 in [1]. The expression $R_0(d, s; t)$ differs from the term (10) of [2] in that $s-(k-j)a$ is instead of $s$ in the numerator and $s-(d-j)a$ is instead of $s$ in the denominator.

### 3. The parameter $a$ unknown.

THEOREM 2. *The estimator* $\breve{R}(t) = \mathrm{E}[\bar{R}(t) \mid X_1, D(T), S]$ *is of the form*

$$\breve{R}(t) = \begin{cases} 1 & \text{for } t \leqslant X_1, \\ 1 - \dfrac{R(X_1, D(T), S; t)}{N} & \text{for } X_1 < t \leqslant T, \end{cases} \tag{8}$$

*where*

$$R(X_1, d, s; t) = \mathrm{E}[D(t) \mid X_1 = x_1,\ D(T) = d,\ S = s] \tag{9}$$

$$= \sum_{k=1}^{d} \left[ k \binom{d-1}{k-1} \times \right.$$

$$\left. \times \frac{\sum_{j=0}^{k-1} \sum_{i=0}^{d-k} (-1)^{i+j} \binom{k-1}{j} \binom{d-k}{i} [s-(k-j)x_1-(d-k+j-i)t-(N-d+i)T]_+^{d-2}}{\sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [s-(d-j)x_1-(N-d+j)T]_+^{d-2}} \right]$$

$$\text{for } d = 2, 3, \ldots, N,$$

*and*

$$R(x_1, 1, s; t) = 1 \quad \text{and} \quad R(\cdot, 0, NT; t) = 0 \quad \text{for } 0 \leqslant t \leqslant T.$$

Before giving the proof of this theorem we introduce some notations. The conditional expectation $\mathrm{E}[D(t) \mid X_1, D(T), S]$, $0 \leqslant t \leqslant T$, is obtained as the mean of the conditional distribution

$$p(k \mid x_1, d, s) = \mathrm{P}\{D(t) = k \mid X_1 = x_1,\ D(t) = d,\ S = s\}. \tag{10}$$

Moreover, we use the following notations:

$p(x_1, \ldots, x_d \mid d)$ — the conditional probability distribution function (cpdf) of $X_1, \ldots, X_d$, given $D(T) = d$;

$p(x_1 \mid d)$ — the cpdf of $X_1$, given $D(T) = d$;

$p(x_1, s \mid d)$ — the cpdf of $X_1$ and $S$, given $D(T) = d$;

$\tilde{p}(x_1, d, s)$ — the pdf of the joint distribution of $X_1$, $D(T)$ and $S$;

$\bar{p}(s \mid x_1, k, d)$ — the cpdf of $S$, given $X_1 = x_1$, $D(t) = k$ and $D(T) = d$;

$\bar{p}(x_1, k, d)$ — the pdf of the joint distribution of $X_1$, $D(t)$ and $D(T)$;

$\tilde{p}(x_1, k, d, s)$ — the pdf of the joint distribution of $X_1$, $D(t)$, $D(T)$ and $S$.

Using these notations, we can give (10) in the form

$$p(k \mid x_1, d, s) = \frac{\tilde{p}(x_1, k, d, s)}{\tilde{p}(x_1, d, s)} = \frac{\bar{p}(s \mid x_1, k, d)\bar{p}(x_1, k, d)}{\bar{p}(x_1, s \mid d)\mathrm{P}\{D(T) = d\}}.$$

For the proof of Theorem 2 we need some lemmas. Three of them are obvious.

LEMMA 1. *If* $D(T) = d \geqslant 1$, *then*

$$p(x_1, \ldots, x_d \mid d) = \frac{d!\lambda^d \exp[-\lambda(x_1 + \ldots + x_d)]}{[\exp(-\lambda a) - \exp(-\lambda T)]^d} \tag{11}$$

*for* $a \leqslant x_1 \leqslant \ldots \leqslant x_d \leqslant T$.

LEMMA 2. *The joint distribution of random variables* $X_1$ *and* $D(t)$ *is of the form*

$$g(x_1, k)$$
$$= \begin{cases} \dfrac{N}{(k-1)!(N-k)!}\lambda\exp[-\lambda(x_1-a)][\exp[-\lambda(x_1-a)]-\exp[-\lambda(t-a)]]^{k-1}\times \\ \quad \times\exp[-\lambda(N-k)(t-a)] \quad \text{for } k = 1, 2, \ldots, N \text{ and } a \leqslant x_1 \leqslant t, \\ \lambda N\exp[-\lambda N(x_1 - a)] \quad \text{for } k = 0 \text{ and } t < x_1, \\ 0 \quad \text{otherwise}. \end{cases}$$

LEMMA 3. *If* $D(T) = d \geqslant 1$, *then*

$$p(x_1 \mid d) = d\left[\frac{\exp(-\lambda x_1) - \exp(-\lambda T)}{\exp(-\lambda a) - \exp(-\lambda T)}\right]^{d-1} \frac{\lambda\exp(-\lambda x_1)}{\exp(-\lambda a) - \exp(-\lambda T)} \tag{12}$$

*for* $a \leqslant x_1 \leqslant T$.

Now we prove

LEMMA 4. *If* $D(T) = d > 1$, *then*

$$\bar{p}(x_1, s \mid d) = \frac{d\lambda^d \exp[-\lambda(s - (N-d)T)]}{(d-2)![\exp(-\lambda a) - \exp(-\lambda T)]^d} \times \tag{13}$$
$$\times \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [s - (d-j)x_1 - (N-d+j)T]_+^{d-2}$$

*for* $0 \leqslant x_1 \leqslant T$, $0 \leqslant s < \infty$.

Proof. In view of (12) the pdf (11) can be written in the form

$$p(x_1, \ldots, x_d \mid d) = d\left[\frac{\exp(-\lambda x_1)-\exp(-\lambda T)}{\exp(-\lambda a)-\exp(-\lambda T)}\right]^{d-1} \frac{\lambda \exp(-\lambda x_1)}{\exp(-\lambda a)-\exp(-\lambda T)} \times$$

$$\times \frac{(d-1)!\,\lambda^{d-1}\exp[-\lambda(x_1+\ldots+x_d)]}{[\exp(-\lambda x_1)-\exp(-\lambda T)]^{d-1}}$$

$$= p(x_1 \mid d)\,p(x_2, \ldots, x_d \mid x_1, d) \qquad \text{for } a \leqslant x_1 \leqslant \ldots \leqslant x_d \leqslant T,$$

where

$$(14) \qquad p(x_2, \ldots, x_d \mid x_1, d) = \frac{(d-1)!\,\lambda^{d-1}\exp[-\lambda(x_2+\ldots+x_d)]}{[\exp(-\lambda x_1)-\exp(-\lambda T)]^{d-1}}$$

is the cpdf of $X_2, X_3, \ldots, X_d$, given $D(T) = d > 1$ and $X_1 = x_1$. This cpdf is same as the unconditional pdf of the order statistics from the sample of size $d-1$ having the truncated exponential distribution on $[x_1, T]$. Hence the cpdf $h(z \mid x_1, d)$ of the random variable $Z = X_2+\ldots+X_d$, given $X_1 = x_1$ and $D(T) = d$, is the same as the pdf of the sum of $d-1$ independent identically exponentially distributed random variables ranging over $[x_1, T]$ and, therefore (see [1], Lemma 6),

$$h(z \mid x_1, d) = \frac{\lambda^{d-1}\exp(-\lambda z)}{(d-2)!\,[\exp(-\lambda x_1)-\exp(-\lambda T)]^{d-1}} \times$$

$$\times \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [z-(d-1-j)x_1-jT]_+^{d-2}.$$

The above implies that the conditional distribution of the random variables $X_1$ and $Z$, given $D(T) = d > 1$, has the pdf of the form

$$(15) \qquad \bar{h}(x_1, z \mid d) = h(z \mid x_1, d)\,p(x_1 \mid d)$$

$$= \frac{d\lambda^d \exp[-\lambda(z+x_1)]}{(d-2)!\,[\exp(-\lambda a)-\exp(-\lambda T)]^d} \times$$

$$\times \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [z-(d-1-j)x_1-jT]_+^{d-2}$$

$$\text{for } a \leqslant x_1 \leqslant T \text{ and } (d-1)x_1 \leqslant z \leqslant (d-1)T.$$

Substituting in equation (15) the variable $s = x_1+z+(N-d)T$, we obtain (13).

LEMMA 5. *The pdf of the sufficient statistic $(X_1, D(T), S)$ is of the form*

$$(16)\quad \tilde{p}(x_1, d, s) = \begin{cases} \dfrac{d\binom{N}{d}\lambda^d \exp[-\lambda(s-Na)]}{(d-2)!}\displaystyle\sum_{j=0}^{d-1}(-1)^j\binom{d-j}{j}\times \\ \qquad \times[s-(d-j)x_1-(N-d+j)T]_+^{d-2} & \text{for } d=2,3,\ldots,N, \\ \lambda N\exp[-\lambda(x_1+(N-1)T-Na)]\,\delta_{x_1+(N-1)T}(s) & \text{for } d=1, \\ \exp[-\lambda N(T-a)]\,\delta_{NT}(s) & \text{for } d=0; \end{cases}$$

*$\delta_a(s)$ is the pdf of the random variable with value a with probability* 1.

Proof. If $d \geqslant 2$, then $\tilde{p}(x_1, d, s)$ is obtained by multiplying (13) by

$$P\{D(T) = d\} = \binom{N}{d}[1-\exp[-\lambda(T-a)]]^d \exp[-\lambda(N-d)(T-a)].$$

The other cases, i.e. where $d = 1$ and $d = 0$, are obvious.

The following lemma is also obvious:

LEMMA 6. *If $a \leqslant t \leqslant T$, then*

$$(17)\quad \bar{p}(x_1, k, d) = \begin{cases} \dfrac{N!}{(k-1)!(d-k)!(N-d)!}\lambda\exp[-\lambda(x_1-a)]\times \\ \times[\exp[-\lambda(x_1-a)]-\exp[-\lambda(t-a)]]^{k-1}\times \\ \times[\exp[-\lambda(t-a)]-\exp[-\lambda(T-a)]]^{d-k}\exp[-\lambda(N-d)(T-a)] \\ \qquad \text{for } a\leqslant x_1\leqslant t\leqslant T,\ k=1,2,\ldots,d,\ d=1,2,\ldots,N, \\ \dfrac{N!}{(d-1)!(N-d)!}\lambda\exp[-\lambda(x_1-a)]\times \\ \times[\exp[-\lambda(x_1-a)]-\exp[-\lambda(T-a)]]^{d-1}\exp[-\lambda(N-d)(T-a)] \\ \qquad \text{for } t\leqslant x_1\leqslant T,\ k=0,\ d\geqslant 1, \\ \lambda N\exp[-\lambda N(x_1-a)] \quad \text{for } x_1>T,\ k=0,\ d=0. \end{cases}$$

LEMMA 7. *If $t \leqslant T$, then*

$$(18)\quad \bar{p}(s \mid x_1, k, d) = \frac{\lambda^{d-1}\exp[-\lambda(s-x_1-(N-d)T)]}{(d-2)![\exp(-\lambda x_1)-\exp(-\lambda t)]^{k-1}[\exp(-\lambda t)-\exp(-\lambda T)]^{d-k}}\times$$

$$\times\sum_{j=0}^{k-1}\sum_{i=0}^{d-k}(-1)^{i+j}\binom{k-1}{j}\binom{d-k}{i}[s-(k-j)x_1-(d-k-j-i)t-$$

$$-(N-d+i)T]_+^{d-2} \quad \text{for } k=1,2,\ldots,d-1 \text{ and } 1<k=d,$$

$$(19)\qquad \overline{p}(s \mid x_1, 0, d) = \frac{\lambda^{d-1}\exp\left[-\lambda\left(s-x_1-(N-d)T\right)\right]}{(d-2)!\,[\exp(-\lambda x_1)-\exp(-\lambda T)]^{d-1}} \times$$

$$\times \sum_{i=0}^{k-1} (-1)^i \binom{d-1}{i} [s-(d-i)x_1-(N-d+i)T]_+^{d-2} \quad \text{for } d = 2, \ldots, N,$$

$$(20)\qquad \overline{p}(s \mid x_1, 1, 1) = \delta_{x_1+(N-1)T}(s),$$

$$(21)\qquad \overline{p}(s \mid x_1, 0, 1) = \delta_{x_1+(N-1)T}(s),$$

$$(22)\qquad \overline{p}(s \mid \cdot, 0, 0) = \delta_{NT}(s).$$

Proof. First we consider the case $1 < k < d$. Of course, we have $x_1 < t$. Let $x_1$ be fixed. Then the random variable $W = S-(N-d)T-x_1$ can be given as the sum $W = V_{k-1}+U_{d-k}$, where $V_{k-1}$ is the sum of $k-1$ independently distributed random variables with pdf

$$(23)\qquad f_{(x_1,t)}(v) = \begin{cases} \dfrac{\lambda\exp(-\lambda v)}{\exp(-\lambda x_1)-\exp(-\lambda t)} & \text{for } x_1 \leqslant v \leqslant t, \\ 0 & \text{otherwise}, \end{cases}$$

and $U_{d-k}$ is independent from $V_{k-1}$ and it is the sum of $d-k$ independent identically distributed random variables with pdf $f_{(t,T)}(u)$ of form (23). If follows from Lemma 6 of [1] that the pdf of $V_{k-1}$ is of the form

$$f_{(x_1,t)}^{(k-1)*}(v) = \frac{\lambda^{k-1}\exp(-\lambda v)}{(k-2)!\,[\exp(-\lambda x_1)-\exp(-\lambda t)]^{k-1}} \times$$
$$\times \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} [v-(k-1-j)x_1-jt]_+^{k-2},$$

and the pdf of $U_{d-k}$ is the following:

$$f_{(t,T)}^{(d-k)*}(u) = \frac{\lambda^{d-k}\exp(-\lambda u)}{(d-k-1)!\,[\exp(-\lambda t)-\exp(-\lambda T)]^{d-k}} \times$$
$$\times \sum_{i=0}^{d-k} (-1)^i \binom{d-k}{i} [u-(d-k-i)t-iT]_+^{d-k-i}.$$

Hence the conditional pdf of $W = S-\left(N-D(T)\right)T-x_1$, given $D(T) = d$ and $D(t) = k$, $1 < k < d$, is of the form

$$(24)\quad g(w \mid x_1, k, d) = \int_{-\infty}^{\infty} f_{(x_1,t)}^{(k-1)*}(w-u) f_{(t,T)}^{(d-k)*}(u)\,du$$
$$= C(w) \sum_{j=0}^{k-1} \sum_{i=0}^{d-k} (-1)^{i+j} \binom{k-1}{j} \binom{d-k}{i} \times$$
$$\times \int_{-\infty}^{\infty} [w-u-(k-1-j)x_1 - jt]_+^{k-2} [u-(d-k-i)t - T]_+^{d-k-1} du,$$

where

$$C(w) = \frac{\lambda^{d-1}\exp(-\lambda w)}{(k-2)!(d-k-1)![\exp(-\lambda x_1)-\exp(-\lambda t)]^{k-1}[\exp(-\lambda t)-\exp(-\lambda T)]^{d-k}}.$$

In the sequel we compute the integral

$$I = \int_{-\infty}^{\infty} [w-u-(k-1-j)x_1 - jt]_+^{k-2} [u-(d-k-i)t - iT]_+^{d-k-1} du$$
$$= \int_{-\infty}^{\infty} [b-u]_+^{k-2} [u-a]_+^{d-k-1} du = \int_{a}^{\max(a,b)} (b-u)^{k-2}(u-a)^{d-k-1} du,$$

where $a = (d-k-i)t + iT$ and $b = w-(k-1-j)x_1 - jt$.

Using some transformations of variables and substitutions, we obtain

$$I = c_+^{d-2} \frac{(k-2)!(d-k-1)!}{(d-2)!},$$

where $c_+ = (b-a)_+ = [w-(k-1-j)x_1 - (d-k+j-1)t - iT]_+$.

Returning to the previous notations, from (24) we obtain the pdf (18). It is easy to see that formula (18) is also valid in the cases $k = 1$ and $1 < k = d$.

If $k = 0$ and $d > 1$, i.e. $a \leqslant t < x_1 < T$, then the random variable $W$ is the sum of $d-1$ independent identically distributed random variables with pdf $f_{(x_1,T)}(w)$ of form (23). By virtue of Lemma 6 from [1], the pdf of $W$ is of the form

$$f_{(x_1,T)}^{(d-1)*}(w) = \frac{\lambda^{d-1}\exp(-\lambda w)}{(d-2)![\exp(-\lambda x_1)-\exp(-\lambda T)]^{d-1}} \times$$
$$\times \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} [w-(d-1-i)x_1 - iT]_+^{d-} ,$$

and hence we obtain formula (19). The other formulas, i.e. (20), (21) and (22), are obvious.

From Lemmas 6 and 7 we have immediately

LEMMA 8. *The pdf of the joint distribution of variables $X_1$, $D(t)$, $D(T)$ and $S$ $(t \leqslant T)$ is of the form*

$$\tilde{p}(x_1, k, d, s) = \frac{N!\lambda^d \exp[-\lambda(s-Na)]}{(k-1)!(d-k)!(N-d)!(d-2)!} \times$$

$$\times \sum_{j=0}^{k-1} \sum_{i=0}^{d-k} (-1)^{i+j} \binom{k-1}{j} \binom{d-k}{i} [s-(k-j)x_1-(d-k+j-i)t-$$

$$-(N-d+i)T]_+^{d-2} \quad \textit{for } 1 \leqslant k < d \textit{ and } 1 < k = d,$$

$$\tilde{p}(x_1, k, d, s)$$

$$= \begin{cases} \dfrac{N!\lambda^d \exp[-\lambda(s-Na)]}{(d-1)!(N-d)!(d-2)!} \displaystyle\sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} [s-(d-i)x_1- \\ \qquad -(N-d+i)T]_+^{d-2} \quad \textit{for } d > 1 \textit{ and } k = 0, \\ \delta_{x_1+(N-1)T}(s) N\lambda \exp[-\lambda(x_1-a)] \exp[-\lambda(N-1)(T-a)] \\ \qquad\qquad \textit{for } k < d = 1, \\ \delta_{NT}(s) N\lambda \exp[-\lambda(T-a)] \quad \textit{for } k = 0,\ d = 0 \textit{ and } x_1 > T. \end{cases}$$

Dividing $\tilde{p}(x_1, k, d, s)$ by $\tilde{p}(x_1, d, s)$, we obtain the conditional distribution $p(k \mid x_1, d, s)$.

LEMMA 9. *The conditional distribution of $D(t)$, given $X_1 = x_1$, $D(T) = d$ and $S = s$ $(t \leqslant T)$, is of the form*

$$(25) \quad p(k \mid x_1, d, s)$$

$$= \binom{d-1}{k-1} \frac{\sum_{j=0}^{k-1} \sum_{i=0}^{d-k} (-1)^{i+j} \binom{k-1}{j} \binom{d-k}{i} [s-(k-j)x_1-(d-k+j-i)t-(N-d+i)T]_+^{d-2}}{\sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [s-(d-j)x_1-(N-d+j)T]_+^{d-2}}$$

$$\textit{for } k = 1, 2, \ldots, d \textit{ and } d = 2, \ldots, N,$$

$$p(0 \mid x_1, d, s) = 1 \quad \textit{for } d = 2, 3, \ldots, N$$

*and*

$$p(1 \mid x_1, 1, s) = p(0 \mid \cdot, 0, NT) = 1.$$

**Proof of Theorem 2.** In order to derive the estimator $\check{R}(t)$ it is sufficient to compute the conditional expectation

$$R(x_1, d, s; t) = \mathrm{E}[D(t) \mid X_1 = x_1,\ D(T) = d,\ S = s] \quad \text{for } t \leqslant T.$$

Taking into account the expectation of distribution (25), we obtain $R(x_1, d, s; t)$, and hence $\check{R}(t)$.

**4. Uncompleteness of the statistic $(X_1, D(T), S)$.** Now we prove that the estimator $\check{R}(t)$, being a function of the sufficient statistic $(X_1, D(T), S)$, cannot be regarded as the minimum-variance unbiased estimator of the function $R_0(t) = e^{-\lambda(t-a)}$ for $t \leqslant T$.

THEOREM 3. *The sufficient statistic $(X_1, D(T), S)$ is not complete except for the cases $N = 1$ and $N = 2$.*

Proof. The method of the proof is similar to that of the proof of Theorem 3 in [1]. First, we show that the statistic $(X_1, D(T), S)$ is complete for the trivial cases $N = 1$ and $N = 2$.

Let $\varphi(x_1, d, s)$ be any function of variables $x_1$, $d$ and $s$. The expectation of $\varphi(X_1, D(T), S)$ is of the form

$$\text{(26)}\qquad \mathrm{E}\left[\varphi(X_1, D(T), S)\right] = \sum_{d=0}^{N} \int_0^\infty \int_0^\infty \varphi(x_1\, d, s)\tilde{p}(x_1, d, s)\, dx_1\, ds$$

$$= \exp[-\lambda(T-a)N]\varphi(\cdot, 0, NT) +$$

$$+\int_a^T \varphi\left(x_1, 1, x_1+(N-1)T\right)\lambda N \exp\left[-\lambda\left(x_1+(N-1)T-Na\right)\right]+$$

$$+\sum_{d=2}^{N}\int_a^T \int_{x_1 d+(N-d)T}^{x_1+(N-1)T} \varphi(x_1, d, s)\, \frac{d\binom{N}{d}\lambda^d \exp[-\lambda(s-Na)]}{(d-2)!} \times$$

$$\times \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} [s-(d-j)x_1-(N-d+j)T]_+^{d-2}\, ds\, dx_1.$$

If $N = 1$, then the expression

$$\text{(27)}\qquad \mathrm{E}\left[\varphi(X_1, D(T), S)\right] = 0$$

can be written in the form

$$\varphi(\cdot, 0, T)\exp[-\lambda(T-a)] + \int_a^T \varphi(x_1, 1, x_1)\lambda \exp[-\lambda(x_1-a)]\, dx_1 = 0.$$

After some modifications of this term we obtain

$$\text{(28)}\qquad \int_0^\infty \tilde{\varphi}(z)\exp(-\lambda z)\, dz = \frac{C}{\lambda}\exp[-\lambda(T-a)],$$

where $C = -\varphi(\cdot, 0, T)$ and

$$\tilde{\varphi}(z) = \begin{cases} \varphi(z+a, 1, z+a) & \text{for } 0 \leqslant z \leqslant T-a, \\ 0 & \text{otherwise}. \end{cases}$$

The right-hand side of (28) is the Laplace transform of the function

$$\psi(z) = \begin{cases} 0 & \text{for } 0 \leqslant z \leqslant T-a, \\ C & \text{otherwise.} \end{cases}$$

Hence and from the uniqueness of the Laplace transform it follows that $\varphi(x_1, 1, x_1) = 0$ almost everywhere (a.e.) for $a \leqslant x_1 \leqslant T$, and also that $\varphi(\cdot, 0, NT) = 0$.

If $N = 2$, then expression (27) can be given by

$$\varphi(\cdot, 0, 2N)\exp[-2\lambda(T-a)] + \int_a^T \varphi(x_1, 1, x_1+T) \times$$
$$\times 2\lambda\exp[-\lambda(x_1+T-2a)]dx_1 + \int_a^T \int_{2x_1}^{x_1+T} \varphi(x_1, 2, s)2\lambda^2\exp[-\lambda(s-2a)] \times$$
$$\times([s-2x_1]_+^0 - [s-x_1-T]_+^0)\,ds\,dx_1 = 0.$$

After simple modifications we obtain

$$\int_0^\infty \check{\varphi}(u)\exp(-\lambda u)\,du = \frac{C_1\exp[-2\lambda(T-a)]}{2\lambda^2}, \tag{29}$$

where

$$u = \begin{cases} \frac{1}{\lambda}\varphi(u+a, 1, u+a+T)\exp[-\lambda(T-a)] + \\ + \int_{u+a}^T \varphi(u+a, 2, z+u+a)\exp[-\lambda(z-a)]dz & \text{for } 0 \leqslant u \leqslant T-a, \\ 0 & \text{for } u > T-a. \end{cases}$$

The right-hand side of (29) is the Laplace transform (see [2]) of the function

$$\check{\psi}(u) = \begin{cases} 0 & \text{for } 0 \leqslant u \leqslant 2(T-a), \\ \frac{C_1}{2}[u-2(T-a)] & \text{for } u > 2(T-a). \end{cases}$$

Now, it follows from the uniqueness of the Laplace transform that $C_1[u-2(T-a)] = 0$ and

$$\int_{u+a}^T \varphi(u+a, 2, z+u+a)\exp[-\lambda(z-a)]dz \tag{30}$$
$$= -\frac{\exp[-\lambda(T-a)]}{\lambda}\varphi(u+a, 1, u+T) \quad \text{a.e. for } 0 \leqslant u \leqslant T-a.$$

But $u \neq 2(T-a)$, and hence $C_1 = 0$, i.e. $\varphi(\cdot, 0, 2T) = 0$.

Let $u \in [0, T-a]$ be fixed and let $v = z-a-u$. Then (30) can be written in the form

$$\int_0^{T-a-u} \varphi(u+a, 2, v+2u+2a)\exp(-\lambda v)\,dv = -\frac{\exp[-\lambda(T-a-u)]}{\lambda}\varphi(u+a, 1, u+T).$$

Using a similar argumentation as above, we obtain

$$\varphi(u+a, 2, v+2u+2a) = 0 \quad \text{and} \quad \varphi(u+a, 1, u+a) = 0$$

a.e. in the region $0 \leqslant v \leqslant T-a-u$ and $0 \leqslant u \leqslant T-a$, and also $\varphi(\cdot, 0, 2T) = 0$. This is equivalent to $\varphi(x_1, d, s) = 0$ a.e. for $x_1 d \leqslant s \leqslant (N-d)T + x_1 d$, $a \leqslant x_1 \leqslant T$, $d = 0, 1, \ldots, N$ and $N = 2$.

With respect to clarity we prove the uncompleteness of the statistic $(X_1, D(T), S)$ in the case $N = 3$. If $N > 3$, we can set $\varphi(x_1, d, s) = 0$ for $d > 3$, but this does not change the situation. If $N = 3$, then expression (27) is of the form

$$\text{(31)} \quad \mathrm{E}[\varphi(X_1, D(T), S)] = \varphi(\cdot, 0, 3T)\exp[-3\lambda(T-a)] + \int_a^T \varphi(x_1, 1, x_1+2T)3\lambda\exp[-\lambda(x_1+2T-3a)]\,dx_1 + \int_a^T \int_{2x_1+T}^{2T+x_1} \varphi(x_1, 2, s)6\lambda^2\exp[-\lambda(s-3a)]\,ds\,dx_1 + \int_a^T \int_{3x_1}^{x_1+T} \varphi(x_1, 3, s)3\lambda^2\exp[-\lambda(s-3a)](s-3x_1)\,ds\,dx_1 - 2\int_0^T \int_{2x_1+T}^{2T+x_1} \varphi(x_1, 3, s)3\lambda^3\exp[-\lambda(s-3a)](s-2x_1-T)\,ds\,dx_1 = 0.$$

Making some easy transformations, we get

$$\text{(32)} \quad \int_0^{T-a} \Bigg[\frac{1}{2\lambda^2}\varphi(u+a, 1, u+a+2T)\exp[-2\lambda(T-a)] + \frac{1}{\lambda}\exp[-\lambda(T-a)] \int_{u+a}^{T} \varphi(u+a, 2, z+u+a+T)\exp[-\lambda(z-a)]\,dz + \frac{1}{2}\int_{2(u+a)}^{2T} \varphi(u+a, 3, z+u+a)[z-2(u+a)]\exp[-\lambda(z-2a)]\,dz - \int_{u+a+T}^{2T} \varphi(u+a, 3, z+u+a)(z-u-a-T)\exp[-\lambda(z-2a)]\,dz\Bigg]du = \frac{C_2}{\lambda^3}\exp[-3\lambda(T-a)],$$

where $C_2 = -\frac{1}{6}\varphi(\cdot, 0, 3T)$. The right-hand side of (32) is the Laplace transform (see [2], p. 127) of the function

$$\check{\psi}(u) = \begin{cases} 0 & \text{for } 0 \leqslant u \leqslant 3(T-a), \\ \dfrac{C_2}{2}[u-3(T-a)]^2 & \text{for } u > 3(T-a). \end{cases}$$

Hence, as previously, it implies that $\varphi(\cdot, 0, 3T) = 0$ and, for a.e. $u \in [0, T-a]$, we have

$$(33)\quad \frac{1}{\lambda}\exp[-\lambda(T-a)]\int\limits_{u+a}^{T}\varphi(u+a, 2, z+u+a+T)\exp[-\lambda(z-a)]dz+$$
$$+\frac{1}{2}\int\limits_{2(u+a)}^{2T}\varphi(u+a, 3, z+u+a)(z-2u-2a)\exp[-\lambda(z-2a)]dz-$$
$$-\int\limits_{u+a+T}^{2T}\varphi(u+a, 3, z+u+a)(z-u-a-T)\exp[-\lambda(z-2a)]dz$$
$$= \frac{-\varphi(u+a, 1, u+a+2T)}{2\lambda^2}\exp[-2\lambda(T-a)].$$

Applying similar considerations as previously to the above term, we get at last

$$\int\limits_0^{\infty}\hat{\varphi}(s)\exp(-\lambda s)ds = \frac{C}{2\lambda^2}\exp[-2\lambda(T-u-a)],$$

where $C = -\varphi(u+a, 1, u+a+2T)$ and

$$\hat{\varphi}(s) = \begin{cases} \dfrac{\varphi\big(u+a, 3, s+3(u+a)\big)}{2}s & \text{for } 0 \leqslant s \leqslant T-u-a, \\ \displaystyle\int\limits_0^{s}\varphi\big(u+a, 2, t+3(u+a)\big)dt+\frac{\varphi\big(u+a, 3, s+3(u+a)\big)}{2}\big(2(T-u-a)-s\big) & \\ & \text{for } T-u-a < s \leqslant 2(T-u-a), \\ \displaystyle\int\limits_0^{2(T-u-a)}\varphi(u+a, 2, t+3(u+a)dt & \text{for } s > 2(T-u-a), \end{cases}$$

but $\varphi(u+a, 2, s+3(u+a)) = 0$ for $s \notin (T-u-a, 2(T-u-a))$. The right-hand side of (33) is the Laplace transform of the function

$$\hat{\psi}(s) = \begin{cases} 0 & \text{for } 0 \leqslant s \leqslant 2(T-u-a), \\ \frac{C}{2}[s-2(T-u-a)] & \text{for } s > 2(T-u-a). \end{cases}$$

Hence, similarly as in the proof of Theorem 3 in [1], we obtain $\varphi(u+a, 3, s+3(u+a)) = 0$ a.e. in the region $0 \leqslant s \leqslant T-u-a$ and $0 \leqslant u \leqslant T-a$, but

$$\varphi(u+a, 3, s+3(u+a)) = \frac{2}{s-2(T-u-a)} \int_0^s \varphi(u+a, 2, t+3(u+a))\,dt$$

$$\text{for } T-u-a \leqslant s \leqslant 2(T-u-a) \text{ and } 0 \leqslant u \leqslant T-a,$$

and also

$$C = -\varphi(u+a, 1, u+a+2T)$$

$$= -\frac{2}{s-2(T-u-a)} \int_{T-u-a}^{2(T-u-a)} \varphi(u+a, 2, t+3(u+a))\,dt = 0$$

$$\text{for } s > 2(T-u-a) \text{ and } 0 \leqslant u \leqslant T-a.$$

This contradicts the completeness of the sufficient statistic $(X_1, D(T), S)$.

**4. Acknowledgements.** I wish to express my sincere thanks to Professor B. Kopociński for helpful discussions and comments. I wish also to thank Professor J. Łukaszewicz for his valuable remark concerning the estimator (5).

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*Received on 20. 5. 1974*

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# ESTYMACJA NIEZAWODNOŚCI W PRZYPADKU WYKŁADNICZYM (II)

## STRESZCZENIE

W pracy, która jest kontynuacją [1], rozpatrzony jest problem nieobciążonej estymacji wykładniczej niezawodności $R_0(t) = e^{-\lambda(t-a)}$, $t > a$, dla planu badania bez odnowy, z czasem badania trwającym do ustalonej chwili $T$. Na podstawie twierdzenia Rao-Blackwella wyprowadzone są wzory na nieobciążone estymatory funkcji $R_0(t)$, będące funkcjami statystyk dostatecznych, zarówno wtedy, gdy parametr $a$ jest znany, jak i wtedy, gdy jest nie znany. Dowodzi się, że statystyki dostateczne — poza trywialnymi przypadkami — nie są zupełne, a zatem nie można twierdzić, że otrzymane estymatory są nieobciążone z minimalną wariancją.