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ON THE NECESSARY CONDITION FOR THE OPTIMALITY OF CONTROL IN THE PROBLEM OF EVASION IN DIFFERENTIAL GAMES

1. Introduction. In the present paper we study the problem of evasion for two controlled objects: the chaser, $x \in E^n$, and the evader, $y \in E^n$, whose motions are described by the differential equations and the initial conditions

$$\frac{dx}{dt} = f\big(t, x, u(t)\big), \tag{1.1}$$

$$x(t_0) = x_0 \tag{1.2}$$

and

$$\frac{dy}{dt} = g\big(t, y, v(t)\big), \tag{1.3}$$

$$y(t_0) = y_0, \tag{1.4}$$

where $u(t) \in U \subseteq E^r$ and $v(t) \in V \subseteq E^s$ are the controls of the chaser and evader, respectively. We understand the problem of evasion in the way it has been formulated in [9]-[11].

We consider the necessary condition for the optimality of the evader's control in the form of the maximum principle. The maximum principle is generally valid for the optimal control of the chaser (see [6]). The research of necessary conditions for the optimality of the evader's control in the evasion problem is equivalent to the research of necessary conditions for the optimality of control of the chased in the pursuit problem (cf. [4] and [8]).

In [4] Kelendžeridze has proved that the maximum principle is valid for the optimal control of the evader for the case where $f(t, x, u) \equiv Ax+Bu$ (A and B are constant matrices), the set U is a convex polyhedron placed suitably in the space E^r, and the function $g(t, y, v)$ and the set V are arbitrary. The same author has stated the hypothesis that the maximum principle holds for any function $f(t, x, u)$ and any set U (see [5]). However, Warga proved in [12] this hypothesis to be false.

In [11] Športjuk has proved the maximum principle in the problem of evasion for the case where

$$f(t, x, u) \equiv A(t)x + B(t)u \quad \text{and} \quad g(t, y, v) \equiv C(t)y + D(t)v,$$

$A(t)$, $B(t)$, $C(t)$, and $D(t)$ being continuous matrix-valued functions and the sets U and V being compact and convex.

In the considerations concerning the maximum principle in a problem of evasion, an essential role is played by the properties of the attainable set of the chaser (cf. [6] and [8]). Warga showed in [12] that the closedness of the chaser's attainable set does not suffice for the maximum principle to hold, and at the same time he stated the hypothesis that the closedness and one-connectedness should suffice. In this paper we prove that the maximum principle holds when the chaser's attainable set is closed, convex, and full-dimensional in E^n.

2. Problem of evasion and the necessary condition for the optimality of control. Assume that in the space E^n there are given two controlled objects x and y, the motions of which are described by the differential equations (1.1) and (1.3) with the initial conditions (1.2) and (1.4), where t_0 is a given initial moment, x_0 and y_0 are given initial points, $x_0 \neq y_0$, and the functions $u(t)$, $t_0 \leqslant t \leqslant t_1$, and $v(t)$, $t_0 \leqslant t \leqslant t_1$, called *control functions* or *controls*, are measurable and bounded vector-valued functions in the interval $[t_0, t_1]$ such that $u(t) \in U$ and $v(t) \in V$ for $t_0 \leqslant t \leqslant t_1$, where U is a given subset of E^r and V is a given subset of E^s. We assume that the vector-valued function $f(t, x, u)$ is continuous with continuous derivatives of the first order with respect to the variables x and u on $E^{1+n+r} = \{(t, x, u)\colon t \in E,\ x \in E^n,\ u \in E^r\}$ and the vector-valued function $g(t, y, v)$ is continuous with continuous derivatives of the first order with respect to the variable y on $E^{1+n+s} = \{(t, y, v)\colon t \in E,\ y \in E^n,\ v \in E^s\}$.

Further on, we call the point object x a *chasing object* or *chaser* for short. The control function $u(t)$ defined in the interval $[t_0, t_1]$ and such that the solution $x(t)$ of equation (1.1) with the initial condition (1.2) exists in the interval $[t_0, t_1]$ is called an *admissible* (in the interval $[t_0, t_1]$) *control of the chaser*, and the vector-valued function $x(t)$ is called an *admissible trajectory of the chaser* corresponding to the admissible control $u(t)$. The solution $x(t)$ of equation (1.1) defined in the interval $[t_0, t_1]$ is a vector-valued function $x(t)$ defined and absolutely continuous on $[t_0, t_1]$, and satisfying equation (1.1) almost everywhere in that interval. We call the point object y an *evader object* or *evader* for short, and in a manner similar to that for the chaser we define an admissible (in the interval $[t_0, t_1]$) control $v(t)$ of the evader and the corresponding admissible trajectory $y(t)$.

Let $u(t)$ and $v(t)$ be any admissible (in the interval $[t_0, t_1]$) controls of the chaser and evader and let $x(t)$ and $y(t)$ be the trajectories corresponding to the controls $u(t)$ and $v(t)$, respectively. Let

$$T(\{u(t)\}, \{v(t)\}) = \min\{t \in [t_0, t_1]\colon x(t) = y(t)\} \quad (\min\emptyset = \infty).$$

If $T(\{u(t)\}, \{v(t)\}) < \infty$, then we call $T(\{u(t)\}, \{v(t)\})$ the *moment of meeting* corresponding to the controls $u(t)$ and $v(t)$. Let further

$$T(\{v(t)\}) = \inf_{\{u(t)\}} T(\{u(t)\}, \{v(t)\}),$$

where the infimum is taken over all controls $u(t)$ of the chaser admissible in $[t_0, t_1]$. We assume that there exists a number $t_1 > t_0$ such that $T(\{v(t)\}) < \infty$ for every admissible (in $[t_0, t_1]$) control $v(t)$ of the evader. The admissible (in $[t_0, t_1]$) control $\hat{v}(t)$ of the evader such that

$$T(\{\hat{v}(t)\}) = \max_{\{v(t)\}} T(\{v(t)\}),$$

where the maximum is taken over all admissible (in $[t_0, t_1]$) controls $v(t)$ of the evader, is called the *optimal control of the evader* and the moment $T = T(\{\hat{v}(t)\})$ is said to be the *optimal moment of meeting*.

Let $v(t)$ be any admissible (in $[t_0, t_1]$) control of the evader and let $y(t)$ be the trajectory corresponding to $v(t)$. Let further $\chi = (\chi_1, \ldots, \chi_n)$ be any n-dimensional vector and let

$$H(\chi, t, y, v) = \chi g(t, y, v) \equiv \sum_{k=1}^{n} \chi_k g^k(t, y, v).$$

Let us consider the system of linear differential equations

$$\frac{d\chi_i}{dt} = -\frac{\partial H(\chi, t, y(t), v(t))}{\partial y^i}, \quad i = 1, \ldots, n. \tag{2.1}$$

If the functions $v(t)$ and $y(t)$ are given and defined in the interval $[t_0, t_1]$, then the system of differential equations (2.1) with the condition $\chi(\tau) = \chi_\tau$ determines uniquely the absolutely continuous vector-valued function $\chi(t)$ for $t \in [t_0, t_1]$, where $\tau \in [t_0, t_1]$ and $\chi_\tau \in E^n$ are arbitrary points of $[t_0, t_1]$ and E^n, respectively.

The admissible (in $[t_0, t_1]$) control $v(t)$ of the evader is called an *extremal control* in $[t_0, t_1]$ if in the interval $[t_0, t_1]$ there exists a nonzero solution $\chi(t)$ of the system (2.1) corresponding to the functions $v(t)$ and $y(t)$ such that the condition

$$\max_{v \in V} H(\chi(t), t, y(t), v) = H(\chi(t), t, y(t), v(t)) \tag{2.2}$$

is satisfied for almost all $t \in [t_0, t_1]$. We say that for the problem of evasion the *maximum principle holds* if an optimal control $v(t)$ of the evader is an extremal control in the interval $[t_0, T]$, where T is the optimal moment of meeting.

The aim of the paper is to prove that under some assumptions given in the sequel the necessary condition for the optimality of an evader's control is the validity of the maximum principle.

3. Basic assumptions. Now, we state the assumptions concerning the chaser needed for the proof of the main result.

Assume that the motion of the chaser is described by the equation (1.1) with the initial condition (1.2). We call an *attainable set of the chaser* at the time $t_1 > t_0$ the set Σ_{t_1} of all points $x_1 \in E^n$ such that there exists an admissible (in the interval $[t_0, t_1]$) control $u(t)$ such that the corresponding solution $x(t)$ of equation (1.1) with the initial condition (1.2) satisfies the condition $x(t_1) = x_1$.

We assume that the set U is compact in the space E^r and that either (α) or (β) holds:

(α) The function $f(t, x, u)$ is linear with respect to x, so that it is of the form $f(t, x, u) \equiv A(t)x+b(t, u)$, where $A(t)$ is a matrix-valued function.

(β) The function $f(t, x, u)$ satisfies the following conditions:

1° There exists a constant $C > 0$ such that $xf(t, x, u) \leqslant C(1+|x|^2)$ for any $t \in [t_0, t_1]$ and for every $x \in E^n$.

2° The set $f(t, x, U) = \{f(t, x, u)\colon u \in U\}$ is convex for every t in $[t_0, t_1]$ and for any $x \in E^n$.

3° The condition $f(t, x, U) = f(t, x', U)$ is satisfied for every t in $[t_0, t_1]$ and for any pair of points $x \in \Sigma_t$ and $x' \in \Sigma_t$.

By our assumptions, the attainable set Σ_t of the chaser is a compact and convex subset of E^n for $t \in [t_0, t_1]$. This has been proved in [7] for the case (α) and in [1] and [3] for the case (β). Further, our assumptions imply that the set Σ_t depends continuously on t for $t \in [t_0, t_1]$, i.e., $t \to \Sigma_t$ is a continuous mapping from $[t_0, t_1]$ into the space of compact subsets of E^n endowed with the Hausdorff metric (see [6]).

Moreover, we assume that there exists a point $\hat{u} \in \operatorname{Int} U$ such that $u(t) = \hat{u}$ for $t \in [t_0, T]$ is an admissible (in the interval $[t_0, T]$) control of the chaser and the matrix

$$\int_{t_0}^{T} X^{-1}(t)B(t)B'(t)X^{-1'}(t)\,dt$$

has rank n, where T is the optimal moment of meeting, $X(t)$ is a solution of the matrix differential equation $dX/dt = A(t)X$ with the initial condition $X(t_0) = I$ (I is an identity matrix), the functions

$$A(t) = \frac{\partial f(t, \hat{x}(t), \hat{u})}{\partial x} \quad \text{and} \quad B(t) = \frac{\partial f(t, \hat{x}(t), \hat{u})}{\partial u}$$

are continuous matrix-valued functions, where $\hat{x}(t)$ is a solution of the differential equation $dx/dt = f(t, x, \hat{u})$ with the initial condition $\hat{x}(t_0) = x_0$ (B' is a transposed matrix of B). By these assumptions, the attainable set Σ_T of the chaser is full-dimensional in E^n (see [2]).

4. Some properties of attainable sets and controls of the chaser. If the chaser's attainable set Σ_t is compact, convex, and depends continuously on t for all t under consideration, the following lemmas hold:

LEMMA 4.1. *If* $x_k \in \Sigma_{t_k}$ *for* $k = 1, 2, \ldots$ *and*

$$\lim_{k\to\infty} x_k = x', \quad \lim_{k\to\infty} t_k = t',$$

then $x' \in \Sigma_{t'}$.

Lemma 4.1 follows immediately from the continuous dependence of Σ_t on t and the closedness of Σ_t.

LEMMA 4.2. *If* $x_k \notin \Sigma_{t_k}$ *for* $k = 1, 2, \ldots$ *and*

$$\lim_{k\to\infty} x_k = x', \quad \lim_{k\to\infty} t_k = t' \quad \textit{and} \quad x' \in \Sigma_{t'},$$

then $x' \in \partial\Sigma_{t'}$.

Proof. First we prove that if $x' \in \operatorname{Int}\Sigma_{t'}$, then there exist a ball B with the centre at x' and a number $\delta > 0$ such that $B \subset \Sigma_t$ for $|t-t'| < \delta$.

Assume that $x' \in \operatorname{Int}\Sigma_{t'}$ and $x' = 0$. Then there exists a number $\varepsilon > 0$ such that the n-dimensional open parallelepiped P of the 2^n-vertices $a_i = (\pm\varepsilon, \pm\varepsilon, \ldots, \pm\varepsilon)$, $i = 1, 2, \ldots, 2^n$, is contained in the set $\Sigma_{t'}$. If $\varrho(\Sigma_t, \Sigma_{t'}) < \varepsilon/2$, then for any i ($i = 1, 2, \ldots, 2^n$) there exists a point $b_i \in \Sigma_t$ such that $d(a_i, b_i) < \varepsilon/2$, where ϱ denotes the distance of the Hausdorff metric and d the Euclidean distance. The point b_i lies in the parallelepiped of the diagonal $(\frac{1}{2}a_i, \frac{3}{2}a_i)$, whence $\frac{1}{2}P \subset \operatorname{conv}\{b_i\colon i = 1, 2, \ldots, 2^n\}$, where $\frac{1}{2}P$ is the parallelepiped of the vertices $\frac{1}{2}a_i$. It follows from the convexity of the set Σ_t that $\operatorname{conv}\{b_i\colon i = 1, 2, \ldots, 2^n\} \subset \Sigma_t$. By the continuous dependence of Σ_t on t there exists a ball B satisfying the required conditions.

Now let $\{x_k\}$ and $\{t_k\}$, $k = 1, 2, \ldots$, be sequences satisfying the assumptions of the lemma. Suppose that $x' \notin \partial\Sigma_{t'}$; so $x' \in \operatorname{Int}\Sigma_{t'}$. Then there exist a ball B with the centre at x' and a number $\delta > 0$ such that $B \subset \Sigma_t$ for $|t-t'| < \delta$. Since

$$\lim_{k\to\infty} t_k = t' \quad \text{and} \quad \lim_{k\to\infty} x_k = x',$$

we have $|t_k - t'| < \delta$ and $x_k \in B$ for k sufficiently large. Hence $x_k \in \Sigma_{t_k}$, a contradiction.

Remark. The convexity of the set Σ_t is an essential assumption of Lemma 4.2. Without this assumption Lemma 4.2 is false, which follows from the example given in [12].

Let $v(t)$ be any admissible (in $[t_0, t_1]$) control of the evader. If the attainable set Σ_t of the chaser is compact, convex, and depends continuously on t for $t \in [t_0, t_1]$, then

$$T\big(\{v(t)\}\big) = \min_{\{u(t)\}} T\big(\{u(t)\}, \{v(t)\}\big),$$

e. g. the infimum in the definition of $T\big(\{v(t)\}\big)$ is attained for some admissible (in $[t_0, t_1]$) control $u(t)$ of the chaser, which is an extremal control in the interval $\big[t_0, T\big(\{v(t)\}\big)\big]$ (see [6]). If the control $v(t)$ of the evader is optimal, then the control $u(t)$ such that $T\big(\{u(t)\}, \{v(t)\}\big) = T\big(\{v(t)\}\big)$ is called the *optimal control of the chaser.*

5. Some properties of the evader's controls. Now, we outline shortly the properties of the controls which will be used in the proof of the main result. These properties and the proofs are given in details in [8].

Let $y(t)$ be a solution of equation (1.3) with the initial condition (1.4) defined in the interval $[t_0, T]$ corresponding to the given admissible (in $[t_0, T]$) control $v(t)$. Let η_t denote any vector placed at the point $y(t)$. We define the transformation $A_{t,\tau}$: $\eta_\tau \to \eta_t$ of vectors η_τ, $\tau \in [t_0, T]$, along the trajectory $y(t)$ in the following way:

$$\frac{d}{dt}[A_{t,\tau}(\eta_\tau)]^i = \sum_{a=1}^{n} \frac{\partial g^i\big(t, y(t), v(t)\big)}{\partial y^a} [A_{t,\tau}(\eta_\tau)]^a, \qquad i = 1, \ldots, n,$$

for $t \in [t_0, T]$ and $A_{\tau,\tau}(\eta_\tau) = \eta_\tau$. The transformation $A_{t,\tau}$ is defined for any τ and for every t contained in the interval $[t_0, T]$, and is linear and nonsingular.

Let $\tau \in [t_0, T]$ be any regular point of the control $v(t)$, i.e. let

$$\int_{\tau-\varepsilon}^{\tau} \big|g\big(t, v(t)\big) - g\big(\tau, v(\tau)\big)\big|\, dt = o(\varepsilon)$$

for an arbitrary continuous function $g(t, v)$, where $\varepsilon > 0$ and

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Almost all points in the interval $[t_0, T]$ are regular points of the control $v(t)$. Let $\varepsilon > 0$ be an arbitrary sufficiently small number and consider any ε-modified control $v_\varepsilon(t)$ (defined in [8], p. 98) of the control

$v(t)$ corresponding to the point τ. If $y_\varepsilon(t)$ is the trajectory corresponding to the control $v_\varepsilon(t)$, then $y_\varepsilon(t)$ is determined in the interval $[t_0, T]$ and

$$y_\varepsilon(t) = y(t)+\varepsilon A_{t,\tau}[\Delta y(\tau)]+o(\varepsilon) \tag{5.1}$$

for $\tau \leqslant t \leqslant T$, where $\Delta y(\tau)$ is a given variation corresponding to the ε-modified control $v_\varepsilon(t)$. The set of variations $\{\Delta y(\tau)\}$ placed at the point $y(\tau)$ corresponding to all possible ε-modified controls $v_\varepsilon(t)$ at a fixed point τ is a convex cone with vertex at the point $y(\tau)$ which will be denoted by $K^{(\tau)}$. Let us write $K_T^{(\tau)} = A_{T,\tau}(K^{(\tau)})$. The sets $K_T^{(\tau)}$ are also convex cones with vertices at the point $y(T)$, and if $\tau_1 < \tau_2$, then $K_T^{(\tau_1)} \subset K_T^{(\tau_2)}$. Let

$$\mathscr{K}_T = \bigcup_\tau K_T^{(\tau)},$$

where the sum is taken over all regular points τ belonging to the interval $[t_0, T]$.

LEMMA 5.1. *If there exists a nonzero vector e^* such that for any vector $\Delta y \in \mathscr{K}_T$ we have $e^* \Delta y \leqslant 0$, then the absolutely continuous function $\chi(t)$, which in the interval $[t_0, T]$ is a solution of the system* (2.1) *with the condition $\chi(T) = e^*$, fulfils* (2.2) *for almost all $t \in [t_0, T]$.*

The proof of Lemma 5.1 can be found in [8], p. 92-100.

6. The necessary condition for the optimality of the evader's control. Now, we prove the theorem providing a necessary condition for the optimality of the evader's control in the evasion problem.

THEOREM 6.1. *Let T be the optimal moment of meeting of the chaser and evader. If the attainable set Σ_t of the chaser is compact, convex, and depends continuously on t for $t \in [t_0, T]$ and the set Σ_T is full-dimensional in E^n, then the optimal control $v(t)$ of the evader is extremal in the interval $[t_0, T]$.*

Proof. Let T be the optimal moment of meeting of the chaser and evader, let $u(t)$ and $v(t)$ be the optimal controls of the chaser and evader, and let $x(t)$ and $y(t)$ be their corresponding optimal trajectories, respectively. Denote by Σ_t the attainable set of the chaser at the time t. Thus we have $y(t) \notin \Sigma_t$ for $t_0 \leqslant t < T$ and $y(T) \in \Sigma_T$ and the relation $x(T) = y(T)$ holds. Moreover, from Lemma 4.2 it follows that $y(T) \in \partial\Sigma_T$.

Let us consider in the interval $[t_0, T]$ the optimal control $v(t)$ of the evader. Let τ be any given regular point of the control $v(t)$ such that $\tau < T$. We show that there exists a unit vector e such that

$$A_{T,\tau}[\Delta y(\tau)]e \leqslant 0 \tag{6.1}$$

for a fixed $\Delta y(\tau) \in K^{(\tau)}$ and

$$(x - x(T))e \leqslant 0 \tag{6.2}$$

for all $x \in \Sigma_T$.

Let $\{\varepsilon_k\}$, $k = 1, 2, \ldots$, be a sequence of positive numbers convergent to zero. The ε-modified control of the control $v(t)$ corresponding to $\varepsilon = \varepsilon_k$ and $\Delta y(\tau) \in K^{(\tau)}$ will be denoted by $v_k(t)$, and the trajectory corresponding to $v_k(t)$ by $y_k(t)$. Then the trajectory $y_k(t)$ for $\tau \leqslant t \leqslant T$ takes the form (5.1). Let t_k be the moment of meeting corresponding to the controls $v_k(t)$ and $u(t)$. Then we have

$$y_k(t_k) \in \Sigma_{t_k}, \quad k = 1, 2, \ldots \tag{6.3}$$

It follows from the optimality of the pair of controls $u(t)$ and $v(t)$ that $t_k \leqslant T$. Further, we have

$$\lim_{k\to\infty} t_k = T.$$

Indeed, if this were not true, then passing to the subsequence we would infer that the limit $\lim\limits_{k\to\infty} t_k$ exists and that

$$\lim_{k\to\infty} t_k = \bar{t} < T.$$

As $\lim\limits_{k\to\infty} \varepsilon_k = 0$, by (5.1) we obtain

$$\lim_{k\to\infty} y_k(t_k) = y(\bar{t}).$$

However, using (6.3) and Lemma 4.1, we have $y(\bar{t}) \in \Sigma_{\bar{t}}$, which contradicts the definition of T. Hence, omitting a finite number of initial elements of the sequence $\{t_k\}$, we have

$$\tau < t_k \leqslant T \quad \text{and} \quad \lim_{k\to\infty} t_k = T.$$

Taking now a subsequence of $\{t_k\}$, which will be denoted in the same way, we find that $\tau < t_k < T$ for $k = 1, 2, \ldots$ or $\tau < t_k = T$ for $k = 1, 2, \ldots$

Let us consider the sequence of points $\{y(t_k)\}$, $k = 1, 2, \ldots$, lying on the optimal trajectory of the evader. If $t_k < T$, then $y(t_k) \notin \Sigma_{t_k}$. Let $\Omega_{t_k} = \operatorname{conv}\{y(t_k), \Sigma_{t_k}\}$. Then, by convexity of the set Σ_{t_k}, $y(t_k)$ is a boundary point of the set Ω_{t_k}. Therefore, there exists a supporting hyperplane for the set Ω_{t_k} at the point $y(t_k)$. Let e_k be a unit vector normal to this hyperplane and so oriented that

$$(x - y(t_k))e_k \leqslant 0 \tag{6.4}$$

for $x \in \Omega_{t_k}$, and the more so for $x \in \Sigma_{t_k}$. Since the vectors $\{e_k\}$ lie on a unit sphere, we may choose from them a converging subsequence. Let $\{e_k\}$ be such a subsequence and let

$$e = \lim_{k\to\infty} e_k.$$

By (6.3) and (6.4) we have $\big(y_k(t_k)-y(t_k)\big)e_k \leqslant 0$. Using (5.1) and passing to the limit, we obtain $A_{T,\tau}[\Delta y(\tau)]e \leqslant 0$ for a given $\Delta y(\tau) \in K^{(\tau)}$. Further we show that $\big(x-x(T)\big)e \leqslant 0$ for $x \in \Sigma_T$. Let x be any point belonging to Σ_T. Then there exists a trajectory $\bar{x}(t)$ such that $\bar{x}(T) = x$. However, $\bar{x}(t_k) \in \Sigma_{t_k}$. Hence, taking into account formula (6.4), we have $\big(\bar{x}(t_k)-y(t_k)\big)e_k \leqslant 0$. As $k \to \infty$, we infer from the last inequality that $\big(x-x(T)\big)e = \big(x-y(T)\big)e \leqslant 0$ for all $x \in \Sigma_T$. If $t_k = T$ for $k = 1, 2, \ldots$, then $y_k(t_k) \in \Sigma_T$. As $y(T)$ is a boundary point of the set Σ_T, there exists a supporting hyperplane for the set Σ_T at the point $y(T)$. Let e be a unit vector normal to this hyperplane, so oriented that $\big(x-y(T)\big)e \leqslant 0$ for all $x \in \Sigma_T$. Consequently, as $x(T) = y(T)$, we have $\big(x-x(T)\big)e \leqslant 0$ for all $x \in \Sigma_T$. Hence $\big(y_k(t_k)-x(T)\big)e \leqslant 0$ and, therefore, using (5.1), we obtain $A_{T,\tau}[\Delta y(\tau)]e \leqslant 0$ for $\Delta y(\tau) \in K^{(\tau)}$. Thus we have shown that there exists a unit vector e satisfying (6.1) and (6.2).

Consider a hyperplane passing through the point $x(T)$ and orthogonal to the vector e. From (6.1) and (6.2) it follows that the set Σ_T and the vector $-A_{T,\tau}[\Delta y(\tau)]$ lie on the opposite sides of this hyperplane. Hence the vector $-A_{T,\tau}[\Delta y(\tau)]$ does not pass through any inner points of the set Σ_T. We take any variations $\Delta y(\tau)$ at the given τ corresponding to all possible ε-modified controls of the optimal control $v(t)$. The vectors $A_{T,\tau}[\Delta y(\tau)]$ form a convex cone $K_T^{(\tau)}$ with vertex at the point $x(T)$. Then the vectors $-A_{T,\tau}[\Delta y(\tau)]$ also form a convex cone with vertex at the point $x(T)$ which will be denoted by $-K_T^{(\tau)}$. From the above considerations it follows that the convex cone $-K_T^{(\tau)}$ does not have any common points with the interior of the set Σ_T, and since the set Σ_T is convex and contains inner points, there exists a hyperplane separating these sets. Let e_τ be a unit vector normal to this hyperplane, oriented into the half-space containing the set $-K_T^{(\tau)}$. Thus the set $K_T^{(\tau)}$ lies in one of the half-spaces determined by this hyperplane and

$$\Delta y \cdot e_\tau \leqslant 0 \tag{6.5}$$

for every $\Delta y \in K_T^{(\tau)}$.

Let now $\{\tau_k\}$, $k = 1, 2, \ldots$, be any sequence of regular points of the control $v(t)$ such that

$$\tau_k < T \quad \text{and} \quad \lim_{k\to\infty} \tau_k = T.$$

Further, let e_{τ_k} denote such a unit vector corresponding to τ_k for which (6.5) holds. We may assume, passing to the subsequence, that

$$\lim_{k\to\infty} e_{\tau_k} = e^*.$$

Let $\Delta y \in \mathscr{K}_T$. By the definition of $\mathscr{K}_T$ there exists a $\tau < T$ such that $\Delta y \in K_T^{(\tau)}$. Further

$$\lim_{k\to\infty} \tau_k = T,$$

whence $\tau_k > \tau$ for k sufficiently large. Thus $\Delta y \in K_T^{(\tau_k)}$ because $K_T^{(\tau)} \subset K_T^{(\tau_k)}$. By (6.5) we have $\Delta y \cdot e_{\tau_k} \leqslant 0$. Consequently, passing to the limit, we obtain

$$\Delta y \cdot e^* \leqslant 0. \tag{6.6}$$

Relation (6.6) holds for any $\Delta y \in \mathscr{K}_T$. From (6.6), taking into account Lemma 5.1, we obtain Theorem 6.1.

Remark. As we have observed previously the optimal control $u(t)$ of the chaser is extremal in the interval $[t_0, T]$, e.g. there exists a nonzero solution $\psi(t) = \big(\psi_1(t), \ldots, \psi_n(t)\big)$ of the differential equation system

$$\frac{d\psi_i}{dt} = -\frac{\partial \bar{H}\big(\psi, t, x(t), u(t)\big)}{\partial x^i}, \quad i = 1, \ldots, n,$$

where $\bar{H}(\psi, t, x, u) = \sum_{k=1}^{n} \psi_k f^k(t, x, u)$, such that the relation

$$\max_{u\in U} \bar{H}\big(\psi(t), t, x(t), u\big) = \bar{H}\big(\psi(t), t, x(t), u(t)\big)$$

is valid almost everywhere in the interval $[t_0, T]$. It follows by the proof of Theorem 6.1 that one can put $\psi(T) = \chi(T)$.

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O WARUNKU KONIECZNYM OPTYMALNOŚCI STEROWANIA W PROBLEMIE UNIKANIA SPOTKANIA W GRACH RÓŻNICZKOWYCH

STRESZCZENIE

W pracy przedstawiony jest dowód ekstremalności sterowania optymalnego dla unikającego spotkania przy założeniu, że zbiór osiągalności ścigającego jest zwarty, wypukły i pełnowymiarowy oraz zależy w sposób ciągły od czasu. Warunek ekstremalności jest sformułowany w postaci zasady maksimum. Przy dowodzie wykorzystuje się konstrukcję wariacji sterowań podaną w [8].
