ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
20, 1 (1988), pp. 103–110

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EXISTENCE OF TIME AVERAGES FROM THE TOPOLOGICAL POINT OF VIEW

Abstract. The Birkhoff Ergodic Theorem states that time averages exist on a large set in the sense of measure. This paper gives a negative result as concerns this problem in the sense of topology even if we weaken our question in a natural way and demand the existence of time averages on large sets only for a large set of dynamical systems.

Let T be a homeomorphism of a topological space X and let f be a real continuous function on X. We may interpret X as a phase space of a physical, chemical or biological process, $\{T^n x\}$ as a time evolution of a state x and $\{f(T^n(x))\}$ as a measurement. The problem of existence of the time average of f,

$$\lim_{N\to+\infty} N^{-1} \sum_{n=0}^{N-1} f(T^n(x)),$$

is an important question from the point of view of applications. The object of the paper is to study topological aspects of this problem.

Measure-theoretical aspects of this problem are considered in the Ergodic Theorem of Birkhoff [4]. Birkhoff proved that for a measure preserving transformation T of a probability space $\langle X, \mu\rangle$ and $f \in L^1(X, \mu)$ the time average of f exists for μ-almost every point x. The support of μ may be quite small, and hence the subset of X on which Birkhoff's Theorem guarantees the existence of averages may be small from the topological point of view.

A great progress in this direction is the theorem of Bowen and Ruelle [1], which implies the existence of the time average of continuous f for initial points from a basin of Axiom A attractor except a set of Lebesgue measure 0.

Studying topological aspects of the existence of time averages we can ask whether for fixed T and continuous f the time average of f exists on a residual subset of the phase space. To find examples indicating a negative answer both for discrete and for continuous time is not difficult.

EXAMPLE 1. Let $\Sigma_2 = \{0, 1\}^Z$ denote the space of all doubly infinite sequences of 0's and 1's with the product topology. Let σ be the left shift on

Σ_2 given by $\sigma(\{x_n\}) = \{x'_n\}$, $x'_n = x_{n+1}$, and let f be a function on Σ_2 defined by $f(\{x_n\}) = x_0$. We define the subset $A \subset \Sigma_2$ as follows: $x = \{x_n\} \in A$ iff there exists an infinite sequence of natural numbers $\{n_i\}$, $n_{i+1} > n_i$ such that

$$n_i^{-1} \sum_{j=0}^{n_i - 1} x_j < 1/4 \quad \text{for even } i$$

and

$$n_i^{-1} \sum_{j=0}^{n_i - 1} x_j > 3/4 \quad \text{for odd } i.$$

It is obvious that the time averages do not exist for the points from A. To show that A is residual we prove that A^c is of first category. Define the sets

$$B_N^0 = \Big\{\{x_n\}\colon \text{ for every } j > N,\ j^{-1} \sum_{i=0}^{j-1} x_i \geqslant 1/4\Big\},$$

$$B_N^1 = \Big\{\{x_n\}\colon \text{ for every } j > N,\ j^{-1} \sum_{i=0}^{j-1} x_i \leqslant 3/4\Big\}.$$

It is easy to see that B_N^0 and B_N^1 are closed and nowhere dense. As

$$A^c = \bigcup \{(B_N^1 \cup B_N^0)\colon N \in \boldsymbol{N}\},$$

A is a residual set.

Every element of Σ_2 describes the process of tossing a coin. Of course, considering probabilistic aspects of this process the time average of f exists for almost all $x \in \Sigma_2$ but it does not exist on the residual subset of Σ_2 as we have shown above.

It is an interesting philosophical question why Nature considers large sets or probable events in the sense of measure.

The second example was suggested to us by F. Takens.

Example 2. Let us consider a flow $\{\varphi_t\}$ on $\boldsymbol{R}^2$ whose phase portrait is in Fig. 1. The flow $\{\varphi_t\}$ has an attracting set γ made of two singular points p_1

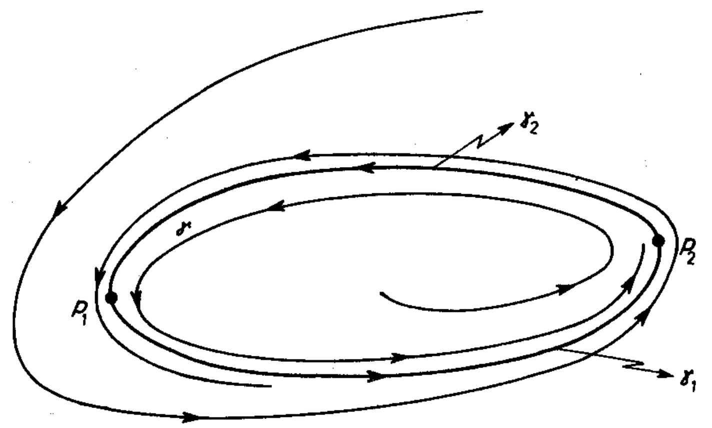

Fig. 1

and p_2 and of two trajectories γ_1 and γ_2 going from p_1 to p_2 and from p_2 to p_1, respectively. Let f be a continuous function on $\mathbf{R}^2$ such that $f(p_1) = 0$ and $f(p_2) = 1$. Roughly speaking, the trajectory of any point p is such that its ω-limit set $\omega(p) = \gamma$ spends larger and larger time intervals in small neighbourhoods of p_1 and p_2. Hence the function

$$S(T) = T^{-1} \int_0^T f(\varphi_s(x))\, ds$$

oscillates and $\lim_{T \to +\infty} S(T)$ does not exist.

Example 2 is in some sense stronger than Example 1, because in Example 2 the time average of f does not exist on an open dense subset of $\mathbf{R}^2$.

These two examples make us weaken the question of the existence of time averages in a topological sense. We formulate it for discrete dynamical systems.

Let X be a compact metric space, let E be a subset of $\mathrm{Hom}(X)$ equipped with a suitable topology, and let $C^0(X, \mathbf{R})$ be the space of continuous maps from X to $\mathbf{R}$ with the topology of uniform convergence. The problem is whether there exist generic sets $D \subset E$ and $S \subset C^0(X, \mathbf{R})$ such that for each $T \in D$ and each $f \in S$ there exists a generic subset $X(T, f)$ of X such that the limit

$$\lim_{N \to +\infty} N^{-1} \sum_{j=0}^{N-1} f(T^j(x))$$

exists for each $x \in X(T, f)$.

For Morse–Smale diffeomorphisms or flows the time average exists for all initial conditions, and this follows from the fact that each trajectory tends asymptotically to a periodic orbit [2]. Hence our hypothesis is true for $E = \mathrm{Diff}(S^1)$ and $E = \mathrm{Flow}^1(M)$, where M is a two-dimensional compact manifold.

Using the idea of Example 1 we construct a simple example indicating a negative answer to our question in the case where X is a compact smooth manifold, $\dim X \geqslant 3$, and $E = \mathrm{Diff}^r(X)$, $1 \leqslant r$. The example which we are going to present is based on Smale's solenoid ([2], [3]), so we remind its construction.

EXAMPLE 3. Let us consider the embedding

$$T\colon S^1 \times D^2 \to S^1 \times D^2,$$

$$S^1 = \{z \in \mathbf{C}\colon |z| = 1\}, \quad D^2 = \{w \in \mathbf{C}\colon |w| \leqslant 1\},$$

defined by

$$T(z, w) = (z^2, z/2 + w/4).$$

It may be shown that T can be extended to a diffeomorphism of $\boldsymbol{R}^3$. If we treat $S^1 \times D^2$ as the solid torus in $\boldsymbol{R}^3$, then

$$T(S^1 \times D^2) \subset \operatorname{Int}(S^1 \times D^2),$$

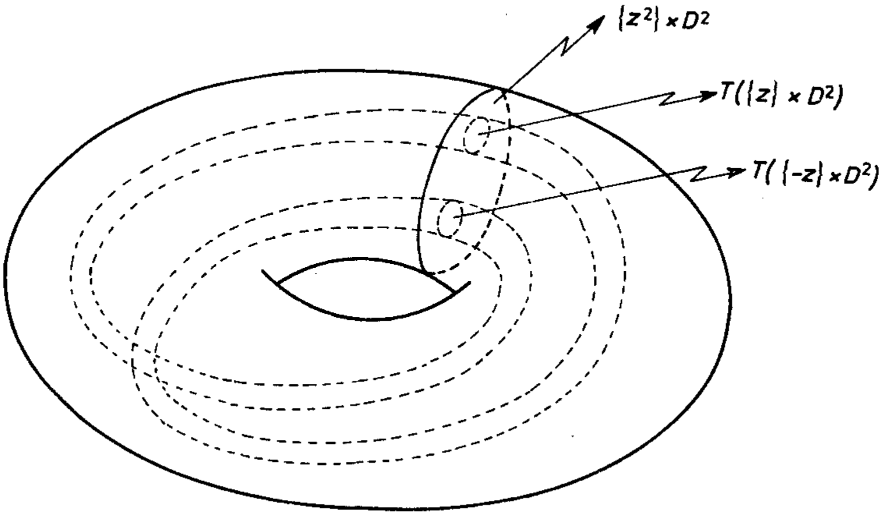

Fig. 2

and T wraps $S^1 \times D^2$ around itself twice as shown in Fig. 2. The torus $T(S^1 \times D^2) \cap \{z^2\} \times D^2$ consists of two disks:

$$T(\{z\} \times D^2) \cup T(\{-z\} \times D^2).$$

The set

$$\Lambda = \bigcap_{n \in N} T^n(S^1 \times D^2)$$

is a hyperbolic attractor [3] and the stable manifold $W^s(x) = \{z \in \boldsymbol{R}^3\colon \lim_{n \to +\infty} d(T^n(x), T^n(z)) = 0\}$ (by d we denote the distance in $\boldsymbol{R}^3$) of every point

$$x = (\exp 2\pi i\varphi, w) \in \Lambda$$

contains the disk $\{\exp 2\pi i\varphi\} \times D^2$.

For our purposes it is important that the mapping T is structurally stable [3]. This means that for every $\bar{T}$ from some C^1-neighbourhood of T there exists a homeomorphism h which conjugates T and $\bar{T}$ on some neighbourhood of Λ.

Let us define the subset $\tilde{\Lambda}$ of Λ as

$$\tilde{\Lambda} = \{x = (\exp 2\pi i\varphi, w)\colon\ \varphi \neq k/2^n \text{ for all } k, n \in N\}$$

and a mapping h_0: $\tilde{\Lambda} \to \{0, 1\}$ by

$$h_0(\exp 2\pi i\varphi, w) = \begin{cases} 0 & \text{if } 0 < \varphi < 1/2, \\ 1 & \text{if } 1/2 < \varphi < 1. \end{cases}$$

Applying h_0 to any $T^n(x)$ for $x \in \tilde{\Lambda}$ we define the map h: $\tilde{\Lambda} \to \Sigma_2$. It is easy to see that the image of h contains all Σ_2 unless the sequences $\{x_n\}$ such that $x_n = 0$ for sufficiently large n or $x_n = 1$ for sufficiently large n. Hence $\tilde{\Sigma}_2 = h(\tilde{\Lambda})$ is a residual subset of Σ_2.

The mapping h conjugates $T|\tilde{\Lambda}$ to the left shift on $\tilde{\Sigma}_2$, i.e., $h \circ T = \sigma \circ h$. This conjugacy will be used to construct the residual set in a neighbourhood of Λ on which the time averages do not exist for f from an open subset of $C^0(\boldsymbol{R}^3, \boldsymbol{R})$.

For $x \in \Sigma_2$ we call a segment $\{x_i\}$, $i = k, \ldots, k+l$, the *j-segment* ($j = 0$ or 1) of x of length $l+1$ if

$$x_k = x_{k+1} = \ldots = x_{k+l} = j \quad \text{and} \quad x_{k+l+1} \neq j, \quad x_{k-1} \neq j.$$

Let $x \in \Sigma_2$ and let k, n ($k \leqslant n$) be natural numbers. We denote by $C_j(x, n)$ the number of all j's in the segment $\{x_i\}$, $i = 0, \ldots, n$, and by $C^k(x, n)$ the number of 0's included in all 0-segments of length $\geqslant k$ of $\{x_i\}$, $i = 0, \ldots, n$, plus the number of 1's included in all 1-segments of length $\geqslant k$.

We say that $x \in \tilde{\Sigma}_2$ satisfies the *condition* $P_0^k(x, n)$ ($P_1^k(x, n)$) if

$$C_0(x, n) - C^k(x, n) > 2/3(n+1) \quad (C_1(x, n) - C^k(x, n) > 2/3(n+1)).$$

For $k > 3$ we define the sets

$A(k) = \{x \in \Sigma_2$: there exists an increasing infinite sequence $\{n_i\}$ of natural numbers (depending on x) such that for even i's $P_0^k(x, n_i)$ holds and for odd i's $P_1^k(x, n_i)$ holds$\}$.

The sets $A(k)$ are residual in Σ_2. To prove this it is enough to show that $\Sigma_2 \setminus A(k)$ is of first category. In fact, let us put

$$B_i(N) = \{x \in \tilde{\Sigma}_2: \text{ for each } n > N, \ C_i(x, n) - C^k(x, n) \leqslant 2/3(n+1)\},$$

$i = 0, 1$. We can write

$$\tilde{\Sigma}_2 \setminus A(k) = \bigcup \{B_0(N) \cup B_1(N): \ N \in \boldsymbol{N}\}.$$

It is easy to see that $B_i(N)$ are closed, nowhere dense subsets of Σ_2.

Using the map h we can identify the set $A(k)$ with some residual subset of Λ. We denote the latter also by Λ and define the set

$$E(k) = \bigcup \{W^s(x): \ x \in A(k)\}.$$

The theorem about a stable and unstable manifold ([2], [3]) implies that $E(k)$ is residual in a neighbourhood of Λ.

Assuming that l is a natural number, we define the sets

$$\Lambda_0(l) = \{(\exp 2\pi i\varphi, w) \in S^1 \times D^2,\ 2^{-l} \leqslant \varphi \leqslant 2^{-1} - 2^{-l}\},$$

$$\Lambda_1(l) = \{(\exp 2\pi i\varphi, w) \in S^1 \times D^2,\ 2^{-1} + 2^{-l} \leqslant \varphi \leqslant 1 - 2^{-l}\}$$

(see Fig. 3). Let $f\colon S^1 \times D^2 \to [0, 1]$ be a continuous function such that

$$f \mid \Lambda_0(l) = 0 \quad \text{and} \quad f \mid \Lambda_1(l) = 1.$$

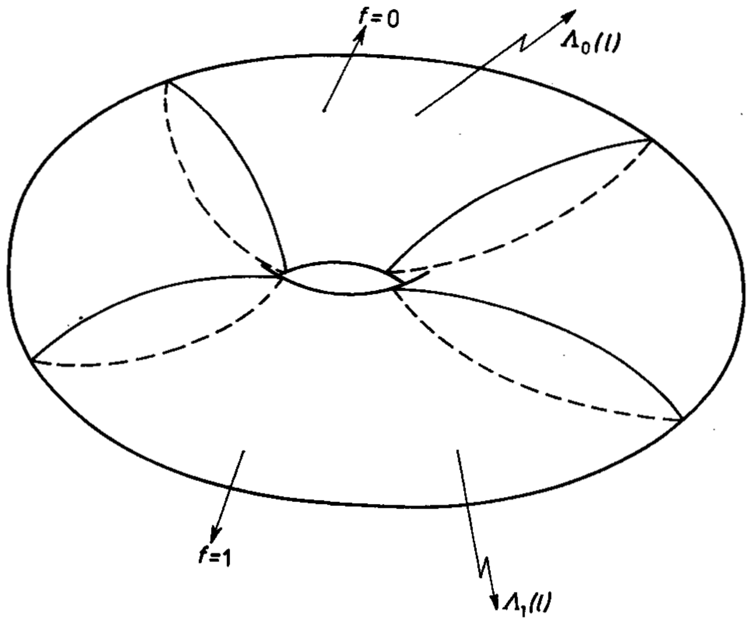

Fig. 3

We suppose that $l > 2k$ and show that

$$\lim_{N \to +\infty} N^{-1} \sum_{i=0}^{N-1} g(T^i(x))$$

does not exist for $x \in E(k)$ and for an arbitrary function g from some small C^0_ε-neighbourhood U of f. In fact, it is easy to show that this limit does not exist for any $x \in A(k)$.

For $x \in A(k)$ and even i we have

$$\begin{aligned}(n_i+1)^{-1} \sum_{j=0}^{n_i} g(T^j(x)) &\leqslant \varepsilon + (n_i+1)^{-1} \sum_{j=0}^{n_i} f(T^j(x)) \\ &\leqslant \varepsilon + (n_i+1)^{-1} \operatorname{card}\{n\colon 0 \leqslant n \leqslant n_i,\ T^n(x) \in S^1 \times D^2 \setminus \Lambda_0(l)\} \\ &\leqslant \varepsilon + (n_i+1)^{-1}[(n_i+1) - C_0(x, n_i) + C^k(x, n_i)] \leqslant 1/3 + \varepsilon.\end{aligned}$$

Analogously, for odd i's we obtain

$$(n_i+1)^{-1} \sum_{j=0}^{n_i} g(T^j(x)) \geqslant 2/3 - \varepsilon.$$

Hence for ε small enough and g, $\|g-f\|_{C^0} < \varepsilon$ and $x \in E(k)$, the time averages do not exist. In this way, for the mapping T we have an open set $U_T \subset C^0(\boldsymbol{R}^3, \boldsymbol{R})$ such that for $g \in U_T$ the average does not exist on a residual subset of $S^1 \times D^2$. Let R denote a small C^1-perturbation of T. The structural stability of T implies that there exists a conjugacy h, $h \circ R = T \circ h$, h being a homeomorphism from a small C^0-neighbourhood of identity [2]. If the time average does not exist for T, g and x, neither it exists for R, $g \circ h$, $h^{-1}(x)$. Thus we can get the sets

$$E_R = h^{-1}(E(k)) \quad \text{and} \quad U_R = \{\tilde{g}\colon \tilde{g} = g \circ h, g \in U_T\}$$

such that the time average does not exist for $x \in E_R$, R and $g \in U_R$. If the C^1-perturbation R of T is sufficiently small, then U_R contains an open neighbourhood W of f. Hence for R from a small neighbourhood of T and $g \in W$ a residual set A_R exists such that for R, g, $x \in A_R$ the time average does not exist.

It was noted by R. Mañé that the crucial point in our example is that the unstable manifold of a periodic orbit is dense in the attractor and that similar examples can be constructed using an arbitrary hyperbolic attractor.

An analogous result for flows can be obtained by using the suspension construction for T. We shall briefly describe that construction. Let T be a diffeomorphism of a manifold M. Denote by $\tilde{M} = M \times [0, 1]$ and $\hat{M}$ the manifolds obtained by identification of the points $(x, 0)$ and $(T(x), 1)$, respectively. This identification transforms the vector field $\partial/\partial t$ tangent to $\{x\} \times [0, 1]$ to a smooth vector field $\hat{X}$ for which $M_0 = M \times \{0\}$ and T are the Poincaré section and the Poincaré map, respectively. Let T and f be the same as in Example 3, $\hat{X}$ be the vector field obtained by the suspension of T, and $\{\hat{\varphi}_t\}$ the flow generated by $\hat{X}$. Define the function

$$F\colon M \times [0, 1-\varepsilon] \to [0, 1]$$

by $F(x, t) = f(x)$ and continuously extend F to $\hat{F}$ defined on the whole $\hat{M}$. The vector field $\hat{X}$ is structurally stable, and so for each small C^1-perturbation $\bar{X}$ of $\hat{X}$ there exist a homeomorphism h which maps trajectories of $\{\hat{\varphi}_t\}$ onto trajectories of $\{\bar{\varphi}_t\}$, where $\{\bar{\varphi}_t\}$ is a flow generated by $\bar{X}$.

The nonexistence of the limit

$$\lim_{T \to +\infty} T^{-1} \int_0^T f\big(\hat{\varphi}_t(h(x))\big)\, dt$$

implies the nonexistence of the limit

$$\lim_{T \to +\infty} T^{-1} \int_0^T f\big(h(\bar{\varphi}_t(x))\big)\, dt.$$

This follows from the observation that the term $\|\bar{X}(h(x))\| - \|\hat{X}(x)\|$ is small, and so the speed of the trajectory $\bar{\varphi}_t(h(x))$ is almost the same as that of $h(\hat{\varphi}_t(x))$.

Thus we have shown that there are an open set V of flows and an open set of continuous functions U such that for $\{\varphi_t\} \in V$ and $g \in U$ the time average does not exist on a residual subset of an open set.

In the last phrases we make several remarks about C^0-perturbations. To be concrete, let us consider C^0-perturbations of the solenoid T. It is easy to show that there are an open C^0-neighbourhood U of T and sets A, B dense in U such that for $S \in A$ and for every continuous function the time average exists on an open dense subset of some neighbourhood of Λ, while for $S \in B$ and every f from an open subset of continuous functions the time average does not exist for points from an open dense subset of some neighbourhood of Λ.

In fact, we can perturb T near a fixed point (0, 0) in such a way that the fixed point is an attractor. The stable manifold of the fixed point is dense, and so the domain of attraction of the fixed point of the perturbed system is an open dense subset of a neighbourhood of Λ. The set A is included among the perturbations of this type.

To construct B it is enough to observe that a small C^0-perturbation of T can create in the vicinity of the fixed point the attractor which was described in Example 2.

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Received on 1986.05.13