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ON A MACHINE SEQUENCING PROBLEM (I)

1. Introduction. In many branches of industry (metallurgical, machine construction) the production process is characterized by the flow of elements (in the technological sequence). These elements are processed on successive machines. The processing times of individual operations are different for different elements and machines. There arises, therefore, the problem of determining such a sequence of elements as to complete all jobs in minimal time. In this paper are presented three variants of the problem, which can be solved by solving the travelling salesman problem.

The considered problem is a particular case of the problem, in which the sequence of elements on the machines can be different for different elements; also many other variants of the objective function can be introduced.

The problem in general form with two objective functions and a branch-and-bound algorithm of solution was presented, for example, in [1].

2. Problem. Consider a given set of m machines $M_1, M_2, \ldots, M_m$ and a set of elements $P_1, P_2, \ldots, P_n$. Each element should be successively processed on machines $M_1, M_2, \ldots, M_m$, thus it can be defined by an ordered sequence of m numbers

$$P_i = \{t_{i1}, t_{i2}, \ldots, t_{im}\} \quad (i = 1, 2, \ldots, n),$$

where t_{ij} is the processing time for element i on machine j ($j = 1, 2, \ldots, m$). The objective is to assign such a processing sequence of elements

$$\{P_{i_1}, P_{i_2}, \ldots, P_{i_n}\},$$

where $\{i_1, i_2, \ldots, i_n\}$ is a permutation of the sequence $\{1, 2, \ldots, n\}$, as to minimize the time required to complete all jobs.

Denote by s_{ij} the process-start time of element i on machine j.

In this section we are intersted in determining the solution of the above-mentioned problem with the following additional constraints:

(a) Every element can be processed on not more than one machine, and any machine can perform not more than one element at any time, therefore,

(1) $$s_{lj} \geqslant s_{kj}+t_{kj} \quad \text{or} \quad s_{kj} \geqslant s_{lj}+t_{lj}$$
$$(j = 1, 2, \ldots, m;\ l, k = 1, 2, \ldots, n;\ l \neq k).$$

(b) The processing of each element must not be interrupted.

(c) The processing sequence of elements $P_1, P_2, \ldots, P_n$ is the same on all machines, i. e. constraints (1) should be of the form

$$s_{lj} \geqslant s_{kj}+t_{kj}\ (j = 1, 2, \ldots, m) \quad \text{or} \quad s_{kj} \geqslant s_{lj}+t_{lj}\ (j = 1, 2, \ldots, m),$$

where $l, k = 1, 2, \ldots, n$ and $l \neq k$.

(d) For each element, the process-end time on any machine is equal to the process-start time on the next machine, i. e.

$$s_{ij} = s_{i,j-1}+t_{i,j-1} \quad (i = 1, 2, \ldots, n; j = 2, \ldots, m).$$

The problem with constraints (a)-(d) appears in many production processes (no bins between successive machines, thermal processing). In Fig. 1 is shown an example of processing 4 elements on 3 machines in the sequence $\{1, 2, 4, 3\}$.

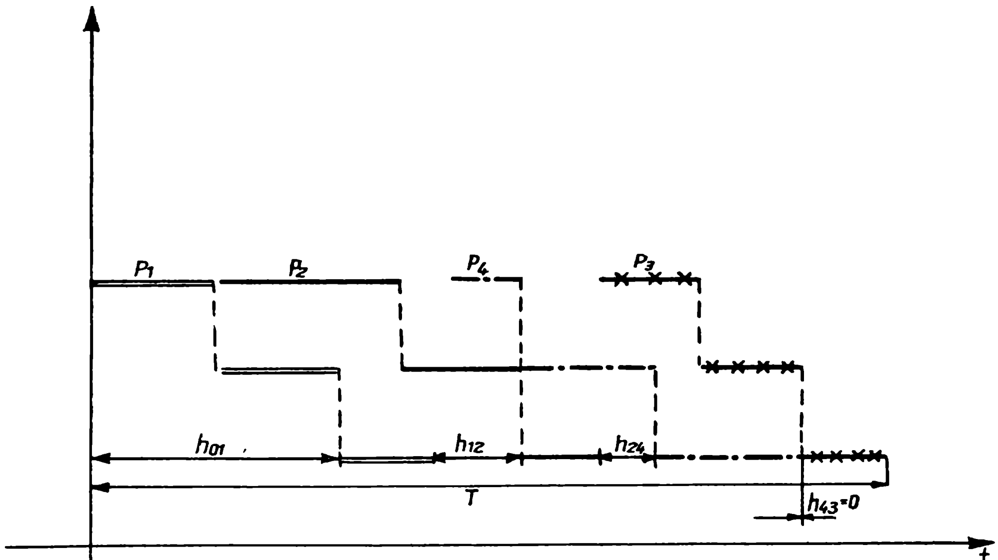

Fig. 1

For this problem, the time required to complete all jobs, which is the criterion of choice of the optimal sequence, is equal to the sum of the processing times and breaks on machine m,

(2) $$T = \sum_{l=1}^{n} t_{k_l m} + \sum_{l=1}^{n-1} h_{k_l k_{l+1}} + h_{0k_1},$$

where h_{ij} is the break on machine m between the i-th and j-th elements $(i, j = 1, 2, \ldots, n)$, h_{0i} — the break (also on machine m) between the process-start time of the element i on the first machine and on the m-th machine, and $\{k_1, k_2, \ldots, k_n\}$ — the permutation of the sequence $\{1, 2, \ldots \ldots, n\}$.

The problem consists in minimizing T over $\{k_1, k_2, \ldots, k_n\}$. Because the value of the first component of the sum on the right-hand side of equation (2) does not depend upon the sequence of elements, the minimization of T is equivalent to the minimization of all breaks on machine m:

$$T_d = \sum_{l=1}^{n-1} h_{k_l k_{l+1}} + h_{0k_1}.$$

Now, let us determine h_{ij}. For the element j, which is the successor of the element i, there are, by constraints (c), satisfied the inequalities

$$(3) \qquad s_{jk} \geqslant s_{ik} + t_{ik} \quad \text{for } k = 1, 2, \ldots, m$$

and, by constraints (e), the equations

$$s_{ik} = s_{i,k-1} + t_{i,k-1} \quad \text{and} \quad s_{jk} = s_{j,k-1} + t_{j,k-1} \quad \text{for } k = 2, 3, \ldots, m.$$

For the given sequence of elements (not only for the optimal sequence) at least one of inequalities (3) is an equality because, otherwise, the time required to complete all jobs could be reduced by the value

$$\min_{1 \leqslant k \leqslant m} \{s_{jk} - (s_{ik} + t_{ik})\}.$$

Hence, for some integer p which satisfies $1 \leqslant p \leqslant m$, there is fulfilled the equation

$$s_{jp} = s_{ip} + t_{ip}.$$

From the definition of h_{ij} we have

$$h_{ij} = s_{jm} - s_{im} + t_{im},$$

and after repeated application of (3) we obtain

$$s_{jm} = s_{jp} + \sum_{k=p}^{m-1} t_{jk}, \qquad s_{im} = s_{jp} + \sum_{k=p+1}^{m-1} t_{ik},$$

hence

$$(4) \qquad h_{ij} = \sum_{k=p}^{m-1} t_{jk} - \sum_{k=p+1}^{m} t_{ik} = \sum_{k=p}^{m} (t_{jk} - t_{ik}) + (t_{ip} - t_{jm}).$$

In numerical computations it is better (with regard to the number of operations) to take $p = \max\{l: s_{jl} = s_{il}+t_{il}\}$. From the definition of h_{0i} and by constraints (b)-(d) we have

$$h_{0i} = \sum_{k=1}^{m-1} t_{ik}. \tag{5}$$

From (4) it is evident that the value of the break h_{ij} depends upon P_i and P_j only.

Let us write the elements of the matrix $D = (d_{ij})$ $(i, j = 0, 1, \ldots, n)$ as follows:

$$d_{ij} = \begin{cases} h_{ij} & \text{for } i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, n;\ i \neq j, \\ 0 & \text{for } i = 1, 2, \ldots, n;\ j = 0, \\ \infty & \text{for } i, j = 0, 1, \ldots, n;\ i = j. \end{cases} \tag{6}$$

The following theorem gives the solution of the considered problem:

THEOREM 1. *The permutation $\{l_1, l_2, \ldots, l_n\}$ minimizes T_d if and only if $\{0, l_1, l_2, \ldots, l_n, 0\}$ is the minimal Hamiltonian cycle of the matrix D.*

Proof. Let W denote the weight of the Hamiltonian cycle $\{0, k_1, k_2, \ldots, k_n, 0\}$ of the matix D. From (6) it follows that

$$W = d_{0k_1} + \sum_{l=1}^{n-1} d_{k_lk_{l+1}} + d_{k_n0} = h_{0k_1} + \sum_{l=1}^{n-1} h_{k_lk_{l+1}}.$$

Therefore, the value of the weight of the Hamiltonian cycle $\{0, k_1, \ldots, k_n, 0\}$ is equal to the value T_d for the permutation $\{k_1, k_2, \ldots, k_n\}$. Simultaneously, there exists the one-to-one mapping

$$\varphi\colon \{0, k_1, k_2, \ldots, k_n, 0\} \leftrightarrow \{k_1, k_2, \ldots, k_n\}$$

of the elements of the set of the Hamiltonian cycles of the matrix D to the elements of the set of the permutations of sequence $\{1, 2, \ldots, n\}$. Therefore,

$$\min_{\{0,k_1,\ldots,k_n,0\}} W = W(\{0, l_1, \ldots, l_n, 0\}) = \min_{\{j_1,\ldots,j_n\}} T_d = T_d(\{l_1, l_2, \ldots, l_n\}).$$

3. Now, assume that N identical n-element groups $(N < \infty)$ are to be processed. Constraints (a)-(d) are still holding. The objective is to assign such a processing sequence of elements in all groups as to minimize the time required to complete all jobs of the N groups. Assuming identical sequences of elements in each group $\{k_1, k_2, \ldots, k_n\}$, the expression for the total time of the breaks on machine m is, analogically to the problem of the section 2,

$$V_d = h_{0k_1}+N\sum_{l=1}^{n} h_{k_l k_{l+1}} - h_{k_n k_1}, \tag{7}$$

or, after change of summation,

$$V_d = h_{0k_1}+\sum_{l=1}^{n-1} h_{k_l k_{l+1}} + (N-1)\sum_{l=1}^{n} h_{k_l k_{l+1}}, \tag{8}$$

where $k_{n+1} = k_1$.

The problem of minimization of V_d over $\{k_1, k_2, \ldots, k_n\}$ is left open.

Let V_d^* denote the minimal value of V_d. Notice that the sum $\sum h_{k_l k_{l+1}}$ in the right-hand side of (7) and of (8) is the weight of the Hamiltonian cycle of the matrix $H = (h_{ij})$ $(i, j = 1, 2, \ldots, n)$. Form (7) of V_d suggests the following solution:

(i) Determine the minimal Hamiltonian cycle of the matrix H; let us denote it by $\{i_1, i_2, \ldots, i_n, i_1\}$.

(ii) Take $\{i_l, i_{l+1}, \ldots, i_n, i_1, \ldots, i_{l-1}\}$ as the sequence of element in each of the N groups, where i_l satisfies the condition

$$\min_{1\leqslant k\leqslant n} (h_{0i_k} - h_{i_{k-1}i_k}) = h_{0i_l} - h_{i_{l-1}i_l}, \qquad \text{where } i_0 = i_n.$$

Let

$$V_d(\{i_l, i_{l+1}, \ldots, i_n, i_1, \ldots, i_{l-1}\}) = V_d^1.$$

Obviously,

$$V_d^1 \geqslant V_d^*.$$

However, suggested by form (8) of V_d, we weaken the assumption of an identical sequence of the elements in all groups, and assume an identical sequence of the elements only in $N-1$ groups. Let P denote the set of permutations of sequence $\{1, 2, \ldots, n\}$, let V_1 be the sum of the first and the second components of (8) and let V_2 be the third component of (8). Assuming an identical sequence of elements in all N groups, we have

$$V_d^* = \min_{p\epsilon P} V_d(p) \tag{9}$$

and, for the weakened assumption, it is necessary to determine

$$V_d^2 = \min_{q\epsilon P} V_1(q) + \min_{r\epsilon P} V_2(r). \tag{10}$$

Obviously,

$$V_d^* \geqslant V_d^2,$$

which results from the form of the right-hand sides of (9) and (10). The determination of the first minimum of (10) is the problem from section 2; let

$$V_1(\{l_1, \ldots, l_n\}) = \min_{q \in P} V_1(q).$$

But

$$V_2(\{i_1, \ldots, i_n\}) = \min_{r \in P} V_2(r),$$

where $\{i_1, \ldots, i_n, i_1\}$ is the minimal Hamiltonian cycle of the matrix H. Joining these solutions, the sequence of elements is as follows:

in the first group $\{l_1, l_2, \ldots, l_n\}$, and in the remaining $N-1$ groups $\{i_j, i_{j+1}, \ldots, i_n, i_1, \ldots, i_{j-1}\}$, where $i_{j-1} = l_n$.

4. From the theoretical point of view, the problem of section 3 can be considered for $N = \infty$. Assuming an identical sequence of elements in all groups, it is seen that, from start of the processing of the first element of this infinite sequence of groups, the breaks on the last machine will be of the form of an infinitely repeating Hamiltonian cycle. Therefore, the instantaneous solution of this problem the permutation $\{i_1, i_2, \ldots, i_n\}$, where

(i) $\{i_1, \ldots, i_n, i_1\}$ is the minimal Hamiltonian cycle of the matrix H,
(ii) $\min_{1 \leqslant j \leqslant n} h_{0j} = h_{0i_1}$.

Reference

[1] H. H. Greenberg, *A branch-bound solution to the general scheduling problem*, Opns, Res. 16 (1968), p. 353-361.

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O PEWNYM ZAGADNIENIU KOLEJNOŚCIOWYM (I)

STRESZCZENIE

W pracy przedstawiono zagadnienie kolejnościowe, odpowiadające procesowi obróbki cieplnej lub obróbki bez składowisk międzyoperacyjnych. W rozdziale 2 zbudowano model matematyczny tego zagadnienia i udowodniono twierdzenie, dające

rozwiązanie problemu kolejności obróbki n-elementowej partii detali na m maszynach. W rozdziałach 3 i 4 podano rozwiązania suboptymalne rozszerzonego zagadnienia obróbki N jednakowych partii n-elementowych dla $N < \infty$ i $N = \infty$. Rozwiązania wyznaczone w rozdziałach 2-4 sprowadzają postawione zagadnienia do problemu wyznaczenia w zadanej macierzy minimalnego cyklu Hamiltona.