ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIII, 4 (1973)

G. SIEGEL (Leipzig)

THE STATIONARY WAITING TIME AND OTHER VARIABLES IN SINGLE-SERVER QUEUES WITH SPECIALITIES AT THE BEGINNING OF A BUSY PERIOD

1. Introduction. We consider two generalizations of the queueing system $GI/G/1$. They are due to Finch and Yeo (see [5] and [11], respectively). The special structure of these single-server systems comes out only if the n-th customer finds the systems vacant. Applying the methods and results from [10], we shall find some important relations between both systems. They make it possible to reduce Yeo's model to Finch's model, although, on the other hand, Finch's model is a special case of Yeo's model. That is why the known results of both these models are very similar (see the results in [2] and [5] in comparison with [11] or the results in [9] compared to [6]). Therefore, it is not necessary to consider both models separately but, for many purposes, we can confine our attention to Finch's model.

One of the most important results of this paper is that the stationary waiting time distribution of Yeo's model can be represented by a convolution of the stationary waiting time distribution W of Lindley's model $GI/G/1$ and another distribution G, provided that we consider only the conditional distribution of the strictly positive waiting time. For G, we give an integral equation.

We compare not only the waiting time distributions of both models but also the idle times, the queue lengths and the busy periods of these models.

In particular, considering Lindley's model as a special case of Yeo's model, we obtain some new interesting results about the factors of the well-known factorization (1.7) (see theorem 4.2).

Let us consider the sequences $\{X_i\}$, $i = 1, 2, \ldots$, and $\{\delta_j\}$, $j = 0, 1, \ldots$, of mutually independent, identically distributed random variables (r.v.) with common distribution functions (d.f.) $K(x)$ and $D(x)$, respectively. Let us suppose that $\{X_i\}$ and $\{\delta_i\}$ are independent.

Further, we consider the sequence of r.v.'s

$$(1.1)\qquad \hat{w}_0 = 0, \qquad \hat{w}_n = \begin{cases} \hat{w}_{n-1}+X_n & \text{if } \hat{w}_{n-1}+X_n > 0,\ \hat{w}_{n-1} > 0, \\ \delta_n & \text{if } \delta_n > 0,\ \hat{w}_{n-1} = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$n = 1, 2, \ldots$ Yeo [11] states without proof that, for $|E\delta_0| < \infty$ and $-\infty < EX_1 < 0$, a stationary d.f. $\hat{W}(x) = \lim_{n\to\infty} P\{\hat{w}_n \leqslant x\}$ exists (1) and that $\hat{W}(x)$ is the solution of the following integral equation:

$$(1.2)\qquad \hat{W}(x) = \begin{cases} K*\hat{W}(x) + \big(D(x)-K(x)\big)\hat{W}(0), & x \geqslant 0 \ (^2), \\ 0, & x < 0. \end{cases}$$

If, for instance, $D(x) \equiv K(x)$, then we obtain from (1.2) and (1.1) Lindley's well-known model (see [8]).

We now study the following queueing system. Let t_n denote the moment of arrival after $t_0 = 0$ of the n-th customer at the single server. The interarrival times $a_i = t_i - t_{i-1}$, $i = 1, 2, \ldots$, are mutually independent, identically distributed with the d.f. $A(x)$. To every customer there correspond two service times β_n and $\tilde{\beta}_n$ with d.f.'s $B(x)$ and $\tilde{B}(x)$, respectively. If the n-th customer finds the system vacant, then his service time is $\tilde{\beta}_n$, otherwise β_n. If $X_n = \beta_{n-1} - a_n$ and $\delta_n = \tilde{\beta}_{n-1} - a_n$, then $\hat{w}_n$ is the waiting time of the n-th customer. If, for instance, $B(x) \equiv \tilde{B}(x)$, we obtain the model $GI/G/1$.

Yeo [11] gives the characteristic function (c.f.) of $\hat{W}(x)$ if the arrivals of the customers form a Poisson process or if $A(x)$ is an Erlang distribution. Further, if the c.f. of $A(x)$ or the c.f.'s of both $B(x)$ and $\tilde{B}(x)$ are rational, then the c.f. of $\hat{W}(x)$ was obtained first by Haeske in [6].

Yeo's model is immediately connected with Finch's model which was studied first in [5]. Its special feature, too, comes out only if the n-th customer finds the system vacant; then he is not served immediately but only after a random delay γ_n (warming up time). Let w_n' denote the waiting time of the n-th customer which is defined by

$$(1.3)\qquad w_0' = \gamma_0, \qquad w_n' = \begin{cases} w_{n-1}'+X_n & \text{if } w_{n-1}'+X_n > 0, \\ \gamma_n & \text{if } w_{n-1}'+X_n \leqslant 0, \end{cases}$$

---

(1) We shall generalize this result in theorem 2.1.

(2) The convolution of the functions F_1 and F_2 is defined by

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)\,dF_2(y).$$

$n = 1, 2, \ldots$, where $\gamma_0, \gamma_1, \ldots$, is a sequence of non-negative, mutually independent and identically distributed r.v.'s. Let us suppose that $\{\gamma_i\}$ and $\{X_i\}$ are independent. We now give some results from [10], which we shall need later.

We say $W'(x)$ is *stationary* if, for $\mathrm{P}\{w_0' \leqslant x\} = W'(x)$, the r.v.'s $w_1', w_2', \ldots$ are identically distributed with the common d.f. $W'(x)$. In this case, however, the limit distribution of $\{w_n'\}$, $n = 0, 1, \ldots$, does not necessarily exist for arbitrary $W_0'(x) = \mathrm{P}\{w_0' \leqslant x\}$ (see remark 2.1). This is an effect which does not appear in Lindley's model.

If we put $\tau_1' = \min\{k\colon w_{k-1}' + X_k \leqslant 0, k \geqslant 1\}$, then we have

LEMMA 1.1. *A stationary d.f.* $W'(x)$ *exists if and only if* $\mathrm{E}\tau_1' < \infty$.

(a) *For* $\mathrm{E}\gamma_0 < \infty$ *and* $-\infty \leqslant \mathrm{E}X_1 < 0$, *we have* $\mathrm{E}\tau_1' < \infty$. *For* $\mathrm{E}\gamma_0 = \infty$ *and* $-\infty < \mathrm{E}X_1 < 0$ *or* $0 \leqslant \mathrm{E}X_1 \leqslant \infty$, *we have* $\mathrm{E}\tau_1' = \infty$.

(b) *The stationary d.f.* $W'(x)$ *is the unique d.f. which solves the integral equation*

$$W'(x) = \begin{cases} K * W'(x) - \big(1 - C(x)\big) K * W'(0), & x \geqslant 0, \\ 0, & x < 0. \end{cases} \tag{1.4}$$

(c) *The d.f.* $W'(x) = \lim_{n\to\infty} \mathrm{P}\{w_n' \leqslant x\}$, *which is independent of* w_0', *exists if and only if*

$$\lim_{n\to\infty} \mathrm{P}\{w_{n-1}' + X_n \leqslant 0\} = \beta > 0. \tag{1.5}$$

In this case we obtain $\beta = 1/\mathrm{E}\tau_1'$. *If* $\mathrm{E}\tau_1' < \infty$, *then the following conditions are sufficient for* (1.5): X_1 *non-lattice and* $K(x) < 1$ *for every* $x < 0$ *or* $C(x) > 0$ *for every* $x > 0$.

LEMMA 1.2. *Let us assume that the stationary d.f.* $W'(x)$ *exists. Putting* $S_i = X_1 + X_2 + \ldots + X_i$ *and* $\Psi_0(x) = C(x)$, $\Psi_n(x) = \mathrm{P}\{S_i + \gamma_0 > 0$, $i = 1, 2, \ldots, n;\ S_n + \gamma_0 \leqslant x\}$, $W'(x)$ *admits the representation*

$$W'(x) = (1/\mathrm{E}\tau_1') \sum_{i=0}^{\infty} \Psi_i(x). \tag{1.6}$$

For $\mathrm{E}\tau_1' = \infty$, *we have* $W'(x) \equiv 0$ *in* (1.6).

In [2], the c.f. of $W'(x)$ is found if $A(x)$ is an Erlang distribution. Rossberg [9] studied the c.f. of $W'(x)$ if $A(x)$ or both $B(x)$ and $C(x)$ have rational c.f.'s. Further, a general solution for $W'(x)$ was given in [7] if $(1-C)/(1-K)$ is of bounded variation.

Let $\varepsilon_0(x)$ be the d.f. concentrated at the origin. Then the d.f. $K(x)$ can be represented by the well-known factorization

$$(\varepsilon_0 - K) = (\varepsilon_0 - U) * (\varepsilon_0 - H), \tag{1.7}$$

where $U(x)$ and $H(x)$ are distributions concentrated on $(-\infty, 0]$ and $(0, \infty)$, respectively (cf. [4], XII, section 3). In particular, in the model $GI/G/1$, U is the d.f. of the negative idle time.

Further, results from [10] are contained in the following lemma:

LEMMA 1.3. *Suppose the stationary d.f. $W'(x)$ exists. If $W(x)$ is the stationary d.f. in Lindley's model $GI/G/1$, then the d.f. $W'(x)$ has the representation*

$$W'(x) = W * G(x), \qquad -\infty < x < \infty, \tag{1.8}$$

where $G(x)$ is the d.f. of a non-negative r.v. and the unique d.f. which is solution of the integral equation

$$G(x) = \begin{cases} G * U(x) - (\eta - C(x))\delta, & x \geq 0, \\ 0, & x < 0,\ \eta = 1,\ \delta = G * U(0). \end{cases} \tag{1.9}$$

The solution of (1.9) *can be represented by*

$$G(x) = \delta\Big(C(x) + \Big(\sum_{i=1}^{\infty} U^{i*}\Big) * C(I)\Big), \qquad I = (0, x],\ x \geq 0. \tag{1.10}$$

δ is uniquely determined by $G(\infty) = 1$. If the stationary d.f. W' does not exist, then the function (1.10) *is also the solution of* (1.9) *for arbitrary δ for a certain η $(0 < \eta \leq 1)$. For $U(0) < 1$, we have*

$$\eta = \mathrm{P}\{\min\{0, S_1, S_2, \ldots\} + \gamma_0 \leq 0\} \leq U(0).$$

2. The waiting time distributions. In this section we study the relations between the waiting times in Yeo's and in Finch's models. Clearly, the model of Yeo is a generalization of Finch's model. In fact, for $\delta_n = \tilde{\beta}_{n-1} - a_n$ and $\tilde{\beta}_n = \beta_n + \gamma_n$, we have $\hat{w}_n = w'_n$ for $\hat{w}_n > 0$ and $\gamma_n = w'_n$ for $\hat{w}_n = 0$. As the following considerations show, we can also reduce Yeo's model to Finch's model if we do not consider the sequence $\{\hat{w}_n\}$ but a random subsequence $\{\hat{w}_{r_n}\} = \{w'_n\}$ of the sequence $\{\hat{w}_n\}$. In our case we obtain $\{w'_n\}$ if we cancel those elements of $\{\hat{w}_n\}$ for which $\hat{w}_n = 0$. Therefore, we investigate k_1 with $\hat{w}_0 = 0$, $\hat{w}_1 = 0, \ldots, \hat{w}_{k_1-1} = 0$, $\hat{w}_{k_1} > 0$ and the index τ'_1 with $\hat{w}_{k_1-1} = 0$, $\hat{w}_{k_1} > 0, \ldots, \hat{w}_{k_1+\tau'_1-1} > 0$, $\hat{w}_{k_1+\tau'_1} = 0$.

Continuing this construction, we obtain the sequences $\{\tau'_i\}$, $i = 1, 2, \ldots$ and $\{k_j\}$, $j = 1, 2, \ldots$, where $\{\tau'_i\}$ and $\{k_j\}$ are sequences of mutually independent and identically distributed r.v.'s, respectively. Moreover, both sequences are mutually independent.

Putting $T'_0 = 0$, $T'_i = \tau'_1 + \tau'_2 + \ldots + \tau'_i$ and $K_0 = 0$, $K_i = k_1 + k_2 + \ldots + k_i$, we can write

$$w'_{T'_i+j} = \hat{w}_{K_{i+1}+T'_i+j}, \qquad j = 0, 1, \ldots, \tau'_{i+1}-1,\ i = 0, 1, 2, \ldots \tag{2.1}$$

We call the sequence $\{w_n'\}$ the *model imbedded in* $\{\hat{w}_n\}$. If $D(0) < 1$, then we have from (1.1) (by induction) a geometric distribution for k_1, i.e. $P\{k_1 = i\} = (1-p)p^{i-1}$, $i = 1, 2, \ldots$, where $p = P\{\delta_0 \leqslant 0\} = D(0)$. From (1.1) and (2.1) we conclude that

$$(2.2) \qquad w_0' = \tilde{\delta}_0, \quad w_n' = \begin{cases} w_{n-1}' + X_n & \text{if } w_{n-1}' + X_n > 0, \\ \tilde{\delta}_n & \text{if } w_{n-1}' + X_n \leqslant 0, \end{cases}$$

$n = 1, 2, \ldots$, where $\{\tilde{\delta}_n\}$ is a random subsequence of the sequence $\{\delta_n\}$ with the d.f.

$$(2.3) \qquad D_+(x) = P\{\delta_n \leqslant x / \delta_n > 0\} = \frac{D(x) - D(0)}{1 - D(0)}, \quad D(0) < 1, \ x \geqslant 0.$$

THEOREM 2.1. *Suppose* $0 < D(0) < 1$.

(a) *The sequence of the events* $\{\hat{w}_n = 0\}$, $n = 0, 1, \ldots$, *is not periodic. For* $-\infty \leqslant EX_1 < 0$ *and* $E\delta_0^+ < \infty$ (³), *the sequence* $\{\hat{w}_n = 0\}$, $n = 0, 1, \ldots$, *is positive persistent with* $\lim_{n\to\infty} P\{\hat{w}_n = 0\} = a > 0$, *where* a *is independent of* $\hat{w}_0$. *For* $-\infty < EX_1 < 0$ *and* $E\delta_0^+ = \infty$ *or* $EX_1 = 0$, *the sequence* $\{\hat{w}_n = 0\}$, $n = 0, 1, \ldots$, *is null-persistent with* $a = 0$. *For* $0 < EX_1 \leqslant \infty$, *the sequence* $\{\hat{w}_n = 0\}$, $n = 0, 1, \ldots$, *is transient with* $a = 0$.

(b) *For* $a > 0$, *the stationary d.f.* $\hat{W}(x) = \lim_{n\to\infty} P\{\hat{w}_n \leqslant x\}$ *exists, is independent of* $\hat{w}_0$ *and is the unique d.f. which is a solution of equation* (1.2). *Further,* $\hat{W}(x)$ *can be represented by*

$$(2.4) \qquad \hat{W}(x) = a\varepsilon_0(x) + (1-a)W'(x),$$

where $W'(x)$ *is the stationary d.f. of the imbedded model* $\{w_n'\}$ *and* a *admits the representation* $a = \beta/[\beta + 1 - D(0)]$, *where* $\beta = K * W'(0)$.

(c) *If* $a = 0$, *then* $\lim_{n\to\infty} P\{\hat{w}_n \leqslant x\} = 0$ *for all* x.

Remark 2.1. Clearly, if $D(0) = 1$, then we obtain from (1.1) $\hat{W}(x) \equiv \varepsilon_0(x)$. If $D(0) = 0$, then $P\{k_1 = 1\} = 0$. The following very simple example proves that theorem 2.1 is not true in this case:

Let us consider the sequence $\{X_i\}$ with

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } q, \ p + q = 1. \end{cases}$$

(³) As usually, we put

$$\delta_0^+ = \begin{cases} \delta_0 & \text{if } \delta_0 > 0, \\ 0 & \text{if } \delta_0 \leqslant 0. \end{cases}$$

Let $EX_1 = p-q < 0$ and $\delta_0 \in \{1, 3, 5, \ldots\}$ with $E\delta_0 < \infty$. Then it is easily verified that $\hat{w}_{2n} \in \{0, 2, 4, \ldots\}$ and $\hat{w}_{2n+1} \in \{1, 3, 5, \ldots\}$, $n = 0, 1, \ldots$ Therefore, the sequence $\hat{W}_n(x) = P\{\hat{w}_n \leqslant x\}$, $n = 0, 1, \ldots$, does not converge. In our example, the limit d.f. $\hat{W}(x)$ does not exist. There exists, however, the stationary d.f. $\hat{W}(x)$, which is defined by (2.4). Using the sufficient conditions in lemma 1.1, part (c), we see that theorem 2.1 is also true for $D(0) = 0$.

Proof of theorem 2.1. (a) Suppose $\hat{w}_0 = 0$. Then $\{\hat{w}_n = 0\}$, $n = 0, 1, \ldots$, is a sequence of recurrent events in the sense of Feller [3], XII, section 4. We denote by $\hat{\tau}_1$ the distance between the first and the second zeros of the sequence $\{\hat{w}_n\}$. Since $P\{\hat{\tau}_1 = 1\} = D(0) > 0$, $\hat{\tau}_1$ is not periodic. Therefore, we have

$$\lim_{n\to\infty} P\{\hat{w}_n = 0\} = 1/E\hat{\tau}_1 = \alpha \geqslant 0. \tag{2.5}$$

If $\hat{w}_1 > 0$, then $\hat{\tau}_1 = \tau_1' + 1$. Otherwise, we have $\hat{\tau}_1 = 1$. Hence we find

$$E\hat{\tau}_1 = P\{\hat{\tau}_1 = 1\} + \sum_{i=2}^{\infty} iP\{\hat{\tau}_1 = i/\hat{\tau}_1 > 1\}P\{\hat{\tau}_1 > 1\}$$

and

$$P\{\hat{\tau}_1 = i/\hat{\tau}_1 > 1\} = P\{\tau_1' = i-1\}.$$

Thus we obtain

$$E\hat{\tau}_1 = \big(1-D(0)\big)E\tau_1' + 1 \tag{2.6}$$

in the sense that if one hand-side exists, so does the other. By the definitions of the sequences $\{\tau_i'\}$ and $\{w_i'\}$, $i = 1, 2, \ldots$, we get $\tau_1' = \min\{k : w_{k-1}' + X_k \leqslant 0,\ k \geqslant 1\}$. If $E\tau_1' < \infty$, then from (2.5) and (2.6) we have $\alpha > 0$. Analogously, from $E\tau_1' = \infty$ we have $\alpha = 0$. Using lemma 1.1, part (a), the proof of (a) is complete.

(b) Suppose $\hat{w}_0 = 0$. We now transform the last term in the equation

$$\hat{W}_n(x) = \hat{W}_n(0)\varepsilon_0(x) + P\{\hat{w}_n \leqslant x,\ \hat{w}_n > 0\}. \tag{2.7}$$

Applying the conditions $\{\hat{w}_{i-1} = 0,\ \delta_i > 0,\ \hat{w}_j > 0,\ j = i+1, i+2, \ldots, n\}$, $i = 1, 2, \ldots, n-1$, and $\{\hat{w}_{n-1} = 0,\ \delta_n > 0\}$, we obtain

$$P\{\hat{w}_n \leqslant x,\ \hat{w}_n > 0\} = \sum_{i=1}^{n} P\{0 < \hat{w}_n \leqslant x,\ \hat{w}_j > 0,\ j = i, i+1, \ldots, n-1,$$
$$\hat{w}_{i-1} = 0,\ \delta_i > 0\}P\{\hat{w}_{i-1} = 0,\ \delta_i > 0\}.$$

Putting $S_n = X_1+X_2+\ldots+X_n$, $n = 1, 2, \ldots$, we have from (1.1)

$$(2.8)\quad \mathrm{P}\{\hat{w}_n \leqslant x,\ \hat{w}_n > 0\} = \Big(D_+(x)+\sum_{i=1}^{n-1} \mathrm{P}\{\delta_i+S_j-S_i > 0,\ j = i+1, i+2, \ldots, n-1,\ 0 < \delta_i+S_n-S_i \leqslant x/\delta_i > 0\}\Big)\mathrm{P}\{\hat{w}_{i-1} = 0,\ \delta_i > 0\}.$$

As $\mathrm{P}\{\hat{w}_{n-1} = 0, \delta_n > 0\} = \mathrm{P}\{\hat{w}_{n-1} = 0\}\big(1-D(0)\big)$, (2.5) and (2.6) yield

$$(2.9)\quad \lim_{n\to\infty} \mathrm{P}\{\hat{w}_{n-1} = 0,\ \delta_n > 0\} = (1/\mathrm{E}\tau_1')\,(1-1/\mathrm{E}\hat{\tau}_1).$$

Obviously, S_j-S_i has the same d.f. as S_{j-i}. If $n\to\infty$ in (2.8), we have from (2.9) and lemma 3.2 in [10] (cf. [4], XI, (1.8))

$$(2.10)\quad \lim_{n\to\infty} \mathrm{P}\{\hat{w}_n \leqslant x,\ \hat{w}_n > 0\} = (1-1/\mathrm{E}\hat{\tau}_1)\,(1/\mathrm{E}\tau_1')\sum_{n=0}^{\infty} \Psi_n(x) = \big(1-\hat{W}(0)\big)W'(x),$$

where $\Psi_0(x) = D_+(x)$ and $\Psi_n(x) = \mathrm{P}\{\tilde{\delta}_0+S_j > 0,\ j = 1, 2, \ldots, n-1,\ 0 < \delta_0+S_n \leqslant x\}$, $n = 1, 2, \ldots$ (cf. lemma 1.2). Lemma 1.2 tells us that $W'(x)$ is a d.f. As $\lim_{n\to\infty} \hat{W}_n(0) = a > 0$, (2.7) and (2.9) hold, we infer that $\hat{W}_n(x)$ tends to a limit distribution $\hat{W}(x)$, and thus (2.4) is proved. (2.6) yields the representation $a = \beta\big(\beta+1-D(0)\big)^{-1}$.

The limit (2.9) and a are independent of $\hat{w}_0$. Therefore, it is easily verified that (2.4) is true also for any $\hat{w}_0$ (cf. [10], proof of theorem 3.2). A very simple consideration shows that $\hat{W}_n(x)$ is defined recursively by

$$\hat{W}_n(x) = \hat{W}_{n-1}*K(x)+\big(C(x)-K(x)\big)\hat{W}_{n-1}(0),\qquad x \geqslant 0,\ n = 1, 2, \ldots$$

Hence $\hat{W}(x)$ is the solution of the integral equation (1.2).

The solution of (1.2) is unique. In fact, let $\tilde{W}(x)$ be another d.f. which is the solution of (1.2). Choose $\mathrm{P}\{\hat{w}_0 \leqslant x\} = \tilde{W}(x)$. From $\hat{W}_n(x) = \tilde{W}(x)$, $n = 1, 2, \ldots$, if $n\to\infty$, it follows that $\tilde{W}(x) = \hat{W}(x)$ as $\hat{W}(x)$ is independent of $\hat{w}_0$ and, therefore, of $\mathrm{P}\{\hat{w}_0 \leqslant x\}$.

(c) If $a = 0$, then we obtain, using $D(0) < 1$ and $\beta = 1/\mathrm{E}\tau_1'$, $\mathrm{E}\tau_1' = \infty$. Applying lemma 1.2 to (2.10), we get $W'(x) \equiv 0$ and hence, by (2.7), $\hat{W}(x) \equiv 0$. The proof is complete.

As simplification, assume in the next sections that a stationary d.f. denotes the same as the limit distribution. Because of lemma 1.3 and (2.4) we are now in a position to give the following result:

THEOREM 2.2. *If a stationary d.f. of the sequence $\{\hat{w}_n\}$ exists, then it can be represented by*

$$\hat{W}(x) = \alpha\varepsilon_0(x)+(1-\alpha)G*W(x), \qquad -\infty < x < \infty, \tag{2.11}$$

in which $W(x)$ is the stationary d.f. in Lindley's model and $G(x)$ is a d.f. of a non-negative r.v. For $G(x)$ we have (1.8) *and* (1.9) *with*

$$C(x) = D_+(x) = \frac{D(x)-D(0)}{1-D(0)}, \qquad x \geqslant 0.$$

Remark 2.2. A particular case of (2.11) was proved first by Haeske [6] if $B(x)$ has a rational c.f. and $\tilde{B}(x)$ is a mixture of negative exponential d.f.'s. Although equation (2.4) is contained also in [6], no relation between the d.f. $W'(x)$ and the model $\{w_n'\}$ is given. But this relation is a fundamental idea of the present paper.

3. The distribution of the idle time. It was shown in [10] that the integral equation (1.4) can be represented for all x by

$$\beta U'(x)+W'(x) = K*W'(x)+C(x)\beta, \qquad -\infty < x < \infty, \tag{3.1}$$

where $\beta = K*W'(0)$, and $U'(x)$ is a d.f., which is concentrated on $(-\infty, 0]$. In the queue, $U'(x)$ is the d.f. of the negative idle time. We now study the d.f. of the idle time in the model $\{\hat{w}_n\}$.

If $\hat{W}(x)$ exists, (1.2) yields $\overline{W}(x)+\hat{W}(x) = K*\hat{W}(x)+\big(D(x)-K(x)\big)\alpha$, $-\infty < x < \infty$, $\overline{W}(-0) \leqslant \alpha$, $\overline{W}(-\infty) = 0$ and $\overline{W}(x) = 0$ for $x \geqslant 0$. $\overline{W}(x)$ is non-decreasing on $(-\infty, 0]$; in fact, $\hat{W}(x)-\varepsilon_0(x)\alpha$ is non--negative and non-decreasing, and hence $K*\big(\hat{W}(x)-\varepsilon_0(x)\alpha\big)$ is also non--decreasing. Clearly, $\hat{U}(x)$ with $\hat{U}(x) = (1/\alpha)\overline{W}(x)+\varepsilon_0(x)$, $-\infty < x < \infty$, is a d.f. Thus we get

$$\alpha\big(\hat{U}(x)-\varepsilon_0(x)\big)+\hat{W}(x) = K*\hat{W}(x)+\big(D(x)-K(x)\big)\alpha, \qquad -\infty < x < \infty. \tag{3.2}$$

Suppose $\hat{w}_0$ has already the d.f. $\hat{W}(x)$. Define two sequences of r.v.'s by

$$Y_n = \begin{cases} \delta_n & \text{if } \hat{w}_{n-1} = 0, \\ X_n & \text{if } \hat{w}_{n-1} > 0, \end{cases}$$

$$z_n = \min\{0, \hat{w}_n+Y_{n+1}\}, \qquad n = 1, 2, \ldots$$

Applying (1.1), we obtain

$$\begin{aligned} \hat{w}_{n-1}+Y_n &= \hat{w}_n+z_{n-1}, \\ \hat{w}_n &= \max\{0, \hat{w}_{n-1}+Y_n\}. \end{aligned} \tag{3.3}$$

We say that the queuing system has an *idle time* ϑ *at the moment* $n \geqslant 1$ if $\hat{w}_{n-1}+Y_n \leqslant 0$. Then the negative length of the idle time is $z_{n-1} = \hat{w}_{n-1}+Y_n \leqslant 0$. Obviously, we have $\hat{w}_{n-1}+Y_n \leqslant 0$ if and only if $\hat{w}_n = 0$. It follows for $x \leqslant 0$ that

$$\begin{aligned}\hat{W}(0)\mathrm{P}\{-\vartheta \leqslant x\} &= \mathrm{P}\{\hat{w}_n = 0\}\mathrm{P}\{z_{n-1} \leqslant x/\hat{w}_n = 0\} \\ &= \mathrm{P}\{\hat{w}_{n-1}+Y_n \leqslant x\} \\ &= \mathrm{P}\{\hat{w}_{n-1}+X_n \leqslant x,\ \hat{w}_{n-1} > 0\}+ \\ &\quad +\mathrm{P}\{w_{n-1}+\delta_n \leqslant x/\hat{w}_{n-1} = 0\}\mathrm{P}\{\hat{w}_{n-1} = 0\}.\end{aligned}$$

Therefore, we obtain

$$\hat{W}(0)\mathrm{P}\{-\vartheta \leqslant x\} = \big(K*\hat{W}(x)-K(x)\hat{W}(0)\big)+D(x)\hat{W}(0), \qquad x \leqslant 0,$$

and, comparing with (3.2),

$$\mathrm{P}\{-\vartheta \leqslant x\} = \hat{U}(x).$$

We now compare $\hat{U}$ with the d.f. U' of the imbedded model. By (3.2) we have

$$\hat{U} = (1/a)\,(W-a\varepsilon_0)*(K-\varepsilon_0)+D \qquad \text{for all } x.$$

By (2.6) we have

$$\frac{1-a}{a} = \frac{1-D(0)}{\beta}, \qquad \beta = 1/\mathrm{E}\tau_1', \ a = 1/\mathrm{E}\hat{\tau}_1.$$

Using $\hat{W} = a\varepsilon_0+(1-a)W'$ (see (2.4)), we get

$$\hat{U} = \frac{1-D(0)}{\beta}W'*(K-\varepsilon_0)+D.$$

On the other hand, we have from (3.1)

$$U' = (1/\beta)W'*(K-\varepsilon_0)+D_+.$$

Comparing both equations, we finally obtain

$$\hat{U}(x) = D(x)+\big(1-D(0)\big)\,U'(x), \qquad x \leqslant 0.$$

We collect all these results in the following theorem:

THEOREM 3.1. *The d.f.* $\hat{U}(x)$, *which is determined by* (3.2), *is the d.f. of the negative idle time in* $\{\hat{w}_n\}$. *For* $\hat{U}$ *we have the relation*

(3.4) $$\hat{U}(x) = D(x)+\big(1-D(0)\big)\,U'(x), \qquad x \leqslant 0,$$

where U' *is the d.f. of the negative idle time in the imbedded model.* U' *is determined by* (3.1).

Remark 3.1. Substituting (2.4) into (3.2), we have

$$\eta\hat{U}(x)+W'(x) = W'*K(x)+\eta D(x), \qquad -\infty < x < \infty, \tag{3.5}$$

where $\eta = \hat{W}(0)/[1-\hat{W}(0)]$. An analogous equation was discussed also by Haeske [6]. In this paper, however, $\hat{U}$ has no relation with the model (cf. remark 2.2).

4. Special cases. In this section we assume that a stationary d.f. $\hat{W}(x)$ exists. We shall investigate some special cases for $G(x)$ in (2.11). First, we compare $G(x)$ with $D(x)$. In [10] we derived

Lemma 4.1. *If $G(x)$ or $C(x)$ is a negative exponential distribution, then we have $G(x) \equiv C(x)$. Conversely, if $G(x) \equiv C(x)$ and $0 < G*U(0) < 1$, then $G(x)$ is a negative exponential distribution.*

Using theorem 2.2 and lemma 4.1, the following result is readily seen:

Corollary 4.1. *Assume that $D(0) < 1$. If $G(x)$ is a negative exponential distribution or $D(x) = 1-ce^{-dx}$, $x \geqslant 0$, $d > 0$, $0 < c \leqslant 1$, then we have*

$$G(x) \equiv D_+(x), \qquad x \geqslant 0. \tag{4.1}$$

Conversely, from (4.1) and $0 < G*U(0) < 1$ it follows that $G(x)$ is a negative exponential distribution.

We now consider the d.f.'s U and H in (1.7).

Theorem 4.1. *The d.f.'s H and U admit the representations*

$$H(x) = \big(1-K(0)\big)M_u*K_+(I), \qquad I = (0, x], \ x \geqslant 0, \tag{4.2}$$

where $M_u(I) = \sum_{i=0}^{\infty} U^{i*}(I)$ for bounded intervals I, and

$$U(x) = K(0)M_h*K_-(x), \qquad x \leqslant 0, \tag{4.3}$$

where $M_h = \sum_{i=0}^{\infty} H^{i*}$. $H(\infty)$ and $U(0)$ admit the formulas

$$H(\infty) = \big(1-K(0)\big)M_u*K_+(\overline{0, \infty}) \tag{4.4}$$

and

$$U(0) = K(0)M_h\dot{*}K_-(0). \tag{4.5}$$

Proof. (1.7) yields $H(x) = H*U(x)-\big(U(0)-K(x)\big)$, $x \geqslant 0$. Using $K(x) = K(0)+\big(1-K(0)\big)K_+(x)$, $x \geqslant 0$, we have

$$\frac{H(x)}{H(\infty)} = \frac{1}{H(\infty)}U*H(x)-\frac{1-K(0)}{H(\infty)}\left(\frac{U(0)-K(0)}{1-K(0)}-K_+(x)\right), \qquad x \geqslant 0.$$

If we put $H(x)/H(\infty) = G(x)$, then we get (1.9) with

$$K_+ = C \quad \text{and} \quad \eta = \frac{U(0)-K(0)}{1-K(0)}.$$

(1.10) yields (4.2). Applying (1.7) to $\hat{K}(x) = \mathrm{P}\{-X_1 \leqslant x\}$, we see from (4.2) that (4.3) is true. We obtain (4.4) and (4.5) if $x \to \infty$ in (4.2) and $x = 0$ in (4.3). The proof is complete.

Remark 4.1. For $x \leqslant 0$, relation (4.3) is equivalent (cf. [4], XII, section 3) with the integral equation

$$U + \Psi = K * \Psi + \varepsilon_0, \qquad \Psi = \sum_0^\infty H^{i*}.$$

An analogous formula to (4.2) is given in [1], p. 165. In this case, Borovkov uses this relation for estimating H.

(4.2) and (4.3) are very interesting. In particular, they show that H is uniquely determined by U and K_+ but for the constant factor $1-K(0)$. Hence, by (1.7), K_- is also uniquely determined by U and K_+. Analogously, U and K_+ are uniquely determined from H and K_-.

THEOREM 4.2. *Assume that $K(x)$ is given. Then*

$$C(x) \equiv K_+(x) \tag{4.6}$$

is necessary and sufficient for

$$H(\infty)G(x) \equiv H(x). \tag{4.7}$$

Proof. (a) Let us assume that (4.6) is true. Comparing (4.2) with (1.10), we obtain $cG(x) \equiv H(x)$, where the constant $c = H(\infty)$ as G is a distribution function without defect.

(b) Let us assume that (4.7) is true. Part (a) shows that this is true for $C(x) \equiv K_+(x)$. We have to prove that under (4.7) C is uniquely determined by K. In fact, U and H are uniquely given by K (see (1.7)). Therefore, under (4.7), G is also determined uniquely. Furthermore, from (1.9) we see that C is uniquely determined by G and U. The proof is complete.

We now apply theorem 4.2 to Yeo's model. In this case we start from (2.11).

COROLLARY 4.2. *Let us assume that $K(x)$ is given. Then*

$$K_+(x) \equiv D_+(x) \tag{4.8}$$

is necessary and sufficient for (4.7). *If the stationary d.f.'s $\hat{W}$ and W exist and* (4.8) *is true, then*

$$\frac{\hat{W}(x)-\hat{W}(0)}{1-\hat{W}(0)} = \frac{W(x)-W(0)}{1-W(0)}, \quad x \geqslant 0. \tag{4.9}$$

In this case we obtain $\hat{W}(x) \equiv W(x)$ *if and only if* $D(0) = K(0)$.

Proof. We obtain the first part from theorem 4.2. As is well known, $W(x)/W(0)$ is the renewal function of $H(x)$ (cf. [4], XII, section 3, (3.10)). Therefore, we have $W(x) = W(0)\varepsilon_0(x) + H * W(x)$, $x \geqslant 0$. From (4.7) and $H(\infty) = 1 - W(0)$ it now follows that

$$W(x) = W(0)\varepsilon_0(x) + \big(1 - W(0)\big)G * W(x), \qquad x \geqslant 0. \tag{4.10}$$

Comparing (2.11) and (4.10), we get (4.9). Since

$$\hat{W}(0) = \beta\big(1 - D(0) + \beta\big)^{-1} \quad \text{and} \quad W(0) = \beta\big(1 - K(0) + \beta\big)^{-1},$$

(cf. theorem 2.1, part (b)), the proof is complete.

We now investigate the d.f. $G(x)$ for

$$U(x) = \begin{cases} e^{\lambda x}, & x < 0, \\ 1, & x \geqslant 0,\ \lambda > 0. \end{cases}$$

In particular, we have this $U(x)$ if the input of the queueing system is a Poisson process. Since $V(x) = 1 - U(-x-0)$ has the renewal function

$$\sum_0^\infty V^{i*}(x) = \lambda x + 1, \qquad x \geqslant 0,$$

we obtain from (1.10) (cf. [10], section 4)

$$G(x) = \delta\Big(C(x) + \lambda \int_0^x \big(1 - C(y)\big)dy\Big),$$

where $\delta = (1 + \lambda m_c)^{-1}$ and $m_c = \int_0^\infty \big(1 - C(y)\big)dy$.

5. The queue length. Suppose the stationary d.f. $\check{W}(x)$ exists. The queue length $Q(t)$ is defined by the number of customers in the system at the moment t. Let customers arrive at the epochs $0 = t_0 < t_1 < t_2 < \ldots$, let $s_0 < s_1 < s_2 < \ldots$ be the instants in which the service of the customers begins, and $r_0, r_1, r_2, \ldots$ the instants in which the customers leave the system.

We introduce the following notation:

$${}^AQ_n = Q(t_n - 0), \quad {}^FQ_n = Q(r_n + 0) \quad \text{and} \quad Q_n = Q(s_n + 0).$$

It was proved by Finch [5] for $\{w_n'\}$ that

$$\begin{aligned} \{{}^FQ_n \geqslant k\} &= \{w_n' + \beta_n \geqslant \alpha_{n+1} + \alpha_{n+2} + \ldots + \alpha_{n+k}\}, \qquad k = 1, 2, \ldots, \\ \{{}^FQ_n = 0\} &= \{w_n' + \beta_n < \alpha_{n+1}\}, \end{aligned} \tag{5.1}$$

and

$$(5.2)\quad \lim_{n\to\infty} \mathrm{P}\{{}^F Q_n = k\} = \lim_{n\to\infty} \mathrm{P}\{{}^A Q_n = k\} = {}^A q'_k, \qquad k = 0, 1, 2, \ldots$$

From (5.1) it is easily verified that

$$(5.3)\qquad {}^A q'_k = \int_0^\infty \big(A^{k*}(x) - A^{(k+1)*}(x)\big)\, d\big(B * W'(x)\big), \qquad k = 0, 1 \ldots$$

Analogously to (5.1) and (5.2), we have formulas for the queue length of the model $\{\hat{w}_n\}$. Using (2.4) and (5.3), a very simple manipulation shows that

$$(5.4)\quad {}^A \hat{q}_k = a \int_0^\infty \big(A^{k*}(x) - A^{(k+1)*}(x)\big)\, d\tilde{B}(x) + (1-a)\, {}^A q'_k, \qquad k = 0, 1, \ldots,$$

where ${}^A \hat{q}_k$ are the stationary probabilities of ${}^A Q_n$ in $\{\hat{w}_n\}$.

The relation between the stationary probabilities of Q_n in $\{\hat{w}_n\}$ and the imbedded model $\{w'_n\}$ is simpler. Let $\{\hat{q}_k\}$ and $\{q'_k\}$ be the stationary probabilities of Q_n in $\{\hat{w}_n\}$ and in the imbedded model $\{w'_n\}$, respectively. We can write (cf. (5.1))

$$(5.5)\qquad \begin{aligned} \{Q_n \geqslant k\} &= \{w'_n \geqslant a_{n+1} + a_{n+2} + \ldots + a_{n+k-1}\}, \\ \{Q_n = 1\} &= \{w'_n < a_{n+1}\}. \end{aligned}$$

It follows that

$$(5.6)\qquad q'_k = \int_0^\infty \big(A^{(k-1)*}(x) - A^{k*}(x)\big)\, dW'(x), \qquad k \geqslant 1.$$

By (5.5) we get, applying (2.4),

THEOREM 5.1. *The stationary probabilities $\{\hat{q}_k\}$ and $\{q'_k\}$, $k = 1, 2, \ldots$, of the queue length Q_n in $\{\hat{w}_n\}$ and in the imbedded model $\{w'_n\}$, respectively, admit the relations*

$$(5.7)\qquad \begin{aligned} \hat{q}_k &= (1-a)\, q'_k, & k &= 2, 3, \ldots, \\ \hat{q}_1 &= (1-a)\, q'_1 + a, & a &= \hat{W}(0). \end{aligned}$$

6. The busy period. Let us assume that $\hat{w}_0 = 0$. We define the first busy period in the imbedded model $\{w'_n\}$ by

$$\Theta'_1 = \beta_{k_1+1} + \beta_{k_1+2} + \ldots + \beta_{k_1+\tau'_1}$$

and the first busy period in $\{\hat{w}_n\}$ by

$$\hat{\Theta}_1 = \begin{cases} \tilde{\beta}_1 + \beta_2 + \ldots + \beta_{\hat{\tau}_1} & \text{for } \hat{\tau}_1 > 1, \\ \tilde{\beta}_1 & \text{for } \hat{\tau}_1 = 1. \end{cases}$$

Let $\hat{L}(x)$ and $L'(x)$ be the d.f.'s of $\hat{\Theta}_1$ and Θ_1', respectively. Note that $\hat{\tau}_1 = 1+\tau_1'$ for $\hat{\tau}_1 > 1$ (cf. proof of theorem 2.1, part (a)). Therefore, from $\{\hat{\tau}_1 > 1\} = \{k_1 = 1\}$ and from $\mathrm{P}\{\tilde{\beta}_1 - a_2 \leqslant 0\} = D(0)$ we get the representation

$$\hat{L}(x) = N(x)D(0) + \mathrm{P}\{\tilde{\beta}_1 + \Theta_1' \leqslant x, \tilde{\beta}_1 - a_2 > 0\}$$

with $N(x) = \mathrm{P}\{\tilde{\beta}_1 \leqslant x / \tilde{\beta}_1 - a_2 \leqslant 0\}$.

Obviously, Θ_1' is independent of k_1, $\tilde{\beta}_1$ and a_2. Therefore, it follows easily

THEOREM 6.1. *The d.f.'s $\hat{L}(x)$ and $L'(x)$ of the busy periods in $\{\hat{w}_n\}$ and in the imbedded model $\{w_n'\}$, respectively, admit the formula*

$$\hat{L}(x) = N(x)D(0) + \big(\tilde{B} - D(0)N\big) * L'(x) \tag{6.1}$$

with

$$N(x) = \begin{cases} A(x) + \tilde{B}(x) \int\limits_x^\infty \big(\tilde{B}(y)\big)^{-1} dA(y) & \text{if } \tilde{B}(x) > 0, \\ 0 & \text{if } \tilde{B}(x) = 0, \end{cases} \tag{6.2}$$

$$D(0) = \int\limits_0^\infty \tilde{B}(x)\,dA(x).$$

Acknowledgement. My thanks are due to Prof. H.-J. Rossberg for many helpful suggestions and criticism.

References

[1] A. A. Borovkov (А. А. Боровков), *Вероятностные процессы в теории массового обслуживания*, Наука, Москва 1972.

[2] W. J. Ewens and P. D. Finch, *A generalized single-server queue with Erlang input*, Biometrika 49 (1962), p. 242-245.

[3] W. Feller, *An introduction to probability theory and its applications*, vol. I, J. Wiley, New York 1950.

[4] — *An introduction to probability theory and its applications*, vol. II, J. Wiley, New York 1960.

[5] P. D. Finch, *A probability limit theorem with application to a generalization of queueing theory*, Acta Math. Acad. Sci. Hung. 10 (1959), p. 317-325.

[6] H.-J. Haeske, *Eine verallgemeinerte Behandlung der Integralgleichung von G. F. Yeo nach einer Methode von H.-J. Rossberg*, Math. Operationsforsch. u. Statist. 2 (1970), p. 31-38.

[7] E. S. Klimova (Э. С. Климова), *Исследование однолинейной системы обслуживания с „разогревом"*, Техническая кибернетика 1 (1968), p. 91-97.

[8] D. V. Lindley, *The theory of queues with a single server*, Proc. Cambridge Philos. Soc. 48 (1952), p. 277-289.

[9] H.-J. Rossberg, *Eine neue Methode zur Behandlung der Integralgleichung von Lindley und ihre Verallgemeinerung durch Finch*, Elektron. Informationsverarbeitung und Kybernetik 3 (1967), p. 213-236.
[10] G. Siegel, *Verallgemeinerung einer Irrfahrt im R^1 und deren Bedeutung für Bedienungssysteme mit verzögertem Bedienungsbeginn*, Math. Operationsforsch. u. Statist., submitted.
[11] G. F. Yeo, *Single server queues with modified service mechanism*, J. Austral. Math. Soc. 2 (1962), p. 499-507.

SEKTION MATHEMATIK
KARL-MARX-UNIVERSITÄT
LEIPZIG, DDR

Received on 26. 9. 1972

G. SIEGEL (Leipzig)

STACJONARNY CZAS OCZEKIWANIA I INNE WIELKOŚCI W JEDNOKANAŁOWYCH SYSTEMACH OBSŁUGI MASOWEJ Z OSOBLIWOŚCIAMI NA POCZĄTKU OKRESU OBSŁUGI

STRESZCZENIE

Autor bada dwa uogólnienia systemu obsługi masowej typu $GI/G/1$ (patrz Finch [5] i Yeo [11]). Struktura tych jednokanałowych systemów charakteryzuje się tym, że obsługa klientów przychodzących w czasie, gdy kanał obsługi jest wolny, zostaje w specjalny sposób opóźniona. Autor pokazuje, że model Yeo można zredukować do modelu Fincha, mimo że ten ostatni jest szczególnym przypadkiem modelu Yeo. Rozkład czasu czekania w modelu Yeo można wyrazić poprzez rozkład czasu czekania w modelu $GI/G/1$ oraz poprzez rozkład dany w twierdzeniu 2.2. Przedstawione są także nowe wyniki dotyczące czynników przy znanym sposobie faktoryzacji różnicy $1-K^*$, gdzie K^* jest dowolną funkcją charakterystyczną.