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THE RELIABILITY OF AN ELEMENT WITH ALTERNATING FAILURE RATE

1. Introduction. In some practical situations the failure rate of the considered elements may be a random process. In [4] examples of such situations are presented and under some assumptions the reliability function of an element and the reliability function of a system are analyzed. In this note, the properties of the reliability function of a single element are presented under the assumption that the failure rate process is a process on a two-element set.

Consider the stochastic process defined as follows:

$$a(t) = \begin{cases} 1, & Z_n'' < t \leqslant Z_{n+1}', \\ 0, & Z_{n+1}' < t \leqslant Z_{n+1}'', \end{cases} \qquad n = 0, 1 \ldots,$$

and

$$a(0) = 1,$$

where $Z_0'' = 0$, $Z_n' - Z_{n-1}'' = X_n$ $(n = 1, 2, \ldots)$ are non-negative random variables with common distribution function F, $Z_n'' - Z_n' = Y_n$ $(n = 1, 2, \ldots)$ are non-negative random variables with common distribution function G, and the random variables $X_1, Y_1, X_2, Y_2, \ldots$ are independent. Moreover, we assume that the expected values $1/\lambda = \mathrm{E}X_1$, $1/\mu = \mathrm{E}Y_1$, and variances $\sigma_1^2 = \mathrm{D}^2X_1$, $\sigma_2^2 = \mathrm{D}^2Y_1$ exist and $\sigma_1^2 + \sigma_2^2 > 0$.

In this note we assume that the failure rate of an element is the random process

(1)
$$\lambda(t) = \theta a(t), \quad t \geqslant 0, \ \theta > 0.$$

The working time of an element with failure rate (1) is denoted by Z, the reliability function of an element by P, and the distribution function of the working time of an element by $\bar{P} = 1 - P$. We define

(2)
$$P(t) = \mathrm{E}\exp\big(-\theta A(t)\big), \quad t \geqslant 0,$$

where

$$A(t) = \int_0^t a(u)\,du, \quad t \geqslant 0. \tag{3}$$

The stochastic process $A(t)$, $t \geqslant 0$, is called the *cumulative process*. The value of this process at moment t is the cumulative time of the process a being at state 1 on the interval $[0, t]$.

2. The distribution function. Now we proceed to analyze the distribution function $\bar{P}$ and the reliability function P.

LEMMA 1. *The distribution function of the working time of an element with failure rate $\lambda(t)$, $t \geqslant 0$, is of the form*

$$\bar{P}(t) = \Pr(U + Y_1 + Y_2 + \ldots + Y_{N(U)} < t), \quad t \geqslant 0, \tag{4}$$

where U is some exponentially distributed random variable with parameter θ independent of $X_1, Y_1, X_2, \ldots$, and N is the renewal process defined for the renewal sequence $X_1 + X_2 + \ldots + X_n$, $n = 1, 2, \ldots$

Proof. Let U denote the working time of an element under the condition that the failure rate θ is constant. Consider the moment t^* at which the cumulative process A achieves the random value U. Obviously, $t^* = U + Y_1 + Y_2 + \ldots + Y_{N(U)}$ and the following equalities are satisfied:

$$\{U + Y_1 + Y_2 + \ldots + Y_{N(U)} \geqslant t\} = \{t^* \geqslant t\} = \{A(t^*) \geqslant A(t)\} = \{U \geqslant A(t)\}.$$

From the property of the random variable U we have

$$\mathrm{E}\exp\big(-\theta A(t)\big) = \Pr\big(U \geqslant A(t)\big) = \Pr(U + Y_1 + Y_2 + \ldots + Y_{N(U)} \geqslant t).$$

Notice the simple interpretation of the random variable U. If a constant failure rate θ of an element is assumed, then the working time of an element is equal to the random variable U defined in the lemma. If we consider the renewal process $X_1 + X_2 + \ldots + X_n$, $n = 1, 2, \ldots$, then in the interval $[0, U)$ there occur $N(U)$ inputs of this process and, if the n-th input prolongs the working time of the element for the random variable Y_n, $n = 1, 2, \ldots, N(U)$, then $U + Y_1 + Y_2 + \ldots + Y_{N(U)}$ is the moment of failure of the element.

COROLLARY 1. *The expected value and the variance of the working time of an element with failure rate $\lambda(t)$, $t \geqslant 0$, are*

$$\mathrm{E}Z = \frac{1}{\theta} + \frac{1}{\mu} h^*(\theta) \tag{5}$$

and

$$\mathrm{D}^2 Z = \frac{1}{\theta^2} + 2\frac{\theta}{\mu}\left(-\frac{d}{d\theta}\frac{h^*(\theta)}{\theta}\right) + \frac{2}{\mu}\left(\frac{1}{\mu} - \frac{1}{\theta}\right) h^*(\theta) + \frac{1}{\mu^2}\big(h^*(\theta)\big)^2, \tag{6}$$

where $h^*(s) = \int_0^\infty e^{-st}dH(t)$ *is the Laplace-Stieltjes transform of the renewal function* $H(t) = \mathrm{E}N(t)$, $t \geqslant 0$.

Proof. From Lemma 1, using the properties of the random variable U, we get

$$\mathrm{E}Z = \mathrm{E}U + \int_0^\infty \theta e^{-\theta u}\mathrm{E}Y_1\mathrm{E}N(u)du = \frac{1}{\theta}+\theta\frac{1}{\mu}\int_0^\infty e^{-\theta u}H(u)du = \frac{1}{\theta}+\frac{1}{\mu}h^*(\theta),$$

$$\mathrm{E}Z^2 = \mathrm{E}U^2+2\mathrm{E}U\sum_{i=1}^{N(U)} Y_i+\mathrm{E}\sum_{i=1}^{N(U)} Y_i^2+2\mathrm{E}\sum_{i=1}^{N(U)}\sum_{j=i+1}^{N(U)} Y_iY_j,$$

$$\mathrm{E}U^2 = 2/\theta^2,$$

$$\mathrm{E}U\sum_{i=1}^{N(U)} Y_i = \int_0^\infty \theta e^{-\theta u}uH(u)\mathrm{E}Y_1du = \frac{\theta}{\mu}\left(-\frac{d}{d\theta}\frac{h^*(\theta)}{\theta}\right),$$

$$\mathrm{E}\sum_{i=1}^{N(U)} Y_i^2 = \mathrm{E}Y_1^2h^*(\theta) = \frac{2}{\mu^2}h^*(\theta),$$

$$\mathrm{E}N(t)\big(N(t)-1\big) = 2\int_0^t H(t-u)dH(u),$$

$$\mathrm{E}\sum_{i=1}^{N(U)}\sum_{j=i+1}^{N(U)} Y_iY_j = \frac{1}{\mu^2}\int_0^\infty \theta e^{-\theta u}\int_0^t H(t-u)dH(u)du = \frac{1}{\mu^2}\big(h^*(\theta)\big)^2.$$

By suitable substitutions we get (5) and (6).

Example 1. If in the process a we put $F(x) = 1-e^{-\lambda x}$, $G(x) = 1-$ $-e^{-\mu x}$, $\lambda > 0$, $\mu > 0$, $x \geqslant 0$, then $H(t) = \lambda t$, $t \geqslant 0$, $h^*(s) = \lambda/s$, $\mathrm{Re}\, s \geqslant 0$, and

$$-\frac{d}{d\theta}\frac{h^*(\theta)}{\theta} = 2\frac{\lambda}{\theta^3}.$$

By (5) and (6) we have

$$\mathrm{E}Z = \frac{1}{\theta}+\frac{1}{\mu\theta},\quad \mathrm{D}^2Z = \frac{1}{\theta^2}\left(1+\frac{\lambda}{\mu}\right)^2+\frac{2\lambda}{\mu^2\theta}. \tag{7}$$

THEOREM 1. *The Laplace-Stieltjes transform of the distribution function* $\bar{P}$ *of the working time of an element with failure rate* $\lambda(t)$, $t \geqslant 0$, *is of the form*

$$\bar{p}^*(s) = \int_0^\infty e^{-sx}d\bar{P}(x) = \frac{\theta}{s+\theta}\,\frac{1-f^*(s+\theta)}{1-f^*(s+\theta)g^*(s)},\quad \mathrm{Re}\, s \geqslant 0, \tag{8}$$

where

$$f^*(s) = \int_0^\infty e^{-sx} dF(x), \qquad g^*(s) = \int_0^\infty e^{-sx} dG(x), \qquad \operatorname{Re} s \geqslant 0.$$

Proof. Notice that the distribution function of the value of the renewal process N takes the form

$$p_n(t) = \Pr(N(t) = n) = F_n(t) - F_{n+1}(t), \qquad n = 0, 1, \ldots,$$

where F_n is the n-th convolution of the distribution function F. Using Lemma 1 we get

$$\begin{aligned}
\bar{p}^*(s) &= \mathrm{E}\, e^{-sZ} = \mathrm{E} \exp\big(-s(U + Y_1 + \ldots + Y_{N(U)})\big) \\
&= \int_0^\infty \mathrm{E} \exp\big(-s(u + Y_1 + \ldots + Y_{N(u)})\big)\, \theta e^{-\theta u} du \\
&= \int_0^\infty \theta e^{-(s+\theta)u} \Big(\sum_{n=0}^\infty \mathrm{E} \exp\big(-s(Y_1 + \ldots + Y_n)\big)\Big)\, p_n(u) du \\
&= \int_0^\infty \theta e^{-(s+\theta)u} \Big(\sum_{n=0}^\infty \big(g^*(s)\big)^n p_n(u)\Big) du \\
&= \theta \sum_{n=0}^\infty \big(g^*(s)\big)^n \left(\frac{\big(f^*(s+\theta)\big)^n}{s+\theta} - \frac{\big(f^*(s+\theta)\big)^{n+1}}{s+\theta}\right) \\
&= \frac{\theta}{s+\theta} \, \frac{1 - f^*(s+\theta)}{1 - f^*(s+\theta) g^*(s)}.
\end{aligned}$$

This result may be obtained also in another way (see [4]).

Example 2. Assume that the conditions of Example 1 hold. We have

$$\bar{P}(t) = 1 - (a/s_1)\exp(-s_1 t) - (b/s_2)\exp(-s_2 t), \tag{9}$$

where

$$s_1 = \tfrac{1}{2}\big(\lambda + \mu + \theta + \sqrt{(\lambda + \mu + \theta)^2 - 4\theta\mu}\big),$$

$$s_2 = \tfrac{1}{2}\big(\lambda + \mu + \theta - \sqrt{(\lambda + \mu + \theta)^2 - 4\theta\mu}\big),$$

$$a = \frac{\theta(-s_1 + \mu)}{-s_1 + s_2}, \qquad b = \frac{\theta(-s_2 + \mu)}{-s_2 + s_1},$$

and

$$s_1 > 0, \qquad s_2 > 0, \qquad a > 0, \qquad b > 0.$$

Indeed, we have $f^*(s) = \lambda/(s+\lambda)$, $g^*(s) = \mu/(s+\mu)$, $\operatorname{Re} s \geqslant 0$. Hence, using (8), we get

$$\bar{p}^*(s) = \frac{\theta(s+\mu)}{s^2+(\lambda+\mu+\theta)s+\mu\theta}, \quad \operatorname{Re} s \geqslant 0. \tag{10}$$

It is easy to see that $s^2+(\lambda+\mu+\theta)s+\mu\theta = (s+s_1)(s+s_2)$, and from the inequality $(\lambda+\mu+\theta)^2-4\mu\theta = (\lambda+\mu-\theta)^2+4\lambda\theta > 0$ it follows that the numbers s_1 and s_2 are real, and $s_1 > 0$, $s_2 > 0$. For the rational function (10) we get

$$\bar{p}^*(s) = \frac{a}{s+s_1}+\frac{b}{s+s_2}, \quad \operatorname{Re} s \geqslant 0, \tag{11}$$

where $a/s_1+b/s_2 = 1$, and $a > 0$, $b > 0$. Inverting the transform (11) we obtain (9).

COROLLARY 1a (see [4]). *If $\theta \to 0$, then the distribution function of the working time of an element is asymptotically exponential*:

$$\lim_{\theta\to 0} \Pr(\theta Z < t) = 1-\exp\left(-\frac{\mu}{\lambda+\mu}t\right), \quad t \geqslant 0.$$

3. Estimations. The application of the exact distribution function $\bar{P}$ in practice is limited because the estimation of the distribution functions F and G is laborious and the inversion of the transform $\bar{p}^*$ may be difficult. Hence some estimations may be of use.

THEOREM 2. *If in the process a we assume $F(x) = 1-e^{-\lambda x}$, $x \geqslant 0$, $\lambda > 0$, and G is a decreasing failure rate, then the reliability function of an element with failure rate $\lambda(t) = \theta a(t)$, $t \geqslant 0$, is NWU (new worse than used).*

We prove Theorem 2 by the method introduced by Barlow and Proschan in [3]. Using the process a we define the auxiliary process $Z(t)$, $t \geqslant 0$, in the following manner:

$$Z(t) = \begin{cases} 0, & Z_n'' < t \leqslant Z_{n+1}', \\ t-Z_{n+1}', & Z_{n+1}' < t \leqslant Z_{n+1}'', \end{cases} \quad n = 0, 1, \ldots$$

Obviously (by [3]), $Z(t)$, $t \geqslant 0$, is a stochastically increasing process and

$$\{Z(s),\ 0 \leqslant s \leqslant t \mid Z(0) = 0\} \leqslant \{Z(s+a),\ 0 \leqslant s \leqslant t \mid Z(0) = 0, \quad \exp(-\theta A(a))\},$$

where $\leqslant$ denotes the stochastic order relation for the process and $a > 0$. Note that the functional $f\{Z(s),\ 0 \leqslant s \leqslant t\} = \exp(-\theta A(t))$ is increasing in the process $Z(\cdot)$ since $f(\cdot)$ is decreasing in $a(\cdot)$, and $a(\cdot)$ is decreasing as a function of $Z(\cdot)$. Hence, using (2), we have

$$P(t) = \mathrm{E}\exp\big(-\theta A(t) \mid Z(0) = 0\big)$$
$$\leqslant \mathrm{E}\Big[\exp\Big(-\theta\big(A(t+a)-A(a)\big)\Big) \mid Z(0) = 0,\ \exp\big(-\theta A(a)\big)\Big].$$

Multiplying this inequality by the random variable $\exp\big(-\theta A(a)\big)$ and taking the expected value under the assumption $Z(0) = 0$, we get

$$P(a)P(t) \leqslant \mathrm{E}\Big[\exp\big(-\theta A(a) \mid Z(0) = 0\big)\mathrm{E}\exp\Big(-\theta\big(A(t)-A(a)\big)\Big) \mid \exp\big(-\theta A(a)\big),\ Z(0) = 0\Big]$$
$$= \mathrm{E}\Big[\exp\big(-\theta A(t+a)\big) \mid Z(0) = 0\Big] = P(t+a), \qquad a \geqslant 0,\ t \geqslant 0.$$

Using the known estimation of the reliability function in the NWU class given the expected value (see [5]), by Theorem 2 we get

COROLLARY 2. *Under the assumptions of Theorem 2 we have the inequality*

$$P(t) \leqslant \frac{\mathrm{E}Z}{t+\mathrm{E}Z}, \qquad t \geqslant 0. \tag{12}$$

Jensen's inequality implies

COROLLARY 3. *The reliability function with failure rate $\lambda(t)$, $t \geqslant 0$, satisfies the inequality*

$$P(t) \geqslant \exp\Big(-\theta\int_0^t p(u)\,du\Big), \qquad t \geqslant 0, \tag{13}$$

where $p(t) = \mathrm{E}\alpha(t)$, $t \geqslant 0$, is the performance coefficient in the process $\alpha(t)$, $t \geqslant 0$.

Remark. For practical purposes we find the approximation of the integral on the right-hand side of (13). If the convolution of probability distribution functions $\varphi = F^* G$ is non-lattice, then

$$\int_0^t p(u)\,du = \frac{\mu t}{\lambda+\mu} + \frac{1}{2\lambda}\Big(\frac{\lambda\mu}{\lambda+\mu}\Big)^2\Big(\sigma_2^2 - \frac{\lambda}{\mu}\sigma_1^2 + \frac{1}{\mu^2} + \frac{1}{\lambda\mu}\Big) + o(1), \tag{14}$$

where $o(1)$ tends to 0 as $t\to\infty$.

Indeed, we have

$$p(t) = 1 - F(t) + \int_0^t \big(1 - F(t-u)\big)\,dH_\varphi(u), \qquad t \geqslant 0,$$

where $H_\varphi(t)$, $t \geqslant 0$, is the renewal function in the renewal process

$Z_n'' = X_1+Y_1+\ldots+X_n+Y_n$, $n = 1, 2, \ldots$ Therefore,

$$(15)\qquad \int_0^t p(u)\,du = \int_0^t \big(1-F(u)\big)\,du - \int_0^t \int_{t-u}^{\infty} \big(1-F(x)\big)\,dx\,dH_\varphi(u) + \frac{1}{\lambda} H_\varphi(t).$$

Hence, using the fundamental renewal theorem and the estimation of the renewal function, we find (see [2])

$$\int_0^t \big(1-F(u)\big)\,du = \frac{1}{\lambda} + o(1),$$

$$\int_0^t \int_{t-u}^{\infty} \big(1-F(x)\big)\,dx\,dH_\varphi(u) = \frac{\lambda u}{\lambda+\mu}\,\frac{1}{2}\left(\sigma_1^2 + \frac{1}{\lambda^2}\right) + o(1),$$

$$H_\varphi(t) = \frac{\lambda\mu t}{\lambda+\mu} + \left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \frac{1}{2}\left[\sigma_1^2+\sigma_2^2+\left(\frac{\mu+\lambda}{\lambda\mu}\right)^2\right] - 1 + o(1).$$

Substituting these approximations into (15) we get (14).

Example 3. Suppose the conditions of Example 1 are satisfied and let $\lambda = \mu = 1$, $\theta = 2$. Then

$$P(t) = \tfrac{1}{2}[\exp(-s_1 t) + \exp(-s_2 t)], \qquad t \geqslant 0,$$

where $s_1 = 2+\sqrt{2}$ and $s_2 = 2-\sqrt{2}$. Moreover, we have $p(t) = \frac{1}{2}+\frac{1}{2}e^{-2t}$. Hence, using Corollary 3, we get

$$P(t) \geqslant \exp\big(-t - \tfrac{1}{4}(1-e^{-2t})\big), \qquad t \geqslant 0.$$

Furthermore, since $\mathrm{E}Z = 1$, by Corollary 2 we have

$$P(t) \leqslant \frac{1}{t+1}, \qquad t \geqslant 0.$$

4. Generalization. Assume that the failure rate of an element is a random process of the form

$$(16)\qquad \lambda_0(t) = \theta a(t) + \theta_0, \qquad t \geqslant 0.$$

It is known that the sum of failure rates appears in the analysis of systems being in series, and then the working time of the system equals the minimum of working times of the elements. Hence the working time of an element with failure rate (16) is of the form $Z_0 = \min(Z, U_0)$, and the random variables Z and U_0 are independent, the probability distribution of the first of them is equal to $\bar{P}$ and of the second one is exponential with parameter θ_0. Therefore, we obtain

$$\Pr(Z_0 \geqslant t) = \Pr(Z \geqslant t)\Pr(U_0 \geqslant t) = P(t)\exp(-\theta_0 t), \qquad t \geqslant 0.$$

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