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EXPLICIT FORMULAS FOR TRANSITION INTENSITIES IN THE QUEUEING SYSTEM $E_2/E_2/n$

1. The problem. Consider a queueing system in which (a) interarrival times are independent random variables each of them having the Erlangian distribution of order 2 and the expected value $2/\lambda$, (b) service times are independent random variables with the Erlangian distribution of order 2 and the expected value $2/\mu$, and (c) there are n independent service channels. If an item arrives when all channels are busy it is lost, otherwise its service begins in any of the free channels.

We will study the stochastic process $N(t)$, defined as the number of items being in the system at moment t. For the defined system $N(t)$ is not a Markov process, however, it is easy to define its appropriate extension which will be Markovian. We note that the interarrival time (service time) having the Erlangian distribution of order 2 and the expected value $2/\lambda$ $(2/\mu)$ may be interpreted as consisting of 2 phases, their durations being mutually independent random variables having identical exponential distributions with parameter λ (μ). Decompose the interarrival time into two consecutive exponential phases: the phase number 2 and the phase number 1. The same will refer also to the consecutive phases of service times.

Let us define a three-dimensional stochastic process

$$X(t) = [M(t), N_1(t), N_2(t)],$$

where $M(t)$ is 2 or 1 according to the number of the interarrival phase at moment t, $N_i(t)$ is the number of items which at the moment t undergo the i-th service phase $(i = 1, 2)$, and $X(t)$ is obviously a Markovian stochastic process and

$$N(t) = N_1(t)+N_2(t). \tag{1}$$

The number S_n of possible states of process $X(t)$ is finite and equals

$$S_n = 2\sum_{k=0}^{n}(k+1) = (n+1)(n+2), \quad n = 1, 2, \ldots, \tag{2}$$

where the coefficient 2 stands for the two possible states of the coordinate $M(t)$ and $k+1$ is the number of possible ways to decompose the number of k items in the system ($k = 0, 1, \ldots, n$) into two non-negative integer components.

To calculate the probabilities of the states of $X(t)$ the order of the states is not essential. If some order is chosen, then the transition intensity matrix

$$Q_n = (q_{ij}^{(n)}), \quad i, j = 1, 2, \ldots, S_n,$$

may be found. The aim of this paper is to introduce a convenient order of the states so as to obtain as simple as possible explicit formulas for the elements of matrix Q_n.

2. Limiting probabilities of $X(t)$ and $N(t)$. The transition intensity matrix may be used to find the limiting probabilities

$$p_i = \lim_{t\to\infty} P\big(X(t) = i\big), \quad i = 1, 2, \ldots, S_n.$$

$X(t)$ is a transitive Markov process, so that the limiting probabilities exist and satisfy the system of equations (see [2], 1; XVII, 9)

$$\sum_{i=1}^{S_n} p_i q_{ij}^{(n)} = 0, \quad j = 1, 2, \ldots, S_n. \tag{3}$$

Since the transition intensities fulfill the relation

$$\sum_{j=1}^{S_n} q_{ij}^{(n)} = 0, \quad i = 1, 2, \ldots, S_n, \tag{4}$$

and we additionally require that

$$\sum_{i=1}^{S_n} p_i = 1, \tag{5}$$

there always exists a non-zero solution of system (3). The limiting probabilities p_i of the process $X(t)$ enable us to find the limiting probabilities P_k of the process $N(t)$. There is

$$P_k = \lim_{t\to\infty} P\big(N(t) = k\big) = \sum_{i\in Z_k} p_i, \quad k = 0, 1, \ldots, n, \tag{6}$$

where Z_k is the set of all the integers i which are the numbers of the states (M, N_1, N_2) of $X(t)$, such that $N_1 + N_2 = k$.

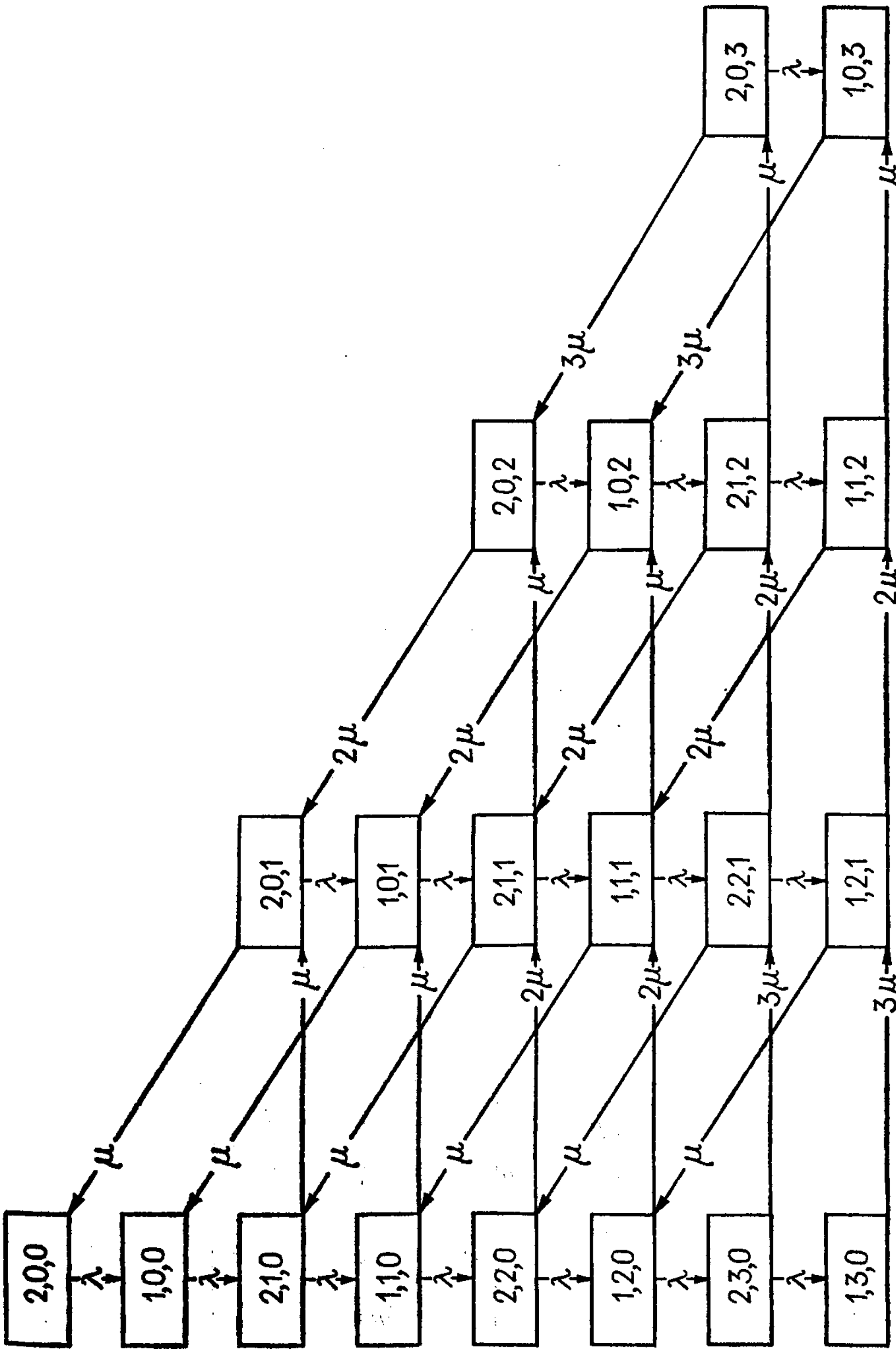

2,0,3
1,0,3
2,0,2
1,0,2
2,1,2
1,1,2
2,0,1
1,0,1
2,1,1
1,1,1
2,2,1
1,2,1
2,0,0
1,0,0
2,1,0
1,1,0
2,2,0
1,2,0
2,3,0
1,3,0

Fig. 1

$$
Q_1 = \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & * & \lambda & & & & \\
2 & & * & \lambda & & & \\
3 & & & * & \mu & \lambda & \\
4 & \mu & & & * & & \lambda \\
5 & & & & & * & \mu \\
6 & & \mu & & & & * \\
\hline
\end{array}
$$

$$
Q_2 = \begin{array}{c|cccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
1 & * & \lambda & & & & & & & & & & \\
2 & & * & \lambda & & & & & & & & & \\
3 & & & * & \mu & \lambda & & & & & & & \\
4 & \mu & & & * & & \lambda & & & & & & \\
5 & & & & & * & \mu & \lambda & & & & & \\
6 & & \mu & & & & * & & \lambda & & & & \\
7 & & & & & & & * & 2\mu & & \lambda & & \\
8 & & & \mu & & & & & * & \mu & & \lambda & \\
9 & & & & 2\mu & & & & & * & & & \lambda \\
10 & & & & & & & & & & * & 2\mu & \\
11 & & & & & \mu & & & & & & * & \mu \\
12 & & & & & & 2\mu & & & & & & * \\
\hline
\end{array}
$$

$$
Q_3 = \begin{array}{c|cccccccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\hline
1 & * & \lambda & & & & & & & & & & & & & & & & & & \\
2 & & * & \lambda & & & & & & & & & & & & & & & & & \\
3 & & & * & \mu & \lambda & & & & & & & & & & & & & & & \\
4 & \mu & & & * & & \lambda & & & & & & & & & & & & & & \\
5 & & & & & * & \mu & \lambda & & & & & & & & & & & & & \\
6 & & \mu & & & & * & & \lambda & & & & & & & & & & & & \\
7 & & & & & & & * & 2\mu & & \lambda & & & & & & & & & & \\
8 & & & \mu & & & & & * & \mu & & \lambda & & & & & & & & & \\
9 & & & & 2\mu & & & & & * & & & \lambda & & & & & & & & \\
10 & & & & & & & & & & * & 2\mu & & \lambda & & & & & & & \\
11 & & & & & \mu & & & & & & * & \mu & & \lambda & & & & & & \\
12 & & & & & & 2\mu & & & & & & * & & & \lambda & & & & & \\
13 & & & & & & & & & & & & & * & 3\mu & & & \lambda & & & \\
14 & & & & & & & \mu & & & & & & & * & 2\mu & & & \lambda & & \\
15 & & & & & & & & 2\mu & & & & & & & * & \mu & & & \lambda & \\
16 & & & & & & & & & 3\mu & & & & & & & * & & & & \lambda \\
17 & & & & & & & & & & & & & & & & & * & 3\mu & & \\
18 & & & & & & & & & & \mu & & & & & & & & * & 2\mu & \\
19 & & & & & & & & & & & 2\mu & & & & & & & & * & \mu \\
20 & & & & & & & & & & & & 3\mu & & & & & & & & * \\
\hline
\end{array}
$$

3. Examples of the transition matrices. Now we shall find the matrices Q_n for $n = 1, 2, 3$. In every case the states of the process $X(t)$ may be written in the form of a triplet (M, N_1, N_2). In the case $n = 3$ the diagram of all states of the process $X(t)$ and all immediate transitions between them is shown in Fig. 1. In diagram 1 immediate transitions are denoted by arrows and the corresponding transition intensities are written along them.

Transition intensities are easy to find here. For example, to find the transition intensity from the state $(2, 2, 0)$ to the state $(2, 1, 1)$ consider the time interval $[t_0, t_0+h]$ of length h and assume that $X(t_0) = (2, 2, 0)$. The transition from this state to state $(2, 1, 1)$ in this interval occurs if exactly one of the items being in the second phase of service at t_0 completes that phase before t_0+h and simultaneously the item being in the second arrival phase remains in that phase up to t_0+h. The probability of this is equal to

$$\binom{2}{1}(1-e^{-\mu h})e^{-\mu h} = 2\mu h + o(h).$$

Other ways of the same transition have probability $o(h)$, hence the corresponding transition intensity equals

$$\lim_{h\to 0}\frac{1}{h}\big(2\mu h + o(h)\big) = 2\mu.$$

Rejecting the two last rows in the diagram for $n = 3$ we obtain the corresponding diagram for the case $n = 2$ and rejecting the four last rows – the diagram for $n = 1$.

To construct a transition matrix Q_n the states have to be ordered in some way. We shall number the states shown in diagram 1 by rows so that, for $n = 3$, $(2, 0, 0)$ will be the state number 1 and $(1, 0, 3)$ will be the state number 20. The complete transition matrices Q_1, Q_2, Q_3 are given on p. 190. For better ilustration of their pattern only some zero transitions have been shown. Asterisks on the main diagonals stand for the negative elements, their absolute values equal to the sums of the remaining elements in the corresponding rows.

The partition of matrices Q_n into separate non-zero submatrices will be used to construct recurrent formulas.

4. Recurrent and explicit formulas. The pattern of matrices Q_1, Q_2, Q_3 sugests the recurrent construction of the next matrices Q_n. Before the deduction of recurrent formulas we shall introduce the notation for non-zero submatrices of Q_n:

$$M_0 = \begin{bmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{bmatrix},$$

$$A_n = \begin{bmatrix} -\lambda-n\mu & n\mu & 0 & \dots & 0 \\ 0 & -\lambda-n\mu & (n-1)\mu & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu \\ 0 & 0 & 0 & \dots & -\lambda-n\mu \end{bmatrix} \text{of size } (n+1)\times(n+1),$$

$$B_n = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 0 \end{bmatrix} \quad \text{of size } n\times(n+1),$$

$$C_n = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda \end{bmatrix} \quad \text{of size } (n+1)\times(n+1),$$

$$D_n = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mu & 0 & \dots & 0 \\ 0 & 2\mu & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n\mu \end{bmatrix} \quad \text{of size } (n+1)\times n,$$

$$E_n = \begin{bmatrix} -n\mu & n\mu & 0 & \dots & 0 & 0 \\ 0 & -n\mu & (n-1)\mu & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -n\mu & \mu \\ 0 & 0 & 0 & \dots & 0 & -n\mu \end{bmatrix} \quad \text{of size } (n+1)\times(n+1).$$

Let us notice that Q_1, Q_2, Q_3 may be written in the form

$$(7) \qquad Q_1 = \begin{array}{|c|c|c|c|} \hline \multicolumn{2}{|c|}{M_0} & B_1 & \\ \hline D_1 & & A_1 & C_1 \\ \hline & D_1 & & E_1 \\ \hline \end{array},$$

$$Q_2 = \left[\begin{array}{cccccc} \multicolumn{2}{c}{M_0} & B_1 & & & \\ D_1 & & A_1 & C_1 & & \\ & D_1 & & A_1 & B_2 & \\ & & D_2 & & A_2 & C_2 \\ & & & D_2 & & E_2 \end{array}\right], \tag{8}$$

$$Q_3 = \left[\begin{array}{cccccccc} \multicolumn{2}{c}{M_0} & B_1 & & & & & \\ D_1 & & A_1 & C_1 & & & & \\ & D_1 & & A_1 & B_2 & & & \\ & & D_2 & & A_2 & C_2 & & \\ & & & D_2 & & A_2 & B_3 & \\ & & & & D_3 & & A_3 & C_3 \\ & & & & & D_3 & & E_3 \end{array}\right]. \tag{9}$$

The following two theorems provide recurrent and explicit formulas for matrices Q_n.

THEOREM 1. *Let M_n be the matrix of size $(n+1)(n+2)\times(n+1)(n+2)$ defined by the recurrent formula*

$$M_n = \left[\begin{array}{ccccc} \multicolumn{3}{c}{M_{n-1}} & B_n & \\ & D_n & & A_n & C_n \\ & & D_n & & A_n \end{array}\right], \quad n = 1, 2, \ldots \tag{10}$$

Then

$$Q_n = \left[\begin{array}{ccccc} \multicolumn{3}{c}{M_{n-1}} & B_n & \\ & D_n & & A_n & C_n \\ & & D_n & & E_n \end{array}\right], \quad n = 1, 2, \ldots \tag{11}$$

THEOREM 2. *The elements* $q_{ij}^{(n)}$ $(i, j = 1, 2, \ldots, S_n)$ *of the matrix* Q_n *are as follows:*

$$q^{(n)}_{i(i-1)+m,i^2+m} = \lambda, \quad m = 1, 2, \ldots, 2i \text{ and } i = 1, 2, \ldots, n, \tag{12}$$

$$q^{(n)}_{(n+1)n+m,(n+1)^2+m} = \lambda, \quad m = 1, 2, \ldots, n+1, \tag{13}$$

$$q^{(n)}_{i(i-1)+m,i(i-1)+m} = -\lambda-(i-1)\mu, \tag{14}$$
$$m = 1, 2, \ldots, 2i \text{ and } i = 1, 2, \ldots n,$$

$$q^{(n)}_{(n+1)n+m,(n+1)n+m} = \begin{cases} -\lambda-n\mu, & m = 1, 2, \ldots, n+1, \\ -n\mu, & m = n+2, n+3, \ldots, 2(n+1), \end{cases} \tag{15}$$

$$q^{(n)}_{i(i-1)+m,i(i-1)+m+1} = (i-m)\mu, \tag{16}$$
$$m = 1, 2, \ldots, i-1 \text{ and } i = 2, 3, \ldots, n+1,$$

$$q^{(n)}_{i^2+m,i^2+m+1} = (i-m)\mu, \tag{17}$$
$$m = 1, 2, \ldots, i-1 \text{ and } i = 2, 3, \ldots, n+1,$$

$$q^{(n)}_{i^2-m+1,(i-1)^2-m+1} = (i-m)\mu, \tag{18}$$
$$m = 1, 2, \ldots, i-1 \text{ and } i = 2, 3, \ldots, n+1,$$

$$q^{(n)}_{i(i+1)-m+1,i(i-1)-m+1} = (i-m)\mu, \tag{19}$$
$$m = 1, 2, \ldots, i-1 \text{ and } i = 2, 3, \ldots, n+1,$$

$$q^{(n)}_{ij} = 0 \text{ otherwise.} \tag{20}$$

Proof of Theorem 1. We proceed by induction. From (7), (8) and (9) it is easy to see that formula (11) holds for $n = 1, 2, 3$. Having assumed that (11) holds for $n = k$, we prove now that Q_{k+1} is of the form

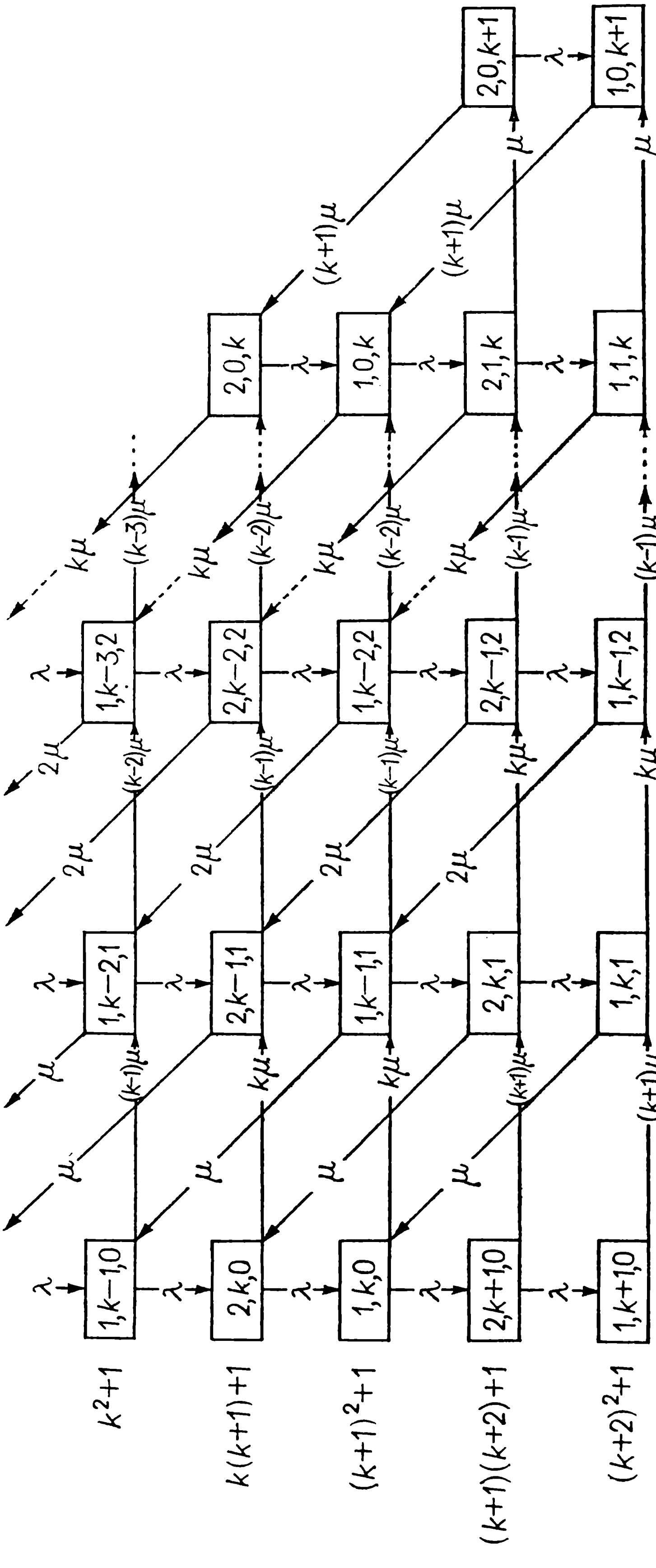
$$Q_{k+1} = \left|\begin{array}{cccccc} \multicolumn{2}{c}{M_{k-1}} & B_k & & & \\ D_k & & A_k & C_k & & \\ & D_k & & A_k & B_{k+1} & \\ & & D_{k+1} & & A_{k+1} & C_{k+1} \\ & & & D_{k+1} & & E_{k+1} \end{array}\right| \tag{21}$$

The transition diagram for the system $E_2/E_2/(k+1)$ may be obtained from the transition diagram for the system $E_2/E_2/k$ adding two rows of the states and the corresponding immediate transitions. The added transitions connect the states of the last four rows of the new diagram. The last five rows of transition diagram for the system $E_2/E_2/(k+1)$ are shown in Fig. 2. In each row there is written the number of the first state in this row.

The transitions between the states not mentioned in Fig. 2 and the states being in the first and second rows of the diagram in Fig. 2 are identical for both the systems $E_2/E_2/k$ and $E_2/E_2/(k+1)$. This justifies the presence of submatrices M_{k-1}, B_k, D_k, A_k and C_k in the two upper rows of the submatrices on the right-hand side of (21). Transition intensities between states of the third row and the states of the fourth row of the diagram form the submatrix B_{k+1}. Transition intensities between states of the third row form overdiagonal elements of the submatrix A_k in the third row of the submatrices in (21). Transition intensities between states of the third row and the states of the first row form submatrix D_k in the third row of the submatrices in (21). Transition intensities between states of the fourth row and the states of the fifth row form the submatrix C_{k+1}. Transition intensities between states of the fourth row form overdiagonal elements of the submatrix A_{k+1}. Transition intensities between the states of the fourth row and the states of the second row form the submatrix D_{k+1} being in the last but one row of the submatrices in (21). Transition intensities between the states of the fifth row and the states of the third row form submatrix D_{k+1} being in the last row of the submatrices in (21). Transition intensities between states of the fifth row form overdiagonal elements of the submatrix E_{k+1}. Diagonal elements of submatrices A_k, A_{k+1} and E_{k+1} are obtained with the help of relation (4).

Proof of Theorem 2. Formulas (12)-(20) in the thesis of theorem 2 follow from theorem 1. Formula (12) for $m = i = 1$ represents the overdiagonal elements of the submatrix M_0 and for other values of m and i the elements of submatrices B_k $(k = 1, 2, \ldots, n)$ and the elements of submatrices C_k $(k = 1, 2, \ldots, n-1)$. Formula (13) represents the elements of submatrix C_n. Formula (14) represents the diagonal elements of submatrix M_0 and the diagonal elements of submatrices A_k $(k = 1, 2, \ldots, n-1)$. Formula (15) represents the diagonal elements of submatrices A_n and E_n. Formulas (16) and (17) represent overdiagonal elements of submatrices A_k $(k = 1, 2, \ldots, n)$ and formulas (18) and (19) the elements of submatrices D_k $(k = 1, 2, \ldots, n)$ (here it is worth to note that the indices of the intensities in formulas (17) and (19) are suitable modifications of those in formulas (16) and (18)).

5. Example of the limiting distributions of $N(t)$. Explicit formulas for elements of the matrix Q_n may be used to build a simple algorithm for the construction and numerical solution of the system of equations (3)



2,0,k+1
1,0,k+1
2,0,k
1,0,k
2,1,k
1,1,k
1,k−3,2
2,k−2,2
1,k−2,2
2,k−1,2
1,k−1,2
1,k−2,1
2,k−1,1
1,k−1,1
2,k,1
1,k,1
1,k−1,0
2,k,0
1,k,0
2,k+1,0
1,k+1,0
k^2+1
k(k+1)+1
(k+1)^2+1
(k+1)(k+2)+1
(k+2)^2+1
λ
μ
2μ
kμ
(k+1)μ
(k−1)μ
(k−2)μ
(k−3)μ

Fig. 2

with condition (5) and the calculation of limiting probabilities P_k ($k = 0, 1, \ldots, n$) for the process $N(t)$. We present here the results of a real application of the theory to analyse the work of railway classification yards. From the analysis of the empirical data in [1] (this problem has been studied also in [3]) follows that the system $E_2/E_2/n$ may be applied. It is assumed here that the expected value of the interarrival times of trains for a given station is equal to $2/\lambda = 37.0$ (min) and the expected value of the service time of a single train is $2/\mu = 195.2$ (min). The results of calculations are presented in Table 1. Obtained distributions indicate that the number $n = 7$ of tracks is sufficient to assure the effective work of the system.

An ALGOL algorithm for the calculation of the limiting distribution of the process $N(t)$ in an $E_2/E_2/n$ queueing system is going to be published in the next issue of Applicationes Mathematicae.

TABLE 1

k \ n	1	2	3	4	5	6
0	0.125	0.037	0.019	0.013	0.011	0.011
1	0.875	0.256	0.127	0.088	0.073	0.068
2		0.707	0.337	0.228	0.189	0.175
3			0.517	0.336	0.275	0.253
4				0.335	0.264	0.241
5					0.187	0.165
6						0.087
$\sum_{k=0}^{n} kP_k$	0.875	1.669	2.352	2.892	3.267	3.489

References

[1] K. Baron, *Opracowanie wyników obserwacji pracy grupy odjazdowej torów kolejowej stacji rozrządowej*, Master's diploma thesis, University of Wrocław, 1971.

[2] W. Feller, *An introduction to probability theory and its applications*, New York 1966.

[3] J. Węgierski, *Metody probabilistyczne w projektowaniu transportu szynowego*, Wydawnictwo Komunikacji i Łączności, Warszawa 1971.

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JAWNE WZORY NA INTENSYWNOŚCI PRZEJŚCIA W SYSTEMIE MASOWEJ OBSŁUGI $E_2/E_2/n$

STRESZCZENIE

W pracy rozpatruje się system masowej obsługi, w którym odstępy między zgłoszeniami do systemu są niezależnymi zmiennymi losowymi o jednakowym rozkładzie Erlanga rzędu 2 i czas obsługi pojedynczej jednostki ma rozkład Erlanga rzędu 2. Zgłoszenia są obsługiwane w n niezależnych liniach obsługi, przy czym jednostka rezygnuje z obsługi, jeśli zastaje wszystkie linie zajęte.

Dla tego systemu analizuje się stan zdefiniowany jako liczba jednostek znajdujących się w systemie przy użyciu metody rozbudowanego procesu Markowa $X(t)$ o skończonej liczbie stanów. Twierdzenie 1 podaje rekurencyjne wzory dla macierzy intensywności przejścia między stanami procesu $X(t)$, natomiast twierdzenie 2 podaje jawne wzory dla tych intensywności.

Podane wzory zastosowano do analizy pracy grupy odjazdowej torów kolejowej stacji rozrządowej, dla której w oparciu o rzeczywiste parametry obliczono rozkłady liczby zajętych torów.