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A SILENT-NOISY VERSUS SILENT DUEL

1. Introduction. Games of timing with several actions on both sides were solved in [1] and [2], where a kind of symmetry in the information pattern was assumed. The asymmetric case has not been generally solved yet. The present paper as well as [3] and [4] gives a solution for simplified asymmetric information pattern duels.

We consider the following model of a duel: two opponents, denoted by A and B, have two bullets and one bullet, respectively. The first shot of A and the shot of B are not heard by the opponent. The second shot of A is heard by B. The probability of hitting the opponent is a function of time, $P(t)$ and $Q(t)$ for A and B, respectively, and it satisfies the usual regularity conditions [4]. The pay-off is defined in a usual way (see [4]).

In Section 2 we present the normal form of the game and the theorem concerning the existence and the analytic form of optimal strategies for both players. The proof of the theorem is divided into three parts. In Section 3 we find a pair of the so-called corresponding equalizer strategies, in Section 4 we prove the existence and uniqueness of the solution for a system of equations describing the strategies, and in Section 5 the optimality of the strategies is proved. A numerical example is given in Section 6.

2. Normal form for the game of timing. According to the description of the game, the set of pure strategies of player A is given by

$$X = \{\bar{x} = (x_1, x_2) \in R^2, 0 \leqslant x_1 \leqslant x_2 \leqslant 1\}.$$

Clearly, player B still having a bullet, after he has heard the second shot of A, shoots at $t = 1$ when he is sure to hit the opponent. Since we shall take it into account in the definition of the pay-off function, we may consider $Y = [0, 1]$ as the set of all pure strategies of B.

The *pay-off function* is an expected value of the pay-off for A evaluated with respect to the probabilities of hitting $P(t)$ and $Q(t)$.

One can easily find that the pay-off function is of the form

$$K(\bar{x}, y) = \tag{1}$$

$$= 1-[1-P(x_1)][1-P(x_2)][1-Q(y)]-\begin{cases} 2Q(y) & \text{if } y < x_1 < x_2, \\ 2Q(y)[1-P(x_1)] & \\ & \text{if } x_1 < y < x_2, \\ [1+Q(y)][1-P(x_1)][1-P(x_2)] & \\ & \text{if } x_1 < x_2 < y. \end{cases}$$

It is easy to check that the game $\Gamma = \langle X, Y, K\rangle$ has no solution in pure strategies, so one has to consider a randomized extension of Γ.

Let us define some classes of mixed strategies for both players in which we shall seek optimal solutions.

Thus, $F(\bar{x}) = F(x_1, x_2) \in \mathcal{F}$ if

$$F(\bar{x}) = F_1(x_1)F_2(x_2) \tag{2a}$$

and

$$F_1(x_1) = \begin{cases} 0 & \text{for } x_1 \leqslant a_1, \\ \int\limits_{a_1}^{x_1} f_1(t)\,dt & \text{for } a_1 \leqslant x_1 \leqslant a_2, \\ 1 & \text{for } a_2 \leqslant x_1, \end{cases} \tag{2b}$$

$$F_2(x_2) = \begin{cases} 0 & \text{for } x_2 \leqslant a_2, \\ \int\limits_{a_2}^{x_2} f_2(t)\,dt & \text{for } a_2 \leqslant x_2 < 1, \\ 1 & \text{for } 1 \leqslant x_2, \end{cases} \tag{2c}$$

where $0 < a_1 < a_2 < 1$; $f_1(t) > 0$, $t \in [a_1, a_2]$; $f_2(t) > 0$, $t \in [a_2, 1]$; and

$$\int\limits_{a_1}^{a_2} f_1(t)\,dt = 1, \tag{2d}$$

$$\int\limits_{a_2}^{1} f_2(t)\,dt = 1-a, \qquad a \in (0,1). \tag{2e}$$

The class of mixed strategies for B is defined as follows: $G \in \mathcal{G}$ if

$$G(y) = \begin{cases} 0 & \text{for } y \leqslant a_1, \\ \int\limits_{a_1}^{y} g(t)\,dt & \text{for } a_1 \leqslant y \leqslant 1, \\ 1 & \text{for } y \geqslant 1, \end{cases} \tag{3}$$

where $0 < a_1 < 1$ and $g(t) > 0$ for every $t \in [a_1, 1]$.

Now, let us introduce some useful notation:

$$D_1 = \int_{a_1}^{a_2} P(x_1)dF_1(x_1), \quad D_2 = \int_{a_2}^{1} P(x_2)dF_2(x_2), \tag{4}$$
$$E_1 = \int_{a_1}^{a_2} Q(y)dG(y), \quad E_2 = \int_{a_2}^{1} Q(y)dG(y), \quad E = E_1 + E_2.$$

In the subsequent sections we prove the following

THEOREM. *The game of timing Γ has a solution in mixed strategies. The optimal mixed strategies for players A and B belong to classes $\mathcal{F}$ and $\mathcal{G}$, respectively, and are described by the following set of equalities:*

$$f_1(t) = \frac{h_1 Q'(t)}{P(t)Q^2(t)}, \tag{5}$$

$$f_2(t) = \frac{2aQ'(t)T(t)}{W(t)}, \tag{6}$$

$$g(y) = \begin{cases} \dfrac{EP(a_1)P'(y)}{Q(y)P^2(y)} & \text{for } y \in [a_1, a_2), \\ \dfrac{(1-E)P'(y)S(y)}{W(y)} & \text{for } y \in [a_2, 1]. \end{cases} \tag{7}$$

The value of the game equals

$$v = 1 - 2[1 - P(a_1)]E. \tag{8}$$

In formulas (5)-(8),

$$W(t) = P(t)Q(t) + P(t) + Q(t) - 1,$$
$$T(t) = \exp\left\{ \int_t^1 \frac{Q'(u)[1+P(u)]}{W(u)} du \right\}, \tag{9}$$
$$S(t) = \exp\left\{ \int_t^1 \frac{P'(u)[1+Q(u)]}{W(u)} du \right\};$$

parameters h_1 and E are given by

$$E^{-1} = 1 + \frac{2P(a_1)}{P(a_2)[1-P(a_2)]S(a_2)}, \tag{10}$$

$$h_1^{-1} = \int_{a_1}^{a_2} \frac{Q'(t)dt}{P(t)Q^2(t)}; \tag{11}$$

and constants a_1, a_2 and a are the unique solutions of the following system of equations:

$$a = \{[1+Q(a_2)]T(a_2)-1\}^{-1}, \tag{12}$$

$$\int_{a_1}^{a_2} \frac{Q'(t)\,dt}{P(t)Q^2(t)} = \frac{1}{Q(a_1)} - \frac{1}{Q(a_2)T(a_2)} + 1, \tag{13}$$

$$\int_{a_1}^{a_2} \frac{P'(t)\,dt}{Q(t)P^2(t)} = \frac{1}{P(a_1)} + \frac{1}{P(a_2)[1-P(a_2)]}\left[\frac{1}{S(a_2)} - P(a_2) - 1\right]. \tag{14}$$

3. Classes of corresponding equalizer strategies. Let us introduce the following notation:

$$M(y) = \int_X K(\bar{x}, y)\,dF(\bar{x}) \quad \text{and} \quad N(\bar{x}) = \int_Y K(\bar{x}, y)\,dG(y),$$

where F and G are arbitrary mixed strategies of A and B, respectively.

F and G are called *corresponding equalizer strategies* [2] if

$$M(y) = \underline{v} \quad \text{for every } y \in \operatorname{supp} G \setminus \{1\},$$

$$N(\bar{x}) = \bar{v} \quad \text{for every } \bar{x} \in \operatorname{supp} F,$$

where $\underline{v}$ and $\bar{v}$ are some constants.

In this section we prove that there exist corresponding equalizer strategies and that they are of the form stated in the Theorem.

Let $y \in \operatorname{supp} G \setminus \{1\}$, $G \in \mathcal{G}$ and $F \in \mathcal{F}$. Then, by (1), (2a)-(2e) and (4), we have

$$M(y) = 1-(1-D_1)(1-D_2) - Q(y)\left[2\int_y^{a_2} P(x_1)f_1(x_1)\,dx_1 + (1-D_1)(1+D_2)\right] \quad \text{for } y \in [a_1, a_2]$$

and

$$M(y) = 1-(1-D_1)(1-D_2) - (1-D_1)\left[\int_{a_2}^{y} \big(1+Q(y)\big)\big(1-P(x_2)\big)f_2(x_2)\,dx_2 + {}\right.$$
$$\left. {} + 2Q(y)\int_y^1 f_2(x_2)\,dx_2 + Q(y)(D_2+2a-1)\right] \quad \text{for } y \in [a_2, 1).$$

We require that the function $M(y)$ be constant in $[a_1, 1)$ or $M'(y) = 0$ for $y \in [a_1, 1)$. Hence we obtain

$$\frac{2P(y)f_1(y)}{2\int_y^{a_2} P(x_1)f_1(x_1)\,dx_1 + (1-D_1)(1+D_2)} = \frac{Q'(y)}{Q(y)} \quad \text{for } y \in [a_1, a_2] \tag{15}$$

and

$$\frac{f_2(y)}{\int\limits_{a_2}^{y}[1-P(x_2)]f_2(x_2)\,dx_2+2\int\limits_{y}^{1}f_2(x_2)\,dx_2+2a-1+D_2} = \frac{Q'(y)}{W(y)} \tag{16}$$

for $y \in [a_2, 1)$.

The integral equation (15) has the unique solution given by (5). From (15) and (5) it follows also that

$$2h_1 = Q(a_2)(1-D_1)(1+D_2). \tag{17}$$

Now, multiplying both sides of (16) by $-[1+P(y)]$ and using (2e), we obtain

$$\frac{-[1+P(y)]f_2(y)}{a+D_2-\int\limits_{a_2}^{y}P(x_2)f_2(x_2)\,dx_2+\int\limits_{y}^{1}f_2(x_2)\,dx_2} = \frac{-Q'(y)[1+P(y)]}{W(y)}. \tag{16'}$$

Next, we use (4) in (16′) and finally we obtain the equation

$$\frac{-[1+P(y)]f_2(y)}{2a+\int\limits_{y}^{1}[1+P(x_2)]f_2(x_2)\,dx_2} = \frac{-Q'(y)[1+P(y)]}{W(y)}$$

and its solution given by (6).

Now, let $F \in \mathscr{F}$ be the strategy with $f_1(x_1)$ and $f_2(x_2)$ defined by (5) and (6), respectively. Since $M(y)$ is continuous at $y = a_2$, we see that

$$M(y) = 1-(1-D_1)(1-D_2)-Q(a_2)(1-D_1)(1+D_2) = \underline{v}$$

for every $y \in [a_1, 1)$.

Now, we deal with $N(\bar{x})$. Using (1) and the fact that $G \in \mathscr{G}$ we find that for $\bar{x} \in \operatorname{supp} F$

$$N(\bar{x}) = 1-2E_1+2\int\limits_{x_1}^{a_2} Q(y)P(x_1)g(y)\,dy-[1-P(x_1)]R(x_2), \tag{18}$$

where

$$R(x_2) = (1-E)[1-P(x_2)]+2\int\limits_{a_2}^{x_2} Q(y)g(y)\,dy+ \tag{19}$$

$$+[1-P(x_2)]\int\limits_{x_2}^{1}[1+Q(y)]g(y)\,dy.$$

We require that the function $N(x_1, x_2)$ be constant for $x_1 \in [a_1, a_2)$ and $x_2 \in [a_2, 1]$ or

$$\frac{\partial N}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial N}{\partial x_2} = -[1-P(x_1)]\frac{\partial R}{\partial x_2} = 0$$

in the corresponding intervals.

The second condition implies that

$$(20) \qquad \frac{-[1+Q(x_2)]g(x_2)}{1-E+\int\limits_{x_2}^{1}[1+Q(y)]g(y)dy} = \frac{-P'(x_2)[1+Q(x_2)]}{W(x_2)} \quad \text{for } x_2 \in [a_2, 1]$$

and that (7) is the solution of (20). From that condition it follows also that $R(x_2)$ is a constant in $[a_2, 1]$. We denote this constant by $2R_0$ and calculate

$$(21) \qquad 2R_0 = R(a_2) = [1-P(a_2)](1-E)S(a_2).$$

On the other hand, using (4) we get

$$(22) \qquad 2R_0 = R(1) = 2E_2.$$

Now, equation (18) takes the form

$$(18') \qquad N(\bar{x}) = 1-2E_1+2P(x_1)\int\limits_{x_1}^{a_2} Q(y)g(y)dy - 2R_0[1-P(x_1)],$$

and the condition $\partial N/\partial x_1 = 0$ for $x_1 \in [a_1, a_2)$ and $x_2 \in [a_2, 1]$ yields

$$(23) \qquad \frac{-Q(x_1)g(x_1)}{R_0+\int\limits_{x_1}^{a_2} Q(y)g(y)dy} = \frac{-P'(x_1)}{P(x_1)}.$$

The solution of (23) is

$$(24) \qquad g(y) = \frac{P(a_2)R_0P'(y)}{Q(y)P^2(y)} \quad \text{for } y \in [a_1, a_2].$$

By (4) we have

$$E_1 = P(a_2)R_0\left[\frac{1}{P(a_1)} - \frac{1}{P(a_2)}\right] = R_0\left[\frac{P(a_2)}{P(a_1)} - 1\right].$$

Hence

$$E_1 + R_0 = R_0\frac{P(a_2)}{P(a_1)},$$

and using (22) we obtain

$$R_0 P(a_2) = EP(a_1). \tag{25}$$

Finally, (25) brings (24) to (7).

Thus, we have proved that $N(\bar{x})$ is constant for every $\bar{x} \in \operatorname{supp} F$ iff $G \in \mathcal{G}$ and $g(y)$ is given by (7).

Putting $x_1 = a_2$ in (18′) and using (22) and (25), we notice that

$$\bar{v} = 1 - 2[1 - P(a_1)]E. \tag{26}$$

Further, by (21) and (25) we have

$$\frac{2E}{1-E} = \frac{P(a_2)[1-P(a_2)]S(a_2)}{P(a_1)}$$

or — after transformation — equality (10).

Now, we may evaluate parameters D_1 and D_2 in (4). We use (5) to obtain

$$D_1 = h_1\left[\frac{1}{Q(a_1)} - \frac{1}{Q(a_2)}\right]. \tag{27}$$

Eliminating D_1 in (17) and (27) we get

$$\frac{1}{\ } = \frac{1}{\ } + \frac{1-D_2}{\ }. \tag{28}$$

Next, by (3) and (7) we obtain

$$(32)\qquad \int_{a_1}^{a_2} \frac{EP(a_1)P'(y)}{Q(y)P^2(y)}dy + \int_{a_2}^{1} \frac{(1-E)P'(y)S(y)}{W(y)}dy = 1,$$

where E is given by (10).

Now, we shall get equivalent forms of (30), (11) and (32) in order to discuss the existence of the solution.

In the sequel we shall use the following relations:

$$(33)\qquad \begin{aligned} \frac{d}{dt}\{[1-P(t)]S(t)\} &= \frac{-2P'(t)Q(t)S(t)}{W(t)},\\ \frac{d}{dt}\{[1-Q(t)]T(t)\} &= \frac{-2Q'(t)P(t)T(t)}{W(t)}. \end{aligned}$$

The integral in (30), by integration by parts and (33), may be reduced to the form

$$\int_{a_2}^{1} \frac{Q'(x_2)}{W(x_2)}T(x_2)dx_2 = -1+T(a_2)\frac{1+Q(a_2)}{2},$$

which results in (12).

We can use equation (12) to find that

$$(34)\qquad \frac{1-aT(a_2)}{T(a_2)} = \frac{T(a_2)Q(a_2)-1}{T(a_2)}.$$

Applying (31) and (34) in (11) we obtain equation (13).

Finally, we discuss equation (32). Let us use (33) in the second integral in (32). Applying integration by parts we have

$$EP(a_1)\int_{a_1}^{a_2} \frac{P'(y)dy}{Q(y)P^2(y)} + (1-E)\left[\frac{1+P(a_2)}{2}S(a_2)-1\right] = 1$$

or

$$(35)\qquad \int_{a_1}^{a_2} \frac{P'(y)dy}{Q(y)P^2(y)} = \frac{2-E}{EP(a_1)} - \frac{(1-E)[1+P(a_2)]S(a_2)}{2EP(a_1)}.$$

Now, combining relations (35) and (10) we get equation (14).

Thus, instead of equations (30), (11) and (32) we consider the system (12), (13) and (14). We prove that this system has a unique solution a_1, a_2 and a.

First, let us define three auxiliary functions:

$$a(z_1) = \{[1+Q(z_1)]T(z_1)-1\}^{-1} \quad \text{for } z_1 \in (t_0, 1], \tag{36}$$

$$H_1(z_1, z_2) = \int_{z_2}^{z_1} \frac{Q'(u)du}{P(u)Q^2(u)} - \frac{1}{Q(z_2)} + \frac{1}{T(z_1)Q(z_2)} - 1 \tag{37}$$
$$\text{for } z_1 \in (t_0, 1], z_2 \in [0, z_1),$$

$$H_2(z_1, z_3) = \int_{z_3}^{z_1} \frac{P'(u)du}{Q(u)P^2(u)} - \frac{1}{P(z_3)} - \tag{38}$$
$$-\frac{1}{P(z_1)[1-P(z_1)]}\left[\frac{4}{S(z_1)} - P(z_1) - 1\right] \quad \text{for } z_1 \in (t_0, 1],\ z_3 \in [0, z_1].$$

We study some properties of these functions which are useful in discussing the existence and uniqueness of the solution for equations (12), (13) and (14). The variables z_1 and z_2, z_3 play a role of a_2 and a_1, respectively.

Thus, we see that $a(z_1)$ is continuous in $(t_0, 1]$, $a(1) = 1$, and

$$\lim_{z_1 \to t_0^+} a(z_1) = 0,$$

$$\frac{da}{dz_1} = \frac{2T(z_1)Q'(z_1)}{W(z_1)\left[T(z_1)[1+Q(z_1)]-1\right]^2} > 0.$$

Hence, there exists a unique value $a \in (0, 1)$ satisfying (12) for every fixed $z_1 = a_2 \in (t_0, 1)$.

Now, we prove that relations (37) and (38) define some functions $z_2 = z_2(z_1)$ and $z_3 = z_3(z_1)$ and we study the properties of the functions.

First, let us remark that the function $H_1(z_1, z_2)$ is continuous in its range of definition together with its partial derivatives

$$\begin{aligned} \frac{\partial H_1}{\partial z_1} &= \frac{Q'(z_1)}{Q^2(z_1)}\left[\frac{1}{P(z_1)} + \frac{1-P(z_1)}{T(z_1)W(z_1)}\right] > 0, \\ \frac{\partial H_1}{\partial z_2} &= -\frac{Q'(z_2)}{Q^2(z_2)}\left[\frac{1}{P(z_2)} - 1\right] < 0. \end{aligned} \tag{39}$$

Next, the function H_1 has the following properties:

$$\begin{aligned} H_1(z_1, z_1) &= -\frac{T(z_1) - 1 + T(z_1)Q(z_1)}{T(z_1)Q(z_1)} < 0, \\ &\lim_{z_2 \to 0^+} H_1(z_1, z_2) = +\infty, \\ &\frac{\partial H_1}{\partial z_2} < 0. \end{aligned} \tag{40}$$

Hence, for every fixed $z_1 \in (t_0, 1)$ there exists a unique value $z_2 \in (0, z_1)$ which satisfies the condition $H_1(z_1, z_2) = 0$.

Using inequalities (39) we find that

$$\frac{dz_2}{dz_1} = -\frac{\partial H_1/\partial z_1}{\partial H_1/\partial z_2} > 0 \quad \text{for } z_1 \in (t_0, 1]. \tag{41}$$

Taking into account (37) we obtain

$$\lim_{z_1 \to 1^-} z_2(z_1) = z_2(1) > 0, \quad \lim_{z_1 \to t_0^+} z_2(z_1) = z_2(t_0), \tag{42}$$

and, by (41),

$$z_2(t_0) < z_2(1).$$

Thus, equation (37) defines z_2 as a unique, differentiable and increasing function of z_1 in $(t_0, 1)$ with $z_2 \in (0, z_1)$ or for every fixed value of $z_1 = a_2 \in (t_0, 1)$ there is a unique value $z_2 = a_1 \in (0, a_2)$ satisfying equation (13).

Similarly we discuss relation (38). The partial derivatives of H_2 are the following:

$$\frac{\partial H_2}{\partial z_1} = \frac{P'(z_1)}{Q(z_1)P^2(z_1)} + \frac{P'(z_1)}{P^2(z_1)} - \frac{2P'(z_1)\big(1-2P(z_1)\big)}{P^2(z_1)\big(1-P(z_1)\big)^2} - \frac{4P'(z_1)\{2P(z_1)Q(z_1)-[1-P(z_1)]W(z_1)\}}{S(z_1)P^2(z_1)[1-P(z_1)]^2W(z_1)}, \tag{43}$$

$$\frac{\partial H_2}{\partial z_3} = -\frac{P'(z_3)[1-P(z_3)]}{Q(z_3)P^2(z_3)} < 0. \tag{44}$$

In the sequel we shall use the fact that the equation

$$P(z_1)S(z_1)-2 = 0$$

has exactly one solution in $(t_0, 1)$.

In order to prove the fact let us notice that

$$[P(z_1)S(z_1)]' = P'(z_1)Q(z_1)\frac{Q(z_1)-1}{W(z_1)} < 0,$$

$$\lim_{z_1 \to t_0^+} P(z_1)S(z_1) = +\infty, \quad \lim_{z_1 \to 1^-} P(z_1)S(z_1) = 1.$$

We shall use also the following inequality:

$$2P(z_1)Q(z_1)-[1-P(z_1)]W(z_1) > 0 \quad \text{for } z_1 \in (t_0, 1]. \tag{45}$$

Inequality (45) may be written as

$$P^2(z_1)[1+Q(z_1)]-[2P(z_1)-1][1-Q(z_1)] > 0,$$

and its validity is easily seen.

Now, we can specify the sign of $\partial H_2/\partial z_1$. By (40), (45) and (43) we obtain

$$\frac{\partial H_2}{\partial z_1} < \frac{P'(z_1)[1-Q(z_1)]\big[(1-P(z_1))W(z_1)-2P(z_1)Q(z_1)\big]}{P^2(z_1)Q(z_1)[1-P(z_1)]W(z_1)} < 0$$

$$\text{for } z_1 \in (t_1, 1),$$

where $P(t_1)S(t_1) = 2$.

Hence, by (44), we have

$$\frac{dz_3}{dz_1} = -\frac{\partial H_2/\partial z_1}{\partial H_2/\partial z_3} < 0 \quad \text{for } z_1 \in (t_1, 1).$$

Notice that the function H_2 has the following properties:

$$H_2(z_1, z_1) = -2\frac{2-P(z_1)S(z_1)}{P(z_1)[1-P(z_1)]} < 0 \quad \text{for } z_1 \in (t_1, 1),$$

$$\lim_{z_1\to 0^+} H_2(z_1, z_3) = +\infty.$$

Thus, for every fixed $z_1 \in (t_1, 1)$ there is a unique value $z_3 \in [0, z_1]$ satisfying the relation $H_2(z_1, z_3) = 0$.

Now, we use relation (38) to find that

$$\text{(46)} \qquad \lim_{z_1\to 1^-} z_3(z_1) = 0 \quad \text{and} \quad \lim_{z_1\to t_1^+} z_3(z_1) = t_1.$$

So, relation (38) defines z_3 as a unique differentiable function increasing in z_1 in $(t_1, 1)$ or for every fixed $z_1 = a_2 \in (t_1, 1)$ there is a unique value $z_3 = a_1 \in (0, a_2)$ satisfying equation (14).

Taking into account the properties of the functions $z_2(z_1)$ and $z_3(z_1)$ and equalities (42) and (46) we see that

$$z_2(t_1) < z_3(t_1), \qquad z_2(1) > \lim_{z_1\to 1^-} z_3(z_1)$$

and that there exists a unique value $z_1^* \in (t_1, 1)$ for which

$$z_2(z_1^*) = z_3(z_1^*).$$

Now, we put $a_2 = z_1^*$, $a_1 = z_2(z_1^*) = z_3(z_1^*)$ and $a = a(z_1^*)$ and obtain a unique solution of (12), (13) and (14). The solution and relations (10) and (11) define completely the strategies described in the previous section. We denote the strategies by T_A and T_B for players A and B, respectively.

To end this section, we notice that one can easily prove that

$$\underline{v} = \bar{v} = v,$$

where v is the value of the game given by (8).

5. Proof of optimality for T_A and T_B. To prove the optimality of T_A and T_B it is sufficient to show that

$$\min_{y\in Y} M(y) = v, \tag{47}$$

$$\max_{\bar{x}\in X} N(\bar{x}) = v. \tag{48}$$

For proving (47) we consider three cases.

1° Let $y \in [0, a_1)$. By (1) we have

$$K(\bar{x}, y) = 1-[1-P(x_1)][1-P(x_2)][1-Q(y)]-2Q(y),$$

and applying (4) we obtain

$$M(y) = D_1+(1-D_1)D_2-[1+D_1+(1-D_1)D_2]Q(y) > M(a_1) = v.$$

2° If $y \in [a_1, 1)$, then $M(y) = v$.

3° Let $y = 1$ and $y_\delta = 1-\delta$, $0 < \delta < 1-a_1$. We use equality (1) to find that

$$K(x_1, x_2; y_\delta)-K(x_1, x_2; 1)$$
$$= \begin{cases} 0 & \text{if } x_2 < y_\delta, \\ -[1-P(x_1)][P(x_2)Q(y_\delta)+P(x_2)+Q(y_\delta)-1] & \text{if } y_\delta < x_2 < 1, \\ -[1-P(x_1)][1-2Q(y_\delta)] & \text{if } x_2 = 1. \end{cases}$$

Now, we may take δ sufficiently small such that $Q(y_\delta) > 0.5$ and

$$P(x_2)Q(y_\delta)+P(x_2)+Q(y_\delta)-1 > W(y_\delta) > 0 \quad \text{for } x_2 \in (1-\delta, 1).$$

After integration we obtain

$$M(1) > M(y_\delta) = v.$$

Thus, taking into account all cases enumerated above we conclude that equality (47) is valid.

Now we discuss equality (48). There are six cases to be considered.

1° Let $0 \leqslant x_1 \leqslant x_2 < a_1$. Then player B applies his pure strategy $y = 1$, since he knows that A has no bullet. Using (1) we observe that $N(x_1, x_2) < N(a_1, a_1)$.

2° Let $a_1 \leqslant x_1 \leqslant x_2 < a_2$. Using (1) and performing some simplifying transformations, we obtain

$$(49)\quad N(x_1, x_2) = 1-2E_1+2P(x_1)\int_{x_1}^{a_2} Q(y)g(y)dy-[1-P(x_1)]U(x_2),$$

where

$$U(x_2) =$$

$$(1-E)[1-P(x_2)]+[1-P(x_2)]\int_{x_2}^{1}[1+Q(y)]g(y)dy-2\int_{x_2}^{a_2} Q(y)g(y)dy.$$

Here we observe that

$$U(x_2)-U(a_2) = (1-E)[P(a_2)-P(x_2)]-2\int_{x_2}^{a_2} Q(y)g(y)dy+$$

$$+[P(a_2)-P(x_2)]\int_{a_2}^{1}[1+Q(y)]g(y)dy+[1-P(x_2)]\int_{x_2}^{a_2}[1+Q(y)]g(y)dy.$$

Now, we use the evident inequality $1+Q(t) > 2Q(t)$ and formula (7) to find that

$$(50)\qquad U(x_2)-U(a_2) > [P(a_2)-P(x_2)](1-E)P(a_2)S(a_2) > 0.$$

From (49) and (50) we conclude that $N(x_1, x_2) < N(x_1, a_2) = v$.

3° Let $0 < x_1 < a_1 \leqslant x_2 < a_2$. Then by (1) we have

$$N(x_1, x_2) = 1-[1-P(x_1)]\Big\{(1-E)[1-P(x_2)]+\int_{a_1}^{x_2} 2Q(y)g(y)dy+$$

$$+\int_{x_2}^{1}[1+Q(y)][1-P(x_2)]g(y)dy\Big\}$$

and it is easy to see that $N(x_1, x_2) < N(a_1, x_2) = v$.

4° Let $x_1 \in [a_1, a_2)$ and $x_2 \in [a_2, 1]$. Then we have already known that $N(x_1, x_2) = v$.

5° Let $a_2 < x_1 \leqslant x_2 \leqslant 1$. Using (1) and performing some simplifying transformations, we obtain

$$N(x_1, x_2) = 1-2E_1-2R_0+2P(x_1)\Big[R_0-\int_{a_2}^{x_1} Q(y)g(y)dy\Big],$$

where R_0 is given by (21). From (7), (21) and (33) we get

$$\int_{a_2}^{x_1} Q(y)g(y)dy = \frac{1-E}{2}\{[1-P(a_2)]S(a_2)-[1-P(x_1)]S(x_1)\}$$

and, therefore,

$$N(x_1, x_2) = 1-2E_1-2R_0+(1-E)[1-P(x_1)]P(x_1)S(x_1).$$

Now, we find that

$$\{[1-P(x_1)]P(x_1)S(x_1)\}' = \frac{P(x_1)S(x_1)}{W(x_1)}\{[1-P(x_1)]W(x_1)-2P(x_1)Q(x_1)\},$$

and from inequality (47) and the fact that $a_2 > t_0$ we infer that $N(x_1, x_2)$ is a decreasing function of x_1 and $N(x_1, x_2) < N(a_2, x_2) = v$.

6° Let $x_1 < a_1$ and $x_2 \in [a_2, 1]$. Similarly as in case 3° we find that

$$N(x_1, x_2) = 1-[1-P(x_1)][2E_1+R(x_2)],$$

where the function $R(x_2)$ is given in (33). Since $R(x_2) = 2R_0 > 0$, we see that $N(x_1, x_2) < N(a_1, x_2) = v$.

Taking into account the six cases considered above we notice that equality (47) is true.

Thus, by (47) and (48) we have the optimality of strategies stated in the Theorem.

6. A numerical example. Let us assume that $P(t) = Q(t) = t$. The system of equations (12), (13) and (14) is of the form

$$\frac{1+a_2}{\sqrt{2W(a_2)}} = \frac{1+a}{2a},$$

$$\frac{1}{2a_1^2}-\frac{1}{2a_2^2} = \frac{1}{a_1}+1-\frac{1}{a_2\sqrt{2/W(a_2)}},$$

$$\frac{1}{2a_1^2}-\frac{1}{2a_2^2} = \frac{1}{a_1}+\frac{1}{a_2(1-a_2)}\left(\frac{4}{\sqrt{2/W(a_2)}}-1-a_2\right),$$

where $W(t) = t^2+2t-1$ and $W(t_0) = 0$ for $t_0 = \sqrt{2}-1$. We find that $a_1 = 0.3061$, $a_2 = 0.5224$ and $a = 0.3543$. Applying (10), (11) and (8) we obtain $E_1 = 0.5057$, $h_1 = 0.4834$ and $v = 0.2929$, respectively. One can compare the value of the considered game with the value for the two-silent versus one-silent duel [2], where it is equal to 0.3065.

Finally, equalities (5), (6) and (7) take the form

$$f_1(t) = h_1t^{-3},$$

$$f_2(t) = 2\sqrt{2}\,a[W(t)]^{-3/2},$$

$$g(y) = \begin{cases} a_1Ey^{-3} & \text{for } y \in [a_1, a_2), \\ (1-E)\sqrt{2}[W(y)]^{-3/2} & \text{for } y \in [a_2, 1], \end{cases}$$

respectively.

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