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ON A MODEL OF A QUEUE WITH DELAYED FEEDBACK

In most models considered in the queueing theory a Poisson input is assumed (1), with constant intensity, depending on time or on the number of customers appearing in the system at a given moment. This last situation arises in models with feedback regulating the input intensity according to the number of customers appearing in the system. In almost all practical situations, however, this feedback acts with some delay. This is so for instance in systems in which the customer decides to enter the queue with a certain probability depending on the length of the queue but enters the system only some time after the moment of making the decision. As an example of such a model imagine a shop with an attractive commodity; the customers of that shop inform their acquaintances of the length of the queue at the moment when they were leaving the shop. A similar situation appears in the stochastic population model (2) in which the intensity of regeneration is proportional to the number of grown-up individuals.

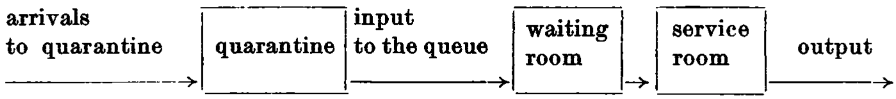
Let us examine a classical situation with a single unlimited queue and with a single service-channel. The customers are served in the order of their arrival and the service-time is a random variable with exponential distribution $F(x) = 1-e^{-\mu x}$. If the input is a Poisson process with intensity $\lambda(t) = \lambda[n(t)]$, $\lambda(t)$ being a function of the actual state of the system $n(t)$, we have to do with an instantaneous feedback, and the state of the system $n(t)$ (number of customers in the system) is a Markov process. Assuming that the intensity of the Poisson input is $\lambda(t) = \lambda[n(t-\tau)]$, we have to do with a feedback with constant delay τ, but the state of the system is not a Markov process. If, in a system with instantaneous feedback the entering customers pass a quarantine whose duration-time is a random variable with exponential distribution $G(x) = 1-e^{-\nu x}$, the instantaneous feedback acts on the input with a certain random delay τ. A system without a quarantine is therefore a system with a random delayed feedback. In such a system $n(t)$ is not a Markov process,

(1) See [1], [3], [4].

(2) See the examples of the birth and death processes [2].

but, adding a second coordinate $m(t)$ (the number of customers in quarantine at the moment t), we get a two-dimensional Markov process $\langle m(t), n(t)\rangle$.

Our model may be represented as follows:



Let (S) denote a system consisting of quarantine, waiting room and service room, and let (s) denote a system without quarantine.

For (S) we suppose that the quarantine may accommodate any number of customers and that the duration-time of the quarantine is a random variable with exponential distribution $G(x) = 1-e^{-\nu x}$. We suppose further that the waiting-room has an unlimited number of places and that the service is in the order of arrival; there is one service-channel with exponential service-time distribution $F(x) = 1-e^{-\mu x}$. We assume also the mutual independence of quarantine, queue, and service.

Now we prove

THEOREM 1. *If in system* (S) *the input is Poissonian with intensity* $\lambda[n(t)]$ *depending on the actual state of system* (s) *then the state of the system* $\langle m(t), n(t)\rangle$ *is a two-dimensional Markov process.*

Proof. The state of the system at time t is fully determined by the state of the system at time t_0 ($t_0 < t$) and by three random factors: (a) moments of arrival for quarantine after time t_0, (b) moments of ending quarantine after time t_0, and (c) moments of finished services after time t_0. None of those factors depends on the past of the process (before t_0). This results from the Poisson input assumed for (a), and from the property of the exponential distribution of quarantine and service times for (b) and (c). This last property states that the quarantine (service) time of a customer after time t_0 does not depend on the duration of the quarantine (service) before t_0. A stochastic process whose development after moment t_0 depends only on its state at time t_0 and not on its past history is a Markov process.

We denote by

$$P_{i,k}(t) = \Pr[\langle m(t) = i,\ n(t) = k\rangle]$$

the probability of the state $\langle m(t) = i, n(t) = k\rangle$ at moment t, by

$$P_{i,j;k,l}(t,\tau) = \Pr[\langle m(t+\tau) = j, n(t+\tau) = l\rangle \mid \langle m(t) = i, n(t) = k\rangle]$$

the probability of transition from state $\langle m(t) = i, n(t) = k\rangle$ at moment t to state $\langle m(t+\tau) = j, n(t+\tau) = l\rangle$ at moment $t+\tau$, and by $\lambda_k = \lambda[n(t)]$ the input intensity for $n(t) = k$.

It is easy to verify the following identities for $i \geqslant 1$, $k \geqslant 1$:

$$
\begin{aligned}
P_{i,i-1;\,k-1,k}(t,\tau) &= i\nu\tau+o(\tau),\\
P_{i-1,i-1;\,k,k-1}(t,\tau) &= \mu\tau+o(\tau),\\
P_{i-1,i;\,k-1,k-1}(t,\tau) &= \lambda_{k-1}\tau+o(\tau),\\
P_{i,i;\,k,k}(t,\tau) &= 1-[\lambda_k+i\nu+\mu]\tau+o(\tau),\\
P_{0,0;\,0,0}(t,\tau) &= 1-\lambda_0\tau+o(\tau),\\
P_{0,0;\,k,k}(t,\tau) &= 1-[\lambda_k+\mu]\tau+o(\tau),\\
P_{i,i;\,0,0}(t,\tau) &= 1-[\lambda_0+i\nu]\tau+o(\tau)
\end{aligned}
$$

and $P_{i,j;\,k;l}(t,\tau) = o(\tau)$ for all remaining cases.

As an example let us verify the first identity; similarly the remaining ones may be proved. A transition from state $\langle m(t) = i, n(t) = k-1\rangle$ to state $\langle m(t+\tau) = i-1, n(t+\tau) = k\rangle$ is possible when the customer leaves the quarantine at the time $[t, t+\tau)$ and when in that time interval no customer asks for quarantine and no customer leaves the service-room. The probability of that event equals $[i\nu\tau+o(\tau)][1-\lambda_k\tau+o(\tau)][1-\mu\tau+ +o(\tau)] = i\nu\tau+o(\tau)$. Besides there are other ways of transition between those states; the cases where at the time $[t, t+\tau)$ k ($k \geqslant 1$) customers ask for quarantine, $k+1$ customers finish the quarantine, and k customers leave the service-room, have a probability of order $o(\tau)$ and we may omit them.

The Kolmogoroff-Chapman system of equations has in our scheme the following form:

$$
\begin{aligned}
P_{0,0}(t+\tau) &= (1-\lambda_0\tau)P_{0,0}(t)+\mu\tau P_{0,1}(t)+o(\tau),\\
P_{i,0}(t+\tau) &= (1-\lambda_0\tau-i\nu\tau)P_{i,0}(t)+\mu\tau P_{i,1}(t)+\lambda_0\tau P_{i-1,0}(t)+o(\tau), \quad i \geqslant 1,\\
P_{0,j}(t+\tau) &= (1-\lambda_j\tau)P_{0,j}(t)+\mu\tau P_{0,j+1}(t)+\nu\tau P_{1,j-1}(t)+o(\tau), \qquad j \geqslant 1,\\
P_{i,j}(t+\tau) &= (1-\lambda_j\tau-i\nu\tau-\mu\tau)P_{i,j}(t)+\mu\tau P_{i,j+1}(t)+(i+1)\nu\tau P_{i+1,j-1}(t)+\\
&\qquad +\lambda_j\tau P_{i-1,j}(t)+o(\tau), \qquad i\cdot j \geqslant 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P'_{0,0}(t) &= -\lambda_0 P_{0,0}(t)+\mu P_{0,1}(t),\\
P'_{i,0}(t) &= -(\lambda_0+i\nu)P_{i,0}(t)+\mu P_{i,1}(t)+\lambda_0 P_{i-1,0}(t), \qquad i \geqslant 1,\\
P'_{0,j}(t) &= -(\lambda_j+\mu)P_{0,j}(t)+\mu P_{0,j+1}(t)+\nu P_{1,j-1}(t), \qquad j \geqslant 1,\\
P'_{i,j}(t) &= -(\lambda_j+i\nu+\mu)P_{i,j}(t)+\mu P_{i,j+1}(t)+(i+1)\nu P_{i+1,j-1}(t)+\\
&\qquad +\lambda_j P_{i-1,j}(t), \qquad i\cdot j \geqslant 1.
\end{aligned} \tag{1}
$$

If we assume that $\lim_{t\to\infty} P_{i,j}(t) = P_{i,j}$ exists for all states and that we may pass to the limit in the unlimited system of equations (1), we get in the limit a steady-state system of equations

$$(2)\qquad \begin{aligned} -\lambda_0 P_{0,0} + \mu P_{0,1} &= 0, \\ -(\lambda_0 + i\nu)P_{i,0} + \mu P_{i,1} + \lambda_0 P_{i-1,0} &= 0, \qquad i \geqslant 1, \\ -(\lambda_j + \mu)P_{0,j} + \mu P_{0,j+1} + \nu P_{1,j-1} &= 0, \qquad j \geqslant 1, \\ -(\lambda_j + i\nu + \mu)P_{i,j} + \mu P_{i,j+1} + (i+1)\nu P_{i+1,j-1} + \lambda_j P_{i-1,j} &= 0, \qquad i\cdot j \geqslant 1, \end{aligned}$$

which can be solved with the norming condition $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{i,j} = 1$.

The state $n(t)$ of the system (s) is the second coordinate of the two-dimensional process $\langle m(t), n(t)\rangle$. Let us denote by $p_k(t) = \Pr[n(t) = k]$ the probability of the state $n(t) = k$ at time t. We have $p_k(t) = \sum_{i=0}^{\infty} P_{i,k}(t)$ and for the limit we get $p_k = \sum_{i=0}^{\infty} P_{i,k}$.

We shall not solve the system (2) in the general case limiting ourselves to the following remarks:

Remark 1. The probabilities p_k satisfy the equation

$$(3)\qquad \sum_{k=0}^{\infty} \lambda_k p_k = \mu(1-p_0).$$

We denote by $L_{i,j}$ the left side of the equation in system (2) which has a negative coefficient at $P_{i,j}$. We calculate $L_k = \sum_{i+j=k} L_{i,j}$ and

$$(4)\qquad L_0 = 0, \quad L_1 - L_0 = 0, \quad \ldots, \quad L_n - L_{n-1} = 0,$$

then we add equations (4) and group the terms in such a way as to obtain

$$-(\lambda_0 p_0 + \lambda_1 p_1 + \ldots) + \mu(p_1 + p_2 + \ldots) = 0.$$

Thus equality (3) is proved.

If we substitute $\lambda_0 = \lambda$, $\lambda_i = 0$ for $i \geqslant 1$ (i.e. customers enter quarantine only if the service-channel is free) then the probability of the channel working is in the limit $1 - p_0 = \lambda/(\lambda+\mu)$.

Remark 2. If $\lambda_i = \lambda$ $(i = 0, 1, \ldots)$ then, independently of the mean quarantine time, we have

$$p_k = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right).$$

In this case we solve system (2) with additional conditions: $P_{i,j} = \overline{p}_i \cdot p_j$, $\sum_{i=0}^{\infty} \overline{p}_i = 1$, $\sum_{j=0}^{\infty} p_j = 1$ and we get

$$\overline{p}_i = \frac{\lambda^i}{\nu^i i!} e^{-\lambda/\nu},$$

$$p_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right).$$

Identical formulas can be obtained for model (s) where the Poissonian input has a constant intensity and in which customers omit the quarantine. For the general system (2) the assumption

$$P_{i,j} = \overline{p}_i \cdot p_j, \quad \sum_{i=0}^{\infty} \overline{p}_i = \sum_{j=0}^{\infty} p_j = 1$$

leads to a contradiction if there exists at least one λ_i, $\lambda_i \neq \lambda_0$. In other words we observe the dependence of quarantine and service only when the length of the queue feeds back the input.

Let us examine how the input intensity depends on time. We denote by $v(t, \tau)$ the probability of a customer arriving in the time interval $[t, t+\tau)$. We define the input intensity $\Lambda(t)$ as $\Lambda(t) = \lim_{\tau \to 0} \frac{v(t, \tau)}{\tau}$.

THEOREM 2. *In system* (s) *the input intensity equals*

$$\Lambda(t) = \nu \int_0^t \lambda[n(t-T)] e^{-\nu T} dT.$$

Proof. We observe that the queue input in system (s) is the quarantine output in system (S). Denote by $\Delta v(t, \tau)$ the probability of a customer arriving at the queue in the time interval $[t, t+\tau)$, that customer having previously arrived at the quarantine in the time interval $[t-T, t-T+\Delta T)$. Quarantine input and duration-length assumptions give

$$\Delta v(t, \tau) = \{\lambda[n(t-T)]\Delta T + o(\Delta T)\}\{\nu\tau + o(\tau) + o(\Delta T)\} e^{-\nu T}.$$

Dividing both sides by ΔT we get

$$\frac{\Delta v(t, \tau)}{\Delta T} = \nu\tau\lambda[n(t-T)] e^{-\nu T} + o(\tau) + o(\Delta T)/\Delta T;$$

this leads to

$$v(t, \tau) = \nu\tau \int_0^t \lambda[n(t-T)] e^{-\nu T} dT + o(\tau),$$

whence

$$\Lambda(t) = \nu \int_0^t \lambda[n(t-T)]e^{-\nu T}dT,$$

which completes the proof.

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O PEWNYM MODELU TEORII KOLEJEK Z OPÓŹNIONYM SPRZĘŻENIEM ZWROTNYM

STRESZCZENIE

Rozpatrywany w pracy model obsługi masowej charakteryzuje się: (a) nieograniczoną kolejką z kolejnością odpowiadającą kolejności zgłoszeń, (b) jednokanałową obsługą z wykładniczym czasem obsługi, (c) poissonowskim procesem zgłoszeń z intensywnością zależną od aktualnej długości kolejki, (d) kwarantanną przed wejściem do kolejki, której czas trwania ma rozkład wykładniczy. W modelu tym sprzężenie zwrotne intensywności strumienia zgłoszeń z długością kolejki działa na wejście do kolejki z opóźnieniem o losowy czas trwania kwarantanny. W systemie tym proces $n(t)$ (ilość osób w kolejce i obsłudze) nie jest procesem Markowa, jednakże po dodaniu drugiej składowej $m(t)$ (liczba osób znajdujących się w kwarantannie) uzyskuje się dwuwymiarowy proces Markowa $\langle m(t), n(t)\rangle$. Prawdopodobieństwa stanów tego procesu spełniają układ równań różniczkowych

$$P'_{0,0}(t) = -\lambda_0 P_{0,0}(t)+\mu P_{0,1}(t),$$

$$P'_{i,0}(t) = -(\lambda_0+i\nu)P_{i,0}(t)+\mu P_{i,1}(t)+\lambda_0 P_{i-1,0}(t), \quad i \geqslant 1,$$

$$P'_{0,j}(t) = -(\lambda_j+\mu)P_{0,j}(t)+\mu P_{0,j+1}(t)+\nu P_{1,j-1}(t), \quad j \geqslant 1,$$

$$P'_{i,j}(t) = -(\lambda_j+i\nu+\mu)P_{i,j}(t)+\mu P_{i,j+1}(t)+(i+1)\nu P_{i+1,j-1}(t)+\lambda_j P_{i-1,j}(t),$$

$$i\cdot j \geqslant 1,$$

gdzie λ_k oznacza intensywność strumienia zgłoszeń, gdy k osób znajduje się w kolejce i obsłudze, $1/\nu$ — średni czas kwarantanny i $1/\mu$ — średni czas obsługi. Jeżeli zało-

żymy dla tego układu możliwość przejścia do granicy, gdy czas dąży do nieskończoności, otrzymamy w granicy układ równań liniowych

$$-\lambda_0 P_{0,0}+\mu P_{0,1} = 0,$$

$$-(\lambda_0+i\nu)P_{i,0}+\mu P_{i,1}+\lambda_0 P_{i-1,0} = 0, \qquad i \geqslant 1,$$

$$-(\lambda_j+\mu)P_{0,j}+\mu P_{0,j+1}+\nu P_{1,j-1} = 0, \qquad j \geqslant 1,$$

$$-(\lambda_j+i\nu+\mu)P_{i,j}+\mu P_{i,j+1}+(i+1)\nu P_{i+1,j-1}+\lambda_j P_{i-1,j} = 0, \qquad i\cdot j \geqslant 1.$$

W pracy podano przejścia do prawdopodobieństw stanów procesu $n(t)$ oraz przedyskutowano te prawdopodobieństwa w szczególnych przypadkach sprzężenia. I tak, jeśli zakładamy, że klienci wchodzą do kwarantanny tylko wtedy, gdy kanał obsługowy jest wolny, to graniczne prawdopodobieństwo pracy kanału nie zależy od średniego czasu kwarantanny. W przypadku gdy intensywność strumienia zgłoszeń nie zależy od stanu kolejki, rozwiązanie przyjmuje postać znanego rozwiązania dla modelu bez kwarantanny. Intensywność procesu zgłoszeń do kolejki wyraża się wzorem

$$\Lambda(t) = \nu \int_0^t \lambda[n(t-T)]e^{-\nu T}dT.$$

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О НЕКОТОРОЙ МОДЕЛИ ТЕОРИИ ОЧЕРЕДНОГО МАССОВОГО ОБСЛУЖИВАНИЯ С ЗАПАЗДЫВАЮЩЕЙ ОБРАТНОЙ СВЯЗЬЮ

РЕЗЮМЕ

Рассматриваемая в работе модель очередного обслуживания характеризуется: a) неограниченной очередью с чередованием, соответствующим чередованию заявок, b) одноканальным обслуживанием с экспоненциальным временем обслуживания, c) пуассоновым процессом заявок, интенсивность которых зависит от актуальной длины очереды и d) карантином, предшествующим вставанию в очередь, длительность которого имеет экспоненциальное распределение. В схарактеризованной таким образом модели обратная связь потока заявок с длиной очереди действует на входе в очередь со случайным запаздыванием. Итак, если пренеб речь карантином, система является системой со случайно запаздывающей обратной связью. В такой системе процесс $n(t)$ (количество ожидающих в очереди и обслуживающих) не является процессом Маркова, но после введения второй компоненты $m(t)$ (число лиц в карантине) получаем двухмерный процесс Маркова $\langle m(t), n(t)\rangle$. В работе дана система уравнений для определения вероятностей состояния процесса $\langle m(t), n(t)\rangle$, переход к вероятностям состояний процесса $n(t)$ и обсуждаются вероятности состояний для процесса $n(t)$ в частных случаях обратной связи. Вычислена также интенсивность процесса заявок в очередь.
