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ALGORITHM 60

K. WRZEŚNIOWSKI and ELŻBIETA WINKEL (Poznań)

DETERMINATION OF THE JORDAN CANONICAL FORM OF A REAL MATRIX

1. Procedure declaration. Procedure *jordanform1* finds the Jordan canonical form of a given real matrix A whose eigenvalues are known.

Data:

n — order of the matrix A;
a[1:n, 1:n] — array containing the elements of the real matrix A (the matrix A is not destroyed in the process);
ne — number of distinct eigenvalues of the matrix A;
lambda[1:ne] — array containing the distinct eigenvalues of A;
eps — input parameter of procedure *matrixrank* (giving the criterion of the linear independence of the columns of the matrix $A-\lambda I$);
inprod — function with heading **real procedure** *inprod* (*i, a, b, x, y*); **value** *a, b*; **integer** *i, a, b*; **real** *x, y*; which for $i = a, a+1, \ldots, b$ accumulates the sum of products $x\times y$ in double precision;
matrixrank — function with heading **integer procedure** *matrixrank* (*n, a, eps*); **value** *n*; **integer** *n*; **real** *eps*; **array** *a*; which calculates the rank of a real matrix.

Results:

e[1:n] — array of type **integer** the elements from *e[2]* to *e[n]* of which give $n-1$ superdiagonal elements of the Jordan canonical form of the matrix A; the element *e[1]* is set to be zero;
d[1:n] — array giving the diagonal elements of the Jordan canonical form of the matrix A.

2. Method used. In general, the structure of the Jordan cononical form of a matrix A can be described as follows.

```
procedure jordanform1(n,a,ne,lambda,eps,inprod,matrixrank)
 result:(e,d);
 value n;
 integer n,ne;
 real eps;
 integer array e;
 array a,lambda,d;
 real procedure inprod;
 integer procedure matrixrank;
 begin
  real lami;
  integer i,j,k,l,m,r,r1,r2,q1,q2,y;
  integer array s[1:n];
  array b,c[1:n,1:n],p[1:n];
  for i:=1 step 1 until n do
   begin
    p[i]:=a[i,i];
    e[i]:=0
   end i;
  y:=0;
  for i:=1 step 1 until ne do
   begin
    lami:=lambda[i];
    for j:=1 step 1 until n do
     for k:=1 step 1 until n do
      if j=k
       then a[j,k]:=b[j,k]:=c[j,k]:=p[j]-lami
       else b[j,k]:=c[j,k]:=a[j,k];
    r1:=matrixrank(n,c,eps);
    q1:=q2:=n-r1;
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```
      j:=0;
      for j:=j+1 while q2>0 do
       begin
        for k:=1 step 1 until n do
         for l:=1 step 1 until n do
          c[l,k]:=inprod(m,1,n,a[l,m],b[m,k]);
        for k:=1 step 1 until n do
         for l:=1 step 1 until n do
          b[k,l]:=c[k,l];
        r2:=matrixrank(n,c,eps);
        q2:=r1-r2;
        r1:=r2;
        r:=s[j]:=q1-q2;
        q1:=q2;
        for k:=1 step 1 until r do
         begin
          m:=y+j-1;
          for l:=y+1 step 1 until m do
           begin
            e[l+1]:=1;
            d[l]:=lami
           end l;
          y:=y+j;
          d[y]:=lami
         end k
       end j
    end i;
   for i:=1 step 1 until n do
    a[i,i]:=p[i]
  end jordanform1
```

The Jordan canonical form of a matrix A is a quasidiagonal matrix

$$J = \begin{bmatrix} J_1 & & & & \\ & \ddots & & 0 & \\ & & J_i & & \\ & 0 & & \ddots & \\ & & & & J_m \end{bmatrix}$$

composed of canonical boxes J_i, corresponding to distinct eigenvalues of A, of the form

$$J_i = \begin{bmatrix} J_{i1} & & & & \\ & \ddots & & 0 & \\ & & J_{ij} & & \\ & 0 & & \ddots & \\ & & & & J_{im_i} \end{bmatrix},$$

where

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

is an elementary Jordan block of order k_{ij},

$$\sum_{j=1}^{m_i} k_{ij} = k_i,$$

and λ_i is an eigenvalue of A of multiplicity k_i. The number of distinct eigenvalues is m, and

$$\sum_{i=1}^{m} k_i = n$$

is the order of A.

In the Jordan canonical form, built by means of procedure *jordanform1*, elementary Jordan blocks J_{ij} are arranged according to non-decreasing orders of k_{ij}.

The method used in procedure *jordanform1* is based on the following theorem stated and proved in [4]:

THEOREM. *Let A be a matrix of complex numbers of order n, and suppose that λ is an eigenvalue of A. For each $j = 1, 2, \ldots, p$ let s_j be the number* *of elementary Jordan blocks of order j in the canonical box associated with λ of the Jordan canonical form of A. Then s_j is given by*

$$s_j = r_{j+1} - 2r_j + r_{j-1}, \quad j = 1, 2, \ldots, p,$$

where

$$r_j = \operatorname{rank} B^j, \quad j = 0, 1, \ldots, p+1, \quad B = A - \lambda I.$$

The dimension of the largest elementary Jordan block is the smallest non-negative integer p for which $r_p = r_{p+1}$.

By convention, let $B^0 = I$ for any square matrix B. Hence $r_0 = \operatorname{rank} B^0 = \operatorname{rank} I = n$.

On the basis of the Theorem the following algorithm has been built to determine the structure of the Jordan canonical form of the matrix A.

For each distinct eigenvalue λ_i repeat the following steps:

Step 1. Compute $B = A - \lambda_i I$.

Step 2. Compute $r_1 = \operatorname{rank} B$ and the first decrease of rank $q_1 = n - r_1$.

Step 3. For each $j = 1, 2, \ldots$ repeat the following three steps terminating as soon as the condition $r_j = r_{j+1}$ (i.e. $q_{j+1} = 0$) is satisfied.

Step 3.1. Compute $r_{j+1} = \operatorname{rank} B^{j+1}$ and the decrease of rank $q_{j+1} = r_j - r_{j+1}$.

Step 3.2. Compute the j-th element of the characteristic $s_j = q_j - q_{j+1}$.

Step 3.3. Build s_j elementary Jordan blocks of order j.

3. Applicability and certification. Procedure *jordanform1* is designed to find in a simple way the Jordan canonical form of a real matrix A with known real eigenvalues. The correctness of the calculation of the Jordan canonical form depends on the accuracy with which the eigenvalues of A are computed. The eigenvalues of the testing matrices were calculated by means of the set of procedures *balance*, *dirhes*, and *hqr* inserted in [5].

In the course of testing calculations, the library procedure *inprod* described in [3] has been used. Furthermore, procedure *matrixrank* containing procedure *svd*, inserted in [1] and [5], has been used. In the testing program, as the input parameter *eps* the estimation $eps = 2^{-t/2+3}\|A\|_E$ has been used, where t denotes the number of binary digits of the mantissa during the calculation of eigenvalues.

Before using procedure *jordanform1*, matrices were scaled to satisfy the condition $0.1n \leqslant \|A\|_E \leqslant 10n$.

Procedure *jordanform1* has extensively been tested on the Odra 1305 computer. The results obtained were correct.

As an illustration, the result obtained for an example taken from [2], p. 110-113, is given.

Data:

$$n = 10,$$

$$a = \begin{bmatrix} 1 & 1 & 1 & -2 & 1 & -1 & 2 & -2 & 4 & -3 \\ -1 & 2 & 3 & -4 & 2 & -2 & 4 & -4 & 8 & -6 \\ -1 & 0 & 5 & -5 & 3 & -3 & 6 & -6 & 12 & -9 \\ -1 & 0 & 3 & -4 & 4 & -4 & 8 & -8 & 16 & -12 \\ -1 & 0 & 3 & -6 & 5 & -4 & 10 & -10 & 20 & -15 \\ -1 & 0 & 3 & -6 & 2 & -2 & 12 & -12 & 24 & -18 \\ -1 & 0 & 3 & -6 & 2 & -5 & 15 & -13 & 28 & -21 \\ -1 & 0 & 3 & -6 & 2 & -5 & 12 & -11 & 32 & -24 \\ -1 & 0 & 3 & -6 & 2 & -5 & 12 & -14 & 37 & -26 \\ -1 & 0 & 3 & -6 & 2 & -5 & 12 & -14 & 36 & -25 \end{bmatrix},$$

$$ne = 3, \quad lambda = [1.0\ 2.0\ 3.0], \quad eps = 5.05_{10}-8.$$

Results:

$$e = [0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1],$$
$$d = [1\ \ 2\ \ 2\ \ 2\ \ 2\ \ 2\ \ 3\ \ 3\ \ 3\ \ 3].$$

Time of computation was equal to 11 seconds.

References

[1] G. H. Golub and C. Reinsch, *Sigular value-decomposition and least squares solutions*, Numer. Math. 14 (1970), p. 403-420.

[2] R. T. Gregory and D. L. Karney, *A collection of matrices for testing computational algorithms*, J. Wiley, New York 1969.

[3] *Procedura inprod: obliczanie z podwójną precyzją sumy ciągu liczb, z których każda jest iloczynem dwóch liczb*, Oprogramowanie maszyn cyfrowych ODRA serii 1300, ALGOL, Procedury pomocnicze, Zeszyt 1, p. 73, Elwro, Wrocław 1973.

[4] W. B. Rubin, *A simple method for finding the Jordan form of a matrix*, IEEE Trans. Autom. Control AC-17 (1972), p. 145-146.

[5] J. H. Wilkinson and C. Reinsch, *Linear algebra*, in: *Handbook for automatic computation*, p. 315-326, 339-371, vol. 2, Springer, Berlin 1971.

TECHNICAL UNIVERSITY
60-965 POZNAŃ

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K. WRZEŚNIOWSKI i ELŻBIETA WINKEL (Poznań)

ALGORYTM 60

OBLICZENIE POSTACI KANONICZNEJ JORDANA MACIERZY RZECZYWISTEJ

STRESZCZENIE

Procedura *jordanform1* znajduje postać kanoniczną Jordana danej macierzy rzeczywistej o znanym rzeczywistym spektrum.

Dane:

n — stopień macierzy A;

$a[1:n, 1:n]$ — tablica zawierająca elementy macierzy rzeczywistej A (macierz A nie zostaje zniszczona w trakcie obliczeń);

ne — liczba różnych wartości własnych macierzy A;

$lambda[1:ne]$ — tablica zawierająca różne wartości własne macierzy A;

eps — parametr wejścia procedury *matrixrank* (podaje kryterium liniowej niezależności kolumn macierzy $A-\lambda I$);

inprod — funkcja o nagłówku **real procedure** *inprod*(i, a, b, x, y); **value** a, b; **integer** i, a, b; **real** x, y; która dla $i = a, a+1, \ldots, b$ kumuluje sumę iloczynów $x \times y$ w podwójnej precyzji;

matrixrank — funkcja o nagłówku **integer procedure** *matrixrank*(n, a, eps); **value** n; **integer** n; **real** eps; **array** a; która oblicza rząd macierzy rzeczywistej.

Wyniki:

$e[1:n]$ — tablica typu **integer**, której elementy od $e[2]$ do $e[n]$ zawierają $n-1$ ponaddiagonalnych elementów postaci kanonicznej Jordana macierzy A; element $e[1] = 0$;

$d[1:n]$ — tablica zawierająca elementy diagonalne postaci kanonicznej Jordana macierzy A.

W procedurze *jordanform1* zastosowano metodę opisaną w § 2, opartą na twierdzeniu sformułowanym i udowodnionym w [4].

W trakcie wykonywania obliczeń korzystano z procedury bibliotecznej *inprod*, opisanej w [3], oraz z procedury *matrixrank*, zawierającej procedurę *svd* zamieszczoną w [1] i [5].

Procedura *jordanform1* była sprawdzana na wielu przykładach na maszynie cyfrowej Odra 1305. Otrzymane wyniki były poprawne.