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ON LOGIC TRANSFORMATIONS OF THE BINARY STOCHASTIC PROCESS

1. Introduction. In this paper we consider the random process $Y(i)$ with discrete parameter, defined as the transformation of a given binary random process,

$$Y(i) = f_n(X_{i-n+1}, X_{i-n+2}, \ldots, X_{i-n+j}, \ldots, X_{i-1}, X_i), \tag{1}$$

where X_i denotes a two-point random variable being the value of the process $X(i)$ for fixed i ($i = -v, \ldots, -r, \ldots, -1, 0, 1, 2, \ldots, v$ and r denoting natural numbers for which $n \leqslant r \leqslant v-n+1$) and f_n is an n-argument logic function ($n \geqslant 1$).

The realizations of the random variable X_i will be denoted by x_i.

Let Y_i be a two-point random variable being the value of the process $Y(i)$ for fixed i ($i = -r, \ldots, -1, 0, 1, 2, \ldots$). Such a process very often occurs in the theory of signal detection in the presence of noise ([1]). In this case $\{X_i\}$ is a sequence of quantized signals mixed with noise. This sequence forms the input stream of a decision device; here f_n is the decision function used for detecting the signal and $\{Y_i\}$ is a sequence of decisions concerning the existence and non-existence of the signal.

For an optimal choice of the function f_n it is necessary to find some probability characteristics of the process $Y(i)$, depending on the process $X(i)$.

The aim of this paper is to present an algorithm for computing probabilities of the events

$$B_J = \{(Y_1 = 1) \cup (Y_2 = 1) \cup \ldots \cup (Y_J = 1)\}, \quad J \geqslant 1,$$

for various functions f_n and distributions of random variables X_i.

For the sake of simplicity let us write:

$$p_i = P\{X_i = 1\}, \quad q_i = P\{X_i = 0\} = 1-p_i,$$

$$a_i = \{Y_i = 0\}, \quad b_i = \{Y_i = 1\},$$

$$A_i = \{a_{-r} \cap a_{-r+1} \cap \ldots \cap a_{i-1} \cap a_i\},$$

$$\bar{A}_i = \{b_{-r} \cup b_{-r+1} \cup \ldots \cup b_{i-1} \cup b_i\},$$

$$B_i = \{b_1 \cup b_2 \cup \ldots \cup b_{i-1} \cup b_i\}.$$

2. A recurrence formula for $P\{B_J\}$.

ASSUMPTIONS. 1° $\{X_i\}$ is a sequence of independent two-point random variables with known probabilities p_i $(i = -v, \ldots, -r, \ldots, -1, 0, 1, 2, \ldots)$; v and r denote natural numbers satisfying the condition $n \leqslant r \leqslant v-n+1$;

2° $p_i = 0$ for $i \leqslant 0$;

3° the stochastic process $Y(i)$ is defined by (1), where f_n is an n-argument logic function $(n \geqslant 1)$;

4° $f_n(0, 0, \ldots, 0) = 0$; $f_n \not\equiv 0$.

The distributions of the random variables X_i $(i = -v, \ldots, -r, \ldots, -1, 0, 1, 2, \ldots)$ completely define the process $Y(i)$. Thus, theoretically, the problem is reduced to classical methods of seeking the distribution of the function with random arguments. However, if J is a large number, classical methods lead to very cumbersome numerical computations (e.g. for $J = 30$ the number of additions would be near to 10^9). These classical methods would require a very long computer working time and could give large numerical errors. It is possible to avoid these difficulties by using suitable recurrent formulae.

Let us consider the relations

$$P\{A_i \cup \bar{A}_i\} = P\{A_i\}+P\{\bar{A}_i\} \leqslant 1,$$
$$P\{A_i \cap \bar{A}_i\} = 0$$

based on the fact that A_i and $\bar{A}_i$ are complementary events. From assumptions 2° and 4° it follows that

(2) $$P\{A_i\} = 1 \quad \text{for } i \leqslant 0.$$

As $P\{A_0\} = 1$, thus

(3) $$P\{B_J|A_0\} = P\{B_J \cap A_0\},$$

(4) $$P\{B_J \cap A_0\} = P\{B_J\}.$$

Similarly

(5) $$P\{A_i|A_0\} = P\{A_i\}.$$

Since $A_i = A_0 \cap \bar{B}_i$; $P\{A_i\} = P\{\bar{B}_i|A_0\} \cdot P\{A_0\} = P\{\bar{B}_i\} = 1-P\{B_i\}$, for every i we have

(6) $$P\{B_i\} = 1-P\{A_i\}.$$

Taking into consideration the above relations we observe that

(7) $$P\{B_J\} = P\{A_0 \cap b_1\}+P\{A_1 \cap b_2\}+\ldots+$$
$$+\ldots+P\{A_{i-1} \cap b_i\}+\ldots+P\{A_{J-1} \cap b_J\}.$$

Let us consider events $\{B_i\}$ and $\{B_{i-1}\}$. From (7) it follows that

(8) $$P\{B_i\} = P\{B_{i-1}\}+P\{A_{i-1} \cap b_i\}.$$

Identities (7) and (8) may be used for deriving a recurrent formula. Define:

$$(9)\quad \begin{cases} \Delta P_i = P\{A_{i-1} \cap b_i\} = P\{a_{-r} \cap \ldots \cap a_0 \cap a_1 \cap \ldots \cap a_{i-1} \cap b_i\}; \\ X_i^{(n)} = (X_{i-n+1}, X_{i-n+2}, \ldots, X_{i-n+j}, \ldots, X_i); \\ X_i^{(2n)} = (X_{i-2n+1}, X_{i-2n+2}, \ldots, X_{i-n}, X_{i-n+1}, \ldots, X_{i-1}, X_i); \\ X_i^{(v+i+1)} = (X_{-v}, \ldots, X_{-r}, \ldots, X_{-1}, X_0, X_1, \ldots, X_{i-1}, X_i). \end{cases}$$

Here $X_i^{(n)}, X_i^{(2n)}$ and $X_i^{(v+i+1)}$ are random vector variables. Their realizations will be denoted by small letters $x_i^{(n)}, x_i^{(2n)}$ and $x_i^{(v+i+1)}$ respectively.

Denote by Γ_i the set of those realizations $x_i^{(v+i+1)}$ which are related to the event $\{A_{i-1} \cap b_i\}$. Hence

$$(10)\quad \Delta P_i = P\{X_i^{(v+i+1)} \in \Gamma_i\}.$$

Taking into consideration that the events $\{X_i^{(v+i+1)} = x_i^{(v+i+1)}\}$ are exclusive, we obtain

$$(11)\quad \Delta P_i = \sum_{x_i^{(v+i+1)} \in \Gamma_i} P\{X_i^{(v+i+1)} = x_i^{(v+i+1)}\}.$$

Let us consider the subsequences

$$x_i^{(n)} = (x_{i-n+1}, x_{i-n+2}, \ldots, x_{i-n+j}, \ldots, x_{i-1}, x_i)$$

of the sequences $x_i^{(v+i+1)} \in \Gamma_i$.

For each $x_i^{(n)}$ the following equality holds:

$$(12)\quad f(x_i^{(n)}) = 1.$$

Now consider the following event:

$$R_i^{(n)} = \{a_{i-n} \cap a_{i-n+1} \cap \ldots \cap a_{i-1} \cap b_i\}.$$

In conformity with (1) this event is realized by correspondent sequences

$$x_i^{(2n)} = (x_{i-2n+1}, x_{i-2n+2}, \ldots, x_{i-n}, x_{i-n+1}, \ldots, x_{i-1}, x_i).$$

Denote by K_i the set of the sequences $x_i^{(2n)}$.

We shall require the following

LEMMA 1. *The sets* K_i, $i = 1, 2, \ldots$, *are identical*: $K_1 = K_2 = \ldots \ldots = K_i = K$.

Proof. Let us consider two arbitrary sets K_g and K_h $(g, h = 1, 2, \ldots)$. Let $x_g^{(2n)} \in K_g$. According to (1) the sequence $x_g^{(2n)}$ realizes the sequence

$$\{y_{g-n} = 0, y_{g-n+1} = 0, \ldots, y_{g-1} = 0, y_g = 1\}$$

which realizes the event $R_g^{(n)}$.

Now, let us consider the sequence

$$x_h^{(2n)} = (x_{h-2n+1}, x_{h-2n+2}, \ldots, x_{h-n}, x_{h-n+1}, \ldots, x_{h-1}, x_h)$$

and let $x_h^{(2n)} = x_g^{(2n)}$. In view of (1) the sequence $x_h^{(2n)}$ realizes the sequence

$$\{y_{h-n} = 0, y_{h-n+1} = 0, \ldots, y_{h-1} = 0, y_h = 1\}$$

which in turn realizes the event $R_h^{(n)}$. Hence $x_h^{(2n)} \in K_h$. Thus

$$\bigwedge_{x^{(2n)}} [x^{(2n)} \in K_g] \Rightarrow [x^{(2n)} \in K_h]$$

and

$$K_g \subset K_h.$$

Similarly, it can be also shown that $K_h \subset K_g$. Hence $K_h = K_g = K$, which completes the proof.

Denote by $\mathfrak{M}_i$ the set of all the different subsequences

$$x_i^{(n)} = (x_{i-n+1}, x_{i-n+2}, \ldots, x_{i-n+j}, \ldots, x_{i-1}, x_i)$$

of the sequences $x_i^{(2n)} \in K_i$.

It follows from Lemma 1 that

$$\mathfrak{M}_1 = \mathfrak{M}_2 = \ldots = \mathfrak{M}_i = \ldots = \mathfrak{M}. \tag{13}$$

Denote card $\{\mathfrak{M}\}$ by M. Then

$$\mathfrak{M} = \{x^{(n)1}, x^{(n)2}, \ldots, x^{(n)m}, \ldots, x^{(n)M}\}, \tag{14}$$

where

$$x^{(n)m} = (x_1^m, x_2^m, \ldots, x_s^m, \ldots, x_n^m).$$

It follows from the definition of the set $\mathfrak{M}$ that this set is unambiguously defined by the function f_n. A simple computer program for finding $\mathfrak{M}$ for various functions f_n has been made by the author.

It follows from the above definitions that if $x^{(n)} = x^{(n)m} \in \mathfrak{M}$ $(m = 1, 2, \ldots, M)$, then $f(x^{(n)}) = 1$. Hence

$$\mathfrak{M} \subset \mathfrak{N}, \tag{15}$$

where $\mathfrak{N}$ is the set of all these values of the vector $x^{(n)}$ for which $f(x^{(n)}) = 1$.

Example. Let $f(x^{(3)}) = \bar{x}_1 x_2 x_3 \vee x_1 x_3 \vee x_1 x_2 \bar{x}_3$. For this function

$$\mathfrak{M} = \{(011), (101)\}, \quad \mathfrak{N} = \{(011), (101), (110), (111)\}.$$

Let us now come back to the sets Γ_i.

If the sequence $x_i^{(v+i+1)} \in \Gamma_i$ ($x_i^{(v+i+1)}$ realizes the event $\{A_{i-1} \cap b_i\}$), then the subsequence of its last $2n$ terms forms the correspondent sequence $x_i^{(2n)} \in K_i$. Hence

$$\{A_{i-1} \cap b_i\} = \{a_{-r} \cap \ldots \cap a_{i-n-1} \cap a_{i-n} \cap \ldots \cap a_{i-1} \cap b_i\}$$
$$= \{a_{-r} \cap \ldots \cap a_{i-n-1} \cap R_i^n\},$$

where, as assumed, $r \geqslant n$. Hence, the last n terms of every sequence $x_i^{(v+i+1)} \in \Gamma_i$ form the subsequence $x_i^{(n)}$ equal to correspondent vector $x^{(n)m} \in \mathfrak{M}$. Thus, we can divide every set Γ_i in separate disjoint subsets Γ_i^m in such a way that each Γ_i^m contains only those sequences $x_i^{(v+i+1)}$ for which their last n terms form the sequences $x_i^{(n)}$ equal to $x^{(n)m}$. Thus

$$\Gamma_i = \bigcup_{m=1}^{M} \Gamma_i^m. \tag{16}$$

It follows from (9), (10), (11) and (16) that

$$\Delta P_i = \sum_{m=1}^{M} \Delta P_i^m, \tag{17}$$

where

$$\Delta P_i^m = \sum_{x_i^{(v+i+1)} \in \Gamma_i^m} P\{X_i^{(v+i+1)} = x_i^{(v+i+1)}\}. \tag{18}$$

From (4) and (7)

$$P\{B_J\} = \sum_{i=1}^{J} \Delta P_i. \tag{19}$$

Thus, seeking the probability $P\{B_J\}$ is reduced to seeking of the probabilities ΔP_i^m ($i = 1, 2, \ldots$ and $m = 1, 2, \ldots, M$).

Let us consider the vectors $x^{(n)m} \in \mathfrak{M}$. Assign to each of these vectors a corresponding binary matrix $[\sigma_{k,j}^m]$, $k = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, n$.

For this purpose the following operators will be defined:

$$L_1^j[x^{(n)}] = (x_1, x_2, \ldots, x_j) \quad (j = 1, 2, \ldots, n), \tag{20}$$

$$L_2^j[x^{(n)}] = (x_{n-j+1}, x_{n-j+2}, \ldots, x_n) \quad (j = 1, 2, \ldots, n), \tag{21}$$

where

$$x^{(n)} = (x_1, x_2, \ldots, x_j, \ldots, x_n).$$

The elements of the matrices $[\sigma_{k,j}^m]$ are defined by

$$\sigma_{k,j}^m = \begin{cases} 1 & \text{for } \{(k, j)\colon L_2^j[x^{(n)k}] = L_1^j[x^{(n)m}]\}, \\ 0 & \text{for the others pairs } (k, j), \end{cases} \tag{22}$$

where $m, k = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, n$.

The general recurrent formula for computing the probabilities ΔP_i^m is defined by the following

THEOREM. *Let f_n be an n-argument logic function and let assumptions* 1°-4° *hold. Then*

$$(23)\quad \Delta P_i^m = $$
$$= P\{A_{i-n}\}\prod_{s=1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)} - \sum_{j=1}^{n-1}\sum_{k=1}^{M}\sigma_{k,j}^m \Delta P_{i-n+j}^k \prod_{s=j+1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)},$$

where x_s^m denote the s-th component of the vector $x^{(n)m}\in\mathfrak{M}$ ($x_s^m\in\{0,1\}$).

Proof. For the sake of simplicity, the event $\{X_i^{(n)} = x_i^{(n)m}\}$ will be denoted in the sequel by $\{x_i^{(n)m}\}$. Similar symbols apply to the other cases, for example the event $\{X_i^{(\nu+i+1)}\in\Gamma_i\}$ will be denoted by $\{\Gamma_i\}$.

Let us consider the event

$$(24)\quad C_i^m = \{a_{-r}\cap\ldots\cap a_{i-n}\cap x_i^{(n)m}\},$$
$$i = 1, 2, \ldots \text{ and } m = 1, 2, \ldots, M,$$

where $x_i^{(n)m} = x^{(n)m}\in\mathfrak{M}$.

From assumption 1° it follows that

$$(25)\qquad P\{C_i^m\} = P\{A_{i-n}\}\prod_{s=1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)}.$$

Consider an event E_i,

$$E_i = \{(b_{i-n+1}\cup\ldots\cup b_{i-1}\cup b_i)\cup(a_{i-n+1}\cap\ldots\cap a_{i-1}\cap a_i)\},$$
$$P\{E_i\} = 1,\quad i = 1, 2, \ldots$$

Hence $P\{C_i^m\cap E_i\} = P\{C_i^m\}$.

It is easy to see that

$$P\{C_i^m\cap(a_{i-n+1}\cap a_{i-n+2}\cap\ldots\cap a_i)\} = 0.$$

Hence

$$P\{C_i^m\} = P\{a_{-r}\cap\ldots\cap a_{i-n}\cap(b_{i-n+1}\cup\ldots\cup b_{i-1}\cup b_i)\cap x_i^{(n)m}\}.$$

But from (7) it follows that

$$\{a_{-r}\cap\ldots\cap a_{i-n}\cap(b_{i-n+1}\cup b_{i-n+2}\cup\ldots\cup b_{i-1}\cup b_i)\}$$
$$= \bigcup_{j=1}^{n}\{a_{-r}\cap\ldots\cap a_{i-n}\cap a_{i-n+1}\cap\ldots\cap a_{i-n+j-1}\cap b_{i-n+j}\}.$$

Thus

$$(26)\qquad P\{C_i^m\} = \sum_{j=1}^{n} P\{\Gamma_{i-n+j}\cap x_i^{(n)m}\}.$$

But, according to (16),

$$\Gamma_{i-n+j} = \bigcup_{k=1}^{M}\Gamma_{i-n+j}^k,\qquad \{\Gamma_{i-n+j}^k\}\cap\{\Gamma_{i-n+j}^m\} = \emptyset\quad \text{for } k\neq m.$$

Hence

$$(27)\qquad P\{\Gamma_{i-n+j}\cap x_i^{(n)m}\} = \sum_{k=1}^{M} P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\}.$$

By definition

$$(28)\qquad \{\Gamma_{i-n+j}^{k}\} \equiv \{\Gamma_{i-n+j}^{k}\cap x_{i-n+j}^{(n)k}\},$$

where $x_{i-n+j}^{(n)k} = x^{(n)k}\in\mathfrak{M}$. Hence

$$(29)\qquad P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = P\{\Gamma_{i-n+j}^{k}\cap x_{i-n+j}^{(n)k}\cap x_i^{(n)m}\}.$$

It is easy to see that

$$(30)\qquad \{x_{i-n+j}^{(n)k}\cap x_i^{(n)m}\} \equiv \{x_{i-n+j}^{(n)k}\cap L_2^j[x_{i-n+j}^{(n)k}]\cap L_1^j[x_i^{(n)m}]\cap L_2^{n-j}[x_i^{(n)m}]\},$$

where

$$L_2^{n-j}[x_i^{(n)m}] = (x_{i-n+j+1}^{m}, x_{i-n+j+2}^{m}, \ldots, x_{i-1}^{m}, x_i^{m}).$$

Thus

$$P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = P\{\Gamma_{i-n+j}^{k}\cap (L_2^j[x_{i-n+j}^{(n)k}]\cap L_1^j[x_i^{(n)m}])\cap L_2^{n-j}[x_i^{(n)m}]\}.$$

Two cases are possible.

1. If $\sigma_{k,j}^{m} = 1$, then $\{L_2^j[x_{i-n+j}^{(n)k}]\} = \{L_1^j[x_i^{(n)m}]\}$ and taking into consideration that

$$(31)\qquad \{\Gamma_{i-n+j}^{k}\cap L_2^j[x_{i-n+j}^{(n)k}]\} \equiv \{\Gamma_{i-n+j}^{k}\}$$

one gets

$$P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = P\{\Gamma_{i-n+j}^{k}\cap L_2^{n-j}[x_i^{(n)m}]\}.$$

2. If $\sigma_{k,j}^{m} = 0$, then $\{L_2^j[x_{i-n+j}^{(n)k}]\}\cap\{L_1^j[x_i^{(n)m}]\} = \emptyset$ (which is an impossible event) and

$$(32)\qquad P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = 0.$$

Thus

$$(33)\qquad P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = \sigma_{k,j}^{m} P\{\Gamma_{i-n+j}^{k}\cap L_2^{n-j}[x_i^{(n)m}]\}.$$

It follows from assumption 1° that the events $\{\Gamma_{i-n+j}^{k}\}$ and $\{L_2^{n-j}[x_i^{(n)m}]\}$ are independent.

Taking into consideration that

$$P\{L_2^{n-j}[x_i^{(n)m}]\} = \prod_{s=j+1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)}$$

and

$$P\{\Gamma_{i-n+j}^{k}\} = \Delta P_{i-n+j}^{k},$$

we infer from (33) that

$$(34)\qquad P\{\Gamma_{i-n+j}^{k}\cap x_i^{(n)m}\} = \sigma_{k,j}^{m}\Delta P_{i-n+j}^{k}\prod_{s=j+1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)}.$$

From (34) and (27) we have

$$(35)\qquad P\{\Gamma_{i-n+j}\cap x_i^{(n)m}\} = \sum_{k=1}^{M} \sigma_{k,j}^{m}\,\Delta P_{i-n+j}^{k} \prod_{s=j+1}^{n} p_{i-n+s}^{x_s^m} q_{i-n+s}^{(1-x_s^m)}.$$

Let us observe that

$$\sigma_{k,n}^{m} = \delta_{m,k} = \begin{cases} 1 & \text{for } k = m,\\ 0 & \text{for } k \neq m.\end{cases}$$

Hence

$$(36)\qquad P\{\Gamma_i \cap x_i^{(n)m}\} = \sum_{k=1}^{M} \delta_{m,k}\,\Delta P_i^k = \Delta P_i^m.$$

From (35), (26), (36) and (25) we obtain (23), which completes the proof.

Now, the probabilities $P\{B_J\}$ can be found from (23), (17) and (19).

3. An example. Let us consider the following function f_n (for $n = 5$):

$$f(x^{(5)}) = x_1x_3x_4 \vee x_2x_3x_4 \vee x_2x_3x_5 \vee x_1x_2x_4x_5.$$

This function has the following set $\mathfrak{N}$:

$$\{01101, 01110, 01111, 10110, 10111, 11011, 11101, 11110, 11111\}.$$

On the other hand, the set $\mathfrak{M}$ consists of the following 5 elements (the equality of M and n is incidental):

m	x_1^5	x_2^5	x_3^5	x_4^5	x_5^5
1	0	1	1	0	1
2	0	1	1	1	0
3	0	1	1	1	1
4	1	0	1	1	0
5	1	0	1	1	1

We have

$$[\sigma_{k,j}^{1}] = \begin{bmatrix} 0&1&0&0&1\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&1&0\\ 0&0&0&0&0 \end{bmatrix},\quad [\sigma_{k,j}^{2}] = \begin{bmatrix} 0&1&0&0&0\\ 1&0&0&0&1\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{bmatrix},\quad [\sigma_{k,j}^{3}] = \begin{bmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{bmatrix},$$

$$[\sigma_{k,j}^{4}] = \begin{bmatrix} 1&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&1\\ 1&0&0&0&0 \end{bmatrix},\quad [\sigma_{k,j}^{5}] = \begin{bmatrix} 1&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&1 \end{bmatrix}.$$

Thus

$$\begin{aligned}
\Delta P_i^1 &= P\{A_{i-5}\}q_{i-4}p_{i-3}p_{i-2}q_{i-1}p_i - [\Delta P_{i-4}^2 + \Delta P_{i-4}^4]p_{i-3}p_{i-2}q_{i-1}p_i - \\
&\qquad - \Delta P_{i-3}^1 p_{i-2}q_{i-1}p_i - \Delta P_{i-1}^4 p_i, \\
\Delta P_i^2 &= P\{A_{i-5}\}q_{i-4}p_{i-3}p_{i-2}p_{i-1}q_i - [\Delta P_{i-4}^2 + \Delta P_{i-4}^4]p_{i-3}p_{i-2}p_{i-1}q_i - \\
&\qquad - \Delta P_{i-3}^1 p_{i-2}p_{i-1}q_i - \Delta P_{i-1}^5 q_i, \\
\Delta P_i^3 &= P\{A_{i-5}\}q_{i-4}p_{i-3}p_{i-2}p_{i-1}p_i - [\Delta P_{i-4}^3 + \Delta P_{i-4}^4]p_{i-3}p_{i-2}p_{i-1}p_i - \\
&\qquad - \Delta P_{i-3}^1 p_{i-2}p_{i-1}p_i - \Delta P_{i-1}^5 p_i, \\
\Delta P_i^4 &= P\{A_{i-5}\}p_{i-4}q_{i-3}p_{i-2}p_{i-1}q_i - \\
&\qquad - [\Delta P_{i-4}^1 + \Delta P_{i-4}^3 + \Delta P_{i-4}^5]q_{i-3}p_{i-2}p_{i-1}q_i - \\
&\qquad - [\Delta P_{i-3}^2 + \Delta P_{i-3}^4]p_{i-2}p_{i-1}q_i - \Delta P_{1-2}p_{i-1}q_i, \\
\Delta P_i^5 &= P\{A_{i-5}\}p_{i-4}q_{i-3}p_{i-2}p_{i-1}p_i - \\
&\qquad - [\Delta P_{i-4}^1 + \Delta P_{i-4}^3 + \Delta P_{i-4}^5]q_{i-3}p_{i-2}p_{i-1}p_i - \\
&\qquad - [\Delta P_{i-3}^2 + \Delta P_{i-3}^4]p_{i-2}p_{i-1}p_i - \Delta P_{i-2}^1 p_{i-1}p_i,
\end{aligned}$$

where

$$P\{A_i\} = 1 - P\{B_i\}, \quad P\{B_i\} = \sum_{w=1}^{i} \Delta P_w \quad (i = 1, 2, \ldots),$$

$$\Delta P_w = \sum_{m=1}^{5} \Delta P_w^m,$$

and for $i \leqslant 0$ we put, according to the assumptions, $P\{A_i\} = 1$ and $\Delta P_i^m = 0$.

The probabilities of the other realizations of the process $Y(i)$ may be computed similarly. The reccurrence method described in this paper may also be used for computing the distributions of various random variables which may be derived from the process $Y(i)$. For example, the distribution of the length of a series may be computed by this method.

Reference

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O TRANSFORMACJACH LOGICZNYCH BINARNEGO PROCESU STOCHASTYCZNEGO

STRESZCZENIE

W pracy rozważany jest binarny proces stochastyczny $Y(i)$ z dyskretnym parametrem, określony jako przekształcenie danego binarnego procesu stochastycznego $X(i)$. Przekształcenie to jest realizowane, zgodnie z wyrażeniem (1), przez n-argumentową funkcję algebry logiki dwuwartościowej.

Celem pracy jest poszukiwanie formuł analitycznych dla wyznaczania wartości miary prawdopodobieństwa dla danych podzbiorów realizacji procesu $Y(i)$ w zależności od charakterystyk probabilistycznych procesu $X(i)$.

Znaleziono formułę rekurencyjną umożliwiającą między innymi wyznaczanie rozkładu prawdopodobieństwa długości serii ciągu $\{Y_i\}$.

Uzyskane wyniki znajdują zastosowanie w telekomunikacji przy poszukiwaniu optymalnych filtrów cyfrowych dla wykrywania sygnałów w szumach odbiorczych.