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$GI/M/1$ QUEUEING SYSTEMS WITH SERVICE RATES DEPENDING ON THE LENGTH OF QUEUE

1. The general case of $GI/M/1$ systems with feedback. Let us consider a single-channel service system with independent interarrival times $\{X_r\}$ having the common distribution

$$G(x) = P\{X_r < x\}, \quad 0 < x < \infty \ (r = 0, 1, \ldots),$$

and service rate $\mu\{n(t)\}$, depending on the actual state $n(t)$ of the system, i.e. on the number of units in the system at the moment t. The conditional probability to complete a service in the interval of time $[t, t+\tau)$, given the number of units $n(t)$ in the system at the beginning of that interval, is equal to $\mu\{n(t)\}\tau + o(\tau)$.

Such systems will be called here $GI/M/1$ service systems with feedback. To simplify further considerations we will assume that $\mu(u) \neq \mu(v)$ for $u \neq v$. Similar results may be obtained without the above restriction, e.g. for systems with a threshold control in which the service rate changes only as an effect of exceeding the control level for $n(t)$, though this would lead to much more complicated formulae.

Let t_r, $r = 0, 1, \ldots, (t_0 = 0)$, be the sequence of the moments of time at which consecutive units enter the system. One can easily see that the sequence of random variables $N_r = n(t_r - 0)$, $r = 0, 1, \ldots$, forms an imbedded Markov chain. The state of the system at $t_{r+1} - 0$ is determined by the state at $t_r - 0$ and the number of services η_r completed during the time interval $[t_r, t_{r+1})$,

$$N_{r+1} = N_r + 1 - \eta_r.$$

Let us denote by

the transition probability of passing in r steps from the state i to the state j and by

$$a_{i,k+i} = P\{\eta_r = i \mid N_r = k+i\}$$

the conditional probability that exactly i services will be completed during the random time interval $[t_r, t_{r+1})$, if $k+i$ units are present in the system at the beginning of that interval. For $i > 0$ and $k > 0$ we have

$$(1) \qquad \begin{aligned} a_{i,k+i} &= \int_0^\infty P\{\eta_r = i \mid N_r = k+i,\, t_{r+1}-t_r = x\}\, dG(x) \\ &= \int_0^\infty P\{Y_{k+i}+Y_{k+i-1}+\ldots+Y_{k+1} \\ &\leqslant x < Y_{k+i}+Y_{k+i-1}+\ldots+Y_k\}\, dG(x) \\ &= \int_0^\infty \Big[\int_0^x b_{k+i,k+1}(y) P\{Y_k > x-y\}\, dy\Big] dG(x) \\ &= \int_0^\infty e^{-\mu_k x}\Big[\int_0^x b_{k+i,\,k+1}(y)\, e^{\mu_k y}\, dy\Big] dG(x), \end{aligned}$$

where $b_{k+i,\,k+1}(y)$ is the probability density function of the sum

$$Z_{k+i,k+1} = Y_{k+i}+Y_{k+i-1}+\ldots+Y_{k+1}$$

of i independent random variables with the common exponential distribution

$$P\{Y_{k+j} < y\} = 1-e^{-\mu_{k+j}y}, \quad j = 1, 2, \ldots, i.$$

Let $b^*_{k+i,\,k+1}(s)$ be the Laplace transform of the density function $b_{k+i,k+1}(y)$. Obviously,

$$(2) \qquad b^*_{k+i,k+1}(s) = \prod_{j=1}^{i} \frac{\mu_{k+j}}{\mu_{k+j}+s},$$

and, according to the assumption that $\mu_k \neq \mu_l$ for $k \neq l$, the inverse transform yields

$$b_{k+i,k+1}(y) = \sum_{j=1}^{i} c_{k,i}\, e^{-\mu_{k+j}y},$$

where

$$c_{k,i} = \Big(\prod_{l=1}^{i} \mu_{k+l}\Big)\Big(\prod_{\substack{l=1 \\ l\neq j}}^{i} [\mu_{k+l}-\mu_{k+j}]\Big)^{-1}.$$

From (1), after simple calculation, we obtain

$$a_{i,k+i} = \sum_{j=1}^{i} \frac{c_{k,i}}{\mu_{k+j}-\mu_k} [G^*(\mu_k)-G^*(\mu_{k+j})], \qquad i>0,\ k>0,$$

where

$$G^*(s) = \int_0^\infty e^{-sx}dG(x) \qquad (\mathrm{Re}(s)>0)$$

is the Laplace-Stieltjes transform of the function $G(x)$. For $i = 0$ and $k>0$ there is

$$\text{(3)} \qquad a_{0,k} = \int_0^\infty P\{\eta_r = 0 \mid N_r = k,\, t_{r+1}-t_r = x\}dG(x)$$

$$= \int_0^\infty P\{Y_k > x\}dG(x) = \int_0^\infty e^{-\mu_k x}dG(x) = G^*(\mu_k).$$

Thus we obtain

$$\text{(4)} \qquad P_{i,j} = P_{i,j}^{(1)} = a_{i-j+1,i+1} \qquad (0<j\leqslant i+1,\, i\geqslant 0)$$

and

$$\text{(5)} \qquad P_{i,0} = P_{i,0}^{(1)} = \int_0^\infty P\{\eta_r = i+1 \mid N_r = i+1,\, t_{r+1}-t_r = x\}dG(x)$$

$$= \int_0^\infty \Big[1-\sum_{l=0}^{i} P\{\eta_r = l \mid N_r = i+1,\, t_{r+1}-t_r = x\}\Big]dG(x)$$

$$= 1-\sum_{l=0}^{i} a_{l,i+1} = a_{i,i+1} \qquad (i\geqslant 0).$$

The transition probability matrix for one-step transitions from the state i to the state j is of the form

$$\text{(6)} \qquad [P_{i,j}] = \begin{array}{c|ccccc} \nearrow & 0 & 1 & 2 & 3 & \ldots \\ \hline 0 & a_{0,1} & a_{0,1} & 0 & 0 & \\ 1 & a_{1,2} & a_{1,2} & a_{0,2} & 0 & \\ 2 & a_{2,3} & a_{2,3} & a_{1,3} & a_{0,3} & \ldots \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

The limiting probabilities of the states of the system,

$$\text{(7)} \qquad p_j = \lim_{r\to\infty} P_{i,j}^{(r)} \qquad \Big(p_j\geqslant 0,\, \sum_j p_j = 1\Big),$$

assuming their existence, have to fulfil the system of equations

(8)
$$p_0 = \sum_{i=0}^{\infty} a_{i,i+1} p_i,$$

$$p_j = \sum_{i=j-1}^{\infty} a_{i-j+1,i+1} p_i \quad (j \geqslant 1).$$

Now we shall state a condition securing the existence of the limiting probabilities (7).

Let

$$\varrho_i = \sum_{l=0}^{i} l a_{l,i} \quad (i \geqslant 1)$$

be the expected number of services completed during the time interval, assuming there are i units in the system at the moment t_r.

THEOREM 1. *The inequality*

$$\varrho = \inf_i \varrho_i > 1$$

provides a sufficient condition for the existence of limiting probabilities (7) *in a $GI/M|1$ service system with feedback $\mu\{n(t)\}$.*

The proof of the theorem follows from a theorem by Foster [1]. A similar result for $M/G/1$ systems with a feedback has been obtained by Harris [3]. Let us notice first that the imbedded Markov chain $\{N_r\}$ is irreducible and aperiodic. This follows from the form of the matrix of transition probabilities (6) and from the independence of services. Now we have to show that the chain $\{N_r\}$ is recurrent non-null. To show this it is sufficient to find a non-negative solution of the inequality

(9)
$$\sum_{j=0}^{\infty} P_{i,j} x_j \leqslant x_i - 1 \quad (i > 0)$$

with the additional condition

(9′)
$$\sum_{j=0}^{\infty} P_{0,j} x_j < \infty.$$

Now, we put

$$x_j = \frac{j}{\varrho - 1}$$

and check that for $i > 0$ there is

$$\sum_{j=0}^{\infty} P_{i,j}x_j = \sum_{j=1}^{i+1} a_{i-j+1,i+1}\frac{j}{\varrho-1} = \sum_{l=0}^{i+1} a_{l,i+1}\frac{i+1-l}{\varrho-1}$$

$$= \frac{i+1}{\varrho-1}\sum_{l=0}^{i+1} a_{l,i+1} - \frac{1}{\varrho-1}\sum_{l=0}^{i+1} l a_{l,i+1}$$

$$= \frac{i+1}{\varrho-1} - \frac{\varrho_{i+1}}{\varrho-1} < \frac{i+1}{\varrho-1} - \frac{\varrho}{\varrho-1} = x_i - 1.$$

We have applied here the obvious equality

$$\sum_{l=0}^{i+1} a_{l,i+1} = 1$$

for the elements of a stochastic matrix (6). Condition (9′) now takes the form

$$\frac{a_{0,1}}{\varrho-1} < \infty$$

and is obviously fulfilled for $\varrho > 1$. This completes the proof of theorem 1.

2. ***GI/M/1* service systems with specified forms of feedback.** We shall now consider six special cases of *GI/M/1* service systems which are displayed in Fig. 1 in form of rectangles numbered 2.1-2.6. Inside any rectangle there is given Kendall's symbol summarizing the assumptions concerning the system and the specification of the form of feedback effects on the service rate.

2.1. *GI/M/1 service system with feedback* $\mu\{n(t)\} = \mu\cdot n(t)$. This system is equivalent to the $GI/M/\infty$ service system with no feedback. For $i \geqslant 0$, $k \geqslant 0$ and $k+i > 0$ we have

$$a_{i,k+i} = \int_0^{\infty} \binom{k+i}{i}(1-e^{-\mu x})^i e^{-\mu k x}\, dG(x) \tag{10}$$

$$= \sum_{j=0}^{i}\int_0^{\infty}\binom{k+i}{i}\binom{i}{j}(-1)^j e^{-\mu(j+k)x}\, dG(x)$$

$$= \binom{k+i}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j} G^*\{\mu(j+k)\}.$$

Let

$$\Phi(s) = \sum_{j=0}^{\infty} p_j s^j$$

be the generating function of the sequence $\{p_j\}$.

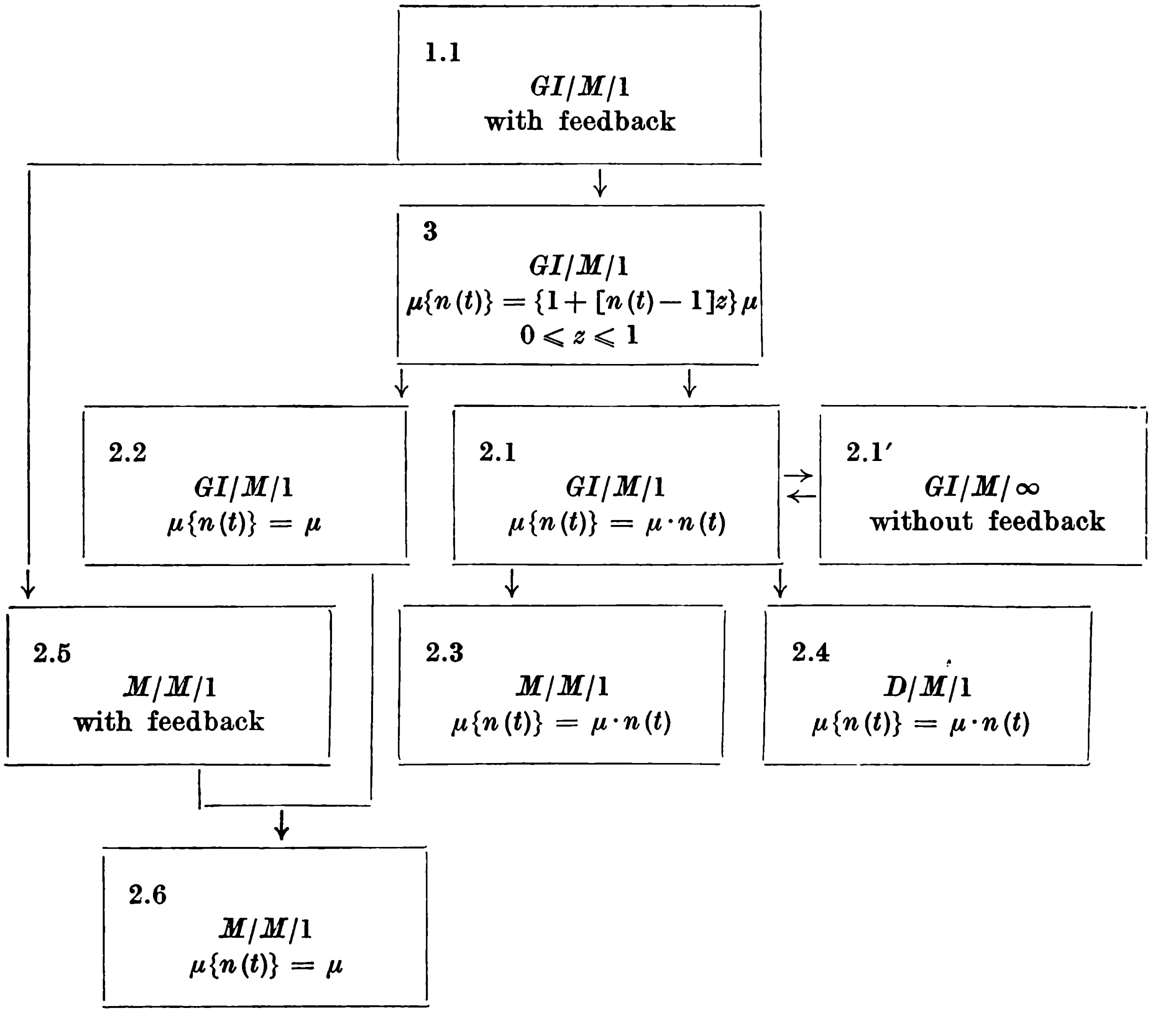

Fig. 1

LEMMA 1. *For the $GI/M/1$ service system with feedback $\mu\{n(t)\} = \mu\cdot n(t)$ the generating function $\Phi(s)$ fulfils the equation*

$$\Phi(s) = \int_0^\infty (P+Qs)\Phi(P+Qs)\,dG(x), \tag{11}$$

where $P = 1-e^{-\mu x}$ and $Q = e^{-\mu x}$.

Proof. The multiplication of appropriate equations of system (8) by factors s^j and the summation over $j = 0, 1, \ldots$ yield

$$\Phi(s) = \sum_{j=0}^{\infty} p_j s^j = p_0[a_{0,1}s^0 + a_{0,1}s^1] +$$
$$+ p_1[a_{1,2}s^0 + a_{1,2}s^1 + a_{0,2}s^2] + \ldots +$$
$$+ p_j[a_{j,j+1}s^0 + a_{j,j+1}s^1 + \ldots + a_{0,j+1}s^{j+1}] + \ldots$$

Substituting $a_{i,k+i}$ from (10) we obtain

$$\Phi(s) = \int_0^\infty \Big\{p_0\Big[\binom{1}{1}P^1Q^0s^0+\binom{1}{0}P^0Q^1s^1\Big]+$$

$$+p_1\Big[\binom{2}{2}P^2Q^0s^0+\binom{2}{1}P^1Q^1s^1+\binom{2}{0}P^0Q^2s^2\Big]+\ldots+$$

$$+p_j\Big[\binom{j+1}{j+1}P^{j+1}Q^0s^0+\binom{j+1}{j}P^jQ^1s^1+\ldots+$$

$$+\binom{j+1}{0}P^0Q^{j+1}s^{j+1}\Big]+\ldots\Big\}dG(x)$$

$$= \int_0^\infty \{p_0[P+Qs]^1+p_1[P+Qs]^2+\ldots+p_j[P+Qs]^j+\ldots\}dG(x)$$

$$= \int_0^\infty (P+Qs)\Phi(P+Qs)\,dG(x),$$

which completes the proof of lemma 1.

COROLLARY 1. *For the GI/M/1 service system with feedback* $\mu\{n(t)\} = \mu\cdot n(t)$ *we have*

$$\frac{\Phi^{(n)}(1)}{n!} = \prod_{k=1}^{n} \frac{G^*(k\mu)}{1-G^*(k\mu)} \qquad (n>0). \tag{12}$$

This follows easily by direct differentiation of formula (11) which yields

$$\Phi^{(n)}(s) = \int_0^\infty Q^n[n\Phi^{(n-1)}(P+Qs)+(P+Qs)\Phi^{(n)}(P+Qs)]\,dG(x), \tag{13}$$

where $P = 1-e^{-\mu x}$ and $Q = e^{-\mu x}$.

Thus we have

$$\Phi^{(n)}(1) = \int_0^\infty Q^n[n\Phi^{(n-1)}(1)+\Phi^{(n)}(1)]\,dG(x)$$

$$= [n\Phi^{(n-1)}(1)+\Phi^{(n)}(1)]G^*(n\mu)$$

and

$$\Phi^{(n)}(1) = n\Phi^{(n-1)}(1)\frac{G^*(n\mu)}{1-G^*(n\mu)} = n!\prod_{k=1}^{n}\frac{G^*(k\mu)}{1-G^*(k\mu)},$$

which completes the proof of (12).

Formula (12) was given by Takács (see [5], p. 166). The values of $\Phi^{(n)}(1)/n!$ are known as *binomial moments* of the distribution of p_j $(j = 0, 1, \ldots)$:

$$\frac{\Phi^{(n)}(1)}{n!} = \sum_{j=n}^{\infty} \binom{j}{n} p_j .$$

The knowledge of binomial moments is sufficient for the determination of the probabilities

$$p_j = \frac{1}{j!} \sum_{i=0}^{\infty} (-1)^i \frac{\Phi^{(i+j)}(1)}{i!} \qquad (j \geqslant 0). \tag{14}$$

2.2. ***GI/M/1 service system with feedback*** $\mu\{n(t)\} = \mu$. This case has been dealt with by Prabhu [4]. For this system the solution is given by

$$p_j = (1-\varsigma)\varsigma^j \qquad (j \geqslant 0),$$

where ς is the root of the equation

$$G^*\{\mu(1-s)\} = s$$

satisfying the condition $0 < \varsigma < 1$.

2.3. ***M/M/1 service system with feedback*** $\mu\{n(t)\} = \mu\cdot n(t)$. Applying corollary 1, one can easily determine the limit distribution for the probabilities of states in the $M/M/1$ system with a feedback of the form $\mu\{n(t)\} = \mu\cdot n(t)$. From (12) and from $G^*(s) = \lambda/(\lambda+s)$ we obtain

$$\Phi^{n)}(1) = n! \prod_{k=1}^{n} \frac{G^*(k\mu)}{1-G^*(k\mu)} = n! \prod_{k=1}^{n} \frac{\lambda/(\lambda+k\mu)}{1-\lambda/(\lambda+k\mu)} = n! \prod_{k=1}^{n} \frac{\lambda}{k\mu} = \varrho^n,$$

where $\varrho = \lambda/\mu$.

Substituting this into (10) we have

$$p_j = \frac{1}{j!} \sum_{i=0}^{\infty} (-1)^i \frac{\varrho^{i+j}}{i!} = \frac{\varrho^j}{j!} e^{-\varrho}.$$

Hence the limiting distribution p_j in the $M/M/1$ service system with a feedback of the form $\mu\{n(t)\} = \mu\cdot n(t)$ is a Poisson distribution with the parameter ϱ

$$p_j = \frac{\varrho^j}{j!} e^{-\varrho} \qquad (j \geqslant 0). \tag{15}$$

This is identical with the t-limiting distribution of the state $n(t)$ of the system $M/M/\infty$ (see Khintchine [6], p. 94), where $n(t)$ is the number of units in the system considered as a stochastic process of the continuous time parameter t. The identity of the limit distributions of continuous time stochastic processes and of imbedded Markov chains for single channel models has been studied by Foster [2].

2.4. *D/M/1 service system with feedback* $\mu\{n(t)\} = \mu\cdot n(t)$. Now we shall find only the limiting probability p_0 which is the probability of immediate service for an entering unit while the system is working under stationary conditions. Now we assume the deterministic distribution of interarrival times

$$G(x) = \begin{cases} 0 & \text{for } x \leqslant 1/\lambda, \\ 1 & \text{for } x > 1/\lambda, \end{cases}$$

whence

$$G^*(k\mu) = e^{-\frac{\mu}{\lambda}k} = e^{-k\varkappa}.$$

Substituting this in (12) we obtain

$$\begin{aligned} \Phi^{(n)}(1) &= n!\frac{e^{-\varkappa(1+2+\ldots+n)}}{(1-e^{-\varkappa})(1-e^{-2\varkappa})\ldots(1-e^{-n\varkappa})} \\ &= n!\frac{a^{n(n+1)/2}}{(1-a)(1-a^2)\ldots(1-a^n)}, \qquad \text{where } a = e^{-\varkappa}. \end{aligned}$$

From (14) it follows that

$$p_0 = 1+\sum_{n=1}^{\infty}\frac{(-1)^n a^{n(n+1)/2}}{(1-a)(1-a^2)\ldots(1-a^n)}.$$

An equivalent formula may be found from (11) which for the considered system has the form

$$\Phi(s) = (1-a+as)\Phi(1-a+as). \tag{16}$$

The consecutive application of (16) yields

$$\begin{aligned} p_0 = \Phi(0) &= (1-a)\Phi(1-a) \\ &= (1-a)(1-a^2)\Phi(1-a^2) \\ & \\ &= \prod_{n=1}^{\infty}(1-a^n). \end{aligned}$$

The direct proof of the equivalence of the two formulae for p_0 seems to be rather difficult.

2.5. *$M/M/1$ service system with feedback $\mu\{n(t)\}$.* From (1) and the exponential distribution of the interarrival times we obtain

$$a_{i,k+i} = \lambda \int_0^\infty \Big[\int_0^x b_{k+i,k+1}(y) P\{Y_k > x-y\}dy\Big] e^{-\lambda x}dx \tag{17}$$

$$= \frac{\lambda}{\lambda+\mu_k} \prod_{j=1}^{i} \frac{\mu_{k+j}}{\lambda+\mu_{k+j}} \quad \text{for } i>0,\ k>0.$$

For $i=0$, and $k>0$ we have

$$a_{0,k} = \int_0^\infty P\{Y_k > x\}dG(x) = G^*(\mu_k) = \frac{\lambda}{\lambda+\mu_k}. \tag{18}$$

The system of equations for the limiting probabilities of states of the system is of the form (8), where (4), (5), (17) and (18) determine the corresponding transition probabilities. A detailed study of this system will be given below for a special case.

2.6. *$M/M/1$ service system without feedback $\mu\{n(t)\} = \mu$.* From (17) and (18) we have

$$a_{i,k+i} = \frac{\lambda}{\lambda+\mu}\Big(\frac{\mu}{\lambda+\mu}\Big)^i = pq^i \quad \text{for } i \geqslant 0,\ k>0,$$

where $p = 1-q = \lambda/(\lambda+\mu)$.

Furthermore,

$$P_{i,j} = a_{i-j+1,i+1} = pq^{i-j+1} \quad (0<j\leqslant i+1,\ i\geqslant 0),$$
$$P_{i,0} = a_{i,i+1} = q^{i+1}.$$

Then the system of equations for the limiting probabilities p_n is of the form

$$p_0 = \sum_{i=0}^\infty q^{i+1}p_i, \tag{19}$$
$$p_j = \sum_{i=j-1}^\infty pq^{i-j+1}p_i \quad (j\geqslant 1).$$

Multiplying the corresponding equations of (19) by s^j and summing them over $j = 1, 2, \ldots$, we obtain

$$\Phi(s)-p_0 = \sum_{j=1}^\infty p_j s^j = \sum_{j=1}^\infty s^j\Big[\sum_{i=j-1}^\infty pq^{i-j+1}p_i\Big] \tag{20}$$

$$= \sum_{i=0}^\infty p_i\Big[\sum_{j=0}^{i} pq^{i-j}s^{j+1}\Big] = ps\sum_{i=0}^\infty p_i q^i \sum_{j=0}^{i}\Big(\frac{s}{q}\Big)^j = \frac{ps}{q-s}\{q\Phi(q)-s\Phi(s)\}.$$

For $s = 1$ this yields

$$p_0 = q\Phi(q) \tag{21}$$

and from (20), (21) we get

$$\Phi(s) = \frac{q}{q-ps} p_0. \tag{22}$$

Once again putting $s = 1$ we obtain

$$p_0 = \frac{q-p}{q} = 1-\varrho, \tag{23}$$

and from (22) and (23) we obtain an explicit formula for the generating function

$$\Phi(s) = \frac{q-p}{q} \frac{1}{1-\frac{p}{q}s} = (1-\varrho)\frac{1}{1-\varrho s} = (1-\varrho)\sum_{i=0}^{\infty} \varrho^i s^i.$$

This, finally, gives the limiting probabilities

$$p_j = (1-\varrho)\varrho^j, \quad j = 0, 1, \ldots$$

3. *GI/M/1* **service systems with feedback of the form** $\mu\{n(t)\} = \{1+[n(t)-1]z\}\mu$**.** Now we assume that in the *GI/M/1* service system there is a feedback of the form

$$\mu\{n(t)\} = \{1+[n(t)-1]z\}\mu, \quad z \in [0, 1]. \tag{24}$$

For $z = 0$ this is a special case discussed in 2.2 (*GI/M/1* system without feedback) and for $z = 1$ a $GI/M/\infty$ system (see 2.1). The service rate (24) may be written in the form

$$\mu\{n(t)\} = \mu_1 + n(t)\mu_2,$$

where $\mu_1 = (1-z)\mu$ and $\mu_2 = z\mu$.

Let us first calculate the density function $b_{k+i,k+1}(y)$ of the random variable

$$Z_{k+i,k+1} = Y_{k+i} + Y_{k+i-1} + \ldots + Y_{k+1},$$

where $Y_{k+i}, \ldots, Y_{k+i}$ are independent random variables with exponential distributions

$$P\{Y_{k+j} < y\} = 1 - e^{-\mu_{k+j} y},$$

and

$$\mu_{k+j} = \mu_1 + (k+j)\mu_2. \tag{25}$$

From (8) we have

$$b^*_{k+i,k+1}(s) = \prod_{j=1}^{i} \frac{\mu_1 + (k+j)\mu_2}{\mu_1 + (k+j)\mu_2 + s},$$

and calculating the inverse transform yields

$$b_{k+i,k+1}(y) = C_{k,i} \sum_{j=1}^{i} \frac{(-1)^{j-1}}{(j-1)!(i-j)!\mu^{i-1}} e^{-[\mu_1+(k+j)\mu_2]y},$$

where

$$C_{k,i} = \prod_{j=1}^{i} [\mu_1 + (k+j)\mu_2].$$

Applying (1), we now easily obtain

(26)

$$a_{i,k+i} = C_{k,i} \sum_{j=1}^{i} \frac{(-1)^{j-1}}{(j-1)!(i-j)!\mu^{i-1}} \{G^*[\mu_1 + k\mu_2] + G^*[\mu_1 + (k+j)\mu_2]\}$$

for $i > 0$, $k > 0$ and, referring to (3),

$$a_{0,k} = G^*(\mu_k) \tag{27}$$

for μ_k given by (25).

Let us consider, as an example, the $D/M/1$ system with feedback (24). For constant interarrival times we have

$$G^*[\mu_1 + (k+j)\mu_2] = e^{-[\mu_1+(k+j)\mu_2]/\lambda}. \tag{28}$$

Making use of the obvious equality

$$\sum_{j=1}^{i} \frac{(-1)^{j-1}}{j!(i-j)!} = \frac{1}{i!}$$

and of (28), we obtain

$$a_{i,k+i} = \frac{\prod_{l=1}^{i} [\mu_1 + (k+l)\mu_2]}{\mu^i i!} [1 - e^{-\mu_2/\lambda}]^i e^{-(\mu_1+k\mu_2)/\lambda}$$

for $i = 0$, $k > 0$ and

Results of numerical calculations for $\lambda = 1$ and $\mu = 1$ are shown in Figs. 2 and 3. Fig. 2 shows the dependence of p_0 on the value of the parameter z. It is interesting to study the speed of convergence of the

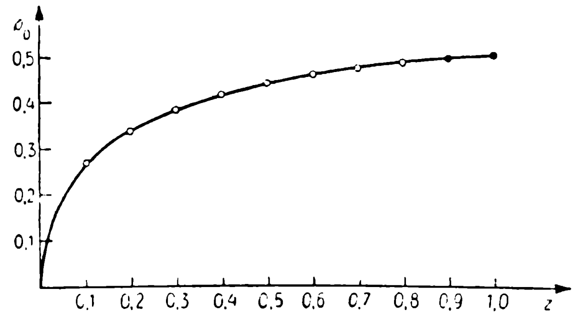

Fig. 2

transient distributions to the limiting distribution. We shall study this, assuming the initial condition

$$P\{N_0 = 0\} = 1, \quad P\{N_0 = j\} = 0, \quad j > 0,$$

and applying the recurrent formula

$$P\{N_{r+1} = j\} = \sum_i{}' P\{N_r = j\} P_{i,j}^{(1)}, \quad j \geqslant 0.$$

Here the transition matrix $P_{i,j}^{(1)}$, determined by (4), (5), (26) and (27), has been truncated to the proper size. The probabilities $P\{N_r = 0\}$ are

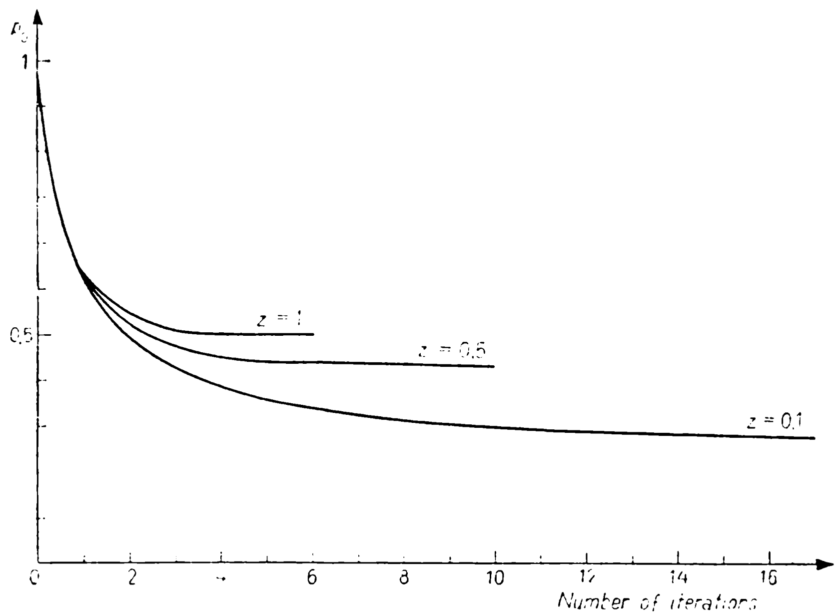

Fig. 3

plotted in Fig. 3 for three values of z and $r = 0, 1, \ldots$ For $z = 1$ it is enough to take $r = 6$ (which corresponds to the 6 iterations in the numerical calculation) to get $P\{N_r = 0\}$ practically equal to the limiting probability $p_0 = 0.5$. For $z = 0.1$ the convergence is much slower and the number of necessary iterations to reach a reasonable approximation of p_0 has to be rather large.

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SYSTEM $GI/M/1$ ZE SPRZĘŻENIEM INTENSYWNOŚCI OBSŁUG Z DŁUGOŚCIĄ KOLEJKI

STRESZCZENIE

W pracy rozpatrzono systemy, w których przedziały oddzielające kolejne momenty zgłoszeń t_r, $r = 0, 1, \ldots$, tworzą ciąg niezależnych, nieujemnych zmiennych losowych $X_r = t_{r+1} - t_r$ o jednakowym rozkładzie $G(x) = P\{X_r < x\}$, $r = 0, 1, \ldots$, natomiast intensywność obsługi $\mu\{n(t)\}$ jest zależna od liczby jednostek $n(t)$ znajdujących się w systemie w chwili t.

Systemy tego typu nazywamy *systemami $GI/M/1$ ze sprzężeniem*. W rozpatrywanych systemach ciąg zmiennych losowych $N_r = n(t_r - 0)$, $r = 0, 1, \ldots$, jest włożonym łańcuchem Markowa. Stan systemu w chwili $t_{r+1} - 0$ jest zdeterminowany przez stan systemu w chwili $t_r - 0$ równością

$$N_{r+1} = N_r + 1 - \eta_r,$$

gdzie η_r jest liczbą zakończonych obsług w przedziale $[t_r, t_{r+1})$. Łańcuch Markowa $\{N_r\}$ w ogólnym systemie jest opisany przez macierz (6), która jest macierzą prawdopodobieństw przejścia ze stanu i do stanu j. W twierdzeniu 1 dany jest warunek wystarczający istnienia granicznych prawdopodobieństw tego łańcucha. Ponadto w pracy przeanalizowano przykłady systemów $GI/M/1$ ze szczególnymi przypadkami sprzężeń. Systemy te są na rys. 1 oznaczone prostokątami o numerach odpowiadających numeracji działów w tej pracy. W każdym z prostokątów znajduje się symbol Kendalla, będący skróconym zapisem założeń systemu, oraz informacja o intensywności obsługi. Dla systemu oznaczonego numerem 3 otrzymano tylko wyniki numeryczne. Przypadek 2.2 był rozpatrywany przez Prabhu [4]. System 2.1 jest równoważny systemowi 2.1′, tzn. systemowi $GI/M/\infty$ bez sprzężenia; system ten był rozważany przez Takácsa [5]. W tej pracy analogiczne rezultaty otrzymano przy użyciu innych

metod. Opis przypadków 2.3 i 2.4 można natychmiast otrzymać przez podstawienie odpowiednich rozkładów do ogólnych wzorów otrzymanych dla systemu 2.1. System 2.5 można analizować bez użycia włożonych łańcuchów Markowa; przy użyciu włożonych łańcuchów Markowa nie udało się w tym przypadku otrzymać bardziej efektywnych rezultatów niż w najogólniejszym przypadku 1. Analizę najprostszego systemu 2.6 można znaleźć w wielu publikacjach z teorii kolejek.

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СИСТЕМА $GI/M/1$ С ОБРАТНОЙ СВЯЗЬЮ ИНТЕНСИВНОСТИ ОБСЛУЖИВАНИЯ С ДЛИНОЙ ОЧЕРЕДИ

РЕЗЮМЕ

В этой работе рассмотрены системы, в которых требования поступают в моменты $t_1 < t_2 < \ldots < t_r < \ldots$, случайные величины $X_r = t_{r+1} - t_r$ независимы в совокупности и обладают одним и тем же законом распределения $G(x) = P\{X_r < x\}, r = 0, 1, \ldots$, а интенсивность обслуживания $\mu\{n(t)\}$ зависит от состояния системы $n(t)$, определенного количеством заявок находящихся в системе в момент времени t.

Системы этого типа называются *системами с обратной связью*. В рассматриваемых системах последовательность случайных величин $N_r = n(t_r - 0), r = 0, 1, \ldots$, является вложенной цепью Маркова. Состояние системы в момент $t_{r+1} - 0$ определено состоянием системы в момент $t_r - 0$,

$$N_{r+1} = N_r + 1 - \eta_r,$$

где η_r — число оконченных обслуживаний в промежутке $[t_r, t_{r+1})$. Цепь Маркова $\{N_r\}$ в общей системе описана матрицей (6), которая является матрицей вероятностей перехода системы из состояния i в состояние j. В теореме 1 дано достаточное условие существования предельных вероятностей этой цепи. Кроме того в работе исследуются примеры систем $GI/M/1$ с частными случаями обратной связи. Эти системы находятся на рис. 1, где они обозначены прямоугольниками с номерами, которые соответствуют номерам разделов этой работы. В каждом из этих прямоугольников находится символ Кендалла, являющийся краткой записью предположений системы, а также и информацией о интенсивности обслуживания. Для системы обозначенной номером 3 получены только численные результаты. Случай 2.2 был рассмотрен Прабху [4]. Система 2.1 эквивалентна системе $GI/M/\infty$ без обратной связи, т.е. системе 2.1′. Эта система была рассмотрена Такачём [5]; в настоящей работе получены аналогические результаты другим методом. Описание случаев 2.3 и 2.4 можно непосредственно получить с помощью подстановки соответствующих распределений в общих формулах, полученных для системы 2.1. Систему 2.5 можно анализировать употребляя вложенные цепи Маркова; при использовании вложенных цепей Маркова не удалось в этом случае получить более эффективных результатов, чем в общем случае 1. Анализ простейшей системы 2.6 можно найти во многих работах по теории массового обслуживания.