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ALGORITHM 46

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# DETERMINATION OF AN INTERPOLATING QUADRATIC SPLINE FUNCTION

**1. Procedure declaration.** For a given number $n > 0$, nodes $x_i$ $(i = 0, 1, \ldots, n)$ and numbers $y_i$ $(i = 0, 1, \ldots, n+1)$, procedure *interp2cf* determines the coefficients of the quadratic spline function $s$ such that

(i) $s \in C^1_{\langle x_0, x_n \rangle}$,

(ii) in each subinterval $\langle x_i, x_{i+1})$ $(i = 0, 1, \ldots, n-1)$, $s$ is a polynomial of degree at most 2,

(iii) $s(x_0) = y_0$, $s(x_n) = y_{n+1}$,

(iv) for $t_i = (x_{i-1}+x_i)/2$, the equality $s(t_i) = y_i$ $(i = 1, 2, \ldots, n)$ holds.

Data:

$n$ — number of nodes of the function $s$ less 1;

$x[0:n]$ — array of nodes of the function $s$ (ordered in an increasing way);

$y[0:n+1]$ — array of ordinates of the function $s$ at nodes $x_0$ and $x_n$, and at the points $t_i$ $(i = 1, 2, \ldots, n)$.

Results:

$a, b, c[0:n-1]$ — arrays of coefficients of the function $s$.

In each subinterval $\langle x_i, x_{i+1})$ $(i = 0, 1, \ldots, n-1)$ the function $s$ is of the form

$$s(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i. \tag{1}$$

**2. Method used.** Let us write

$$s_i = s(x_i) \quad (i = 0, 1, \ldots, n)$$

(in virtue of (iii) we have $s_0 = y_0$, $s_n = y_{n+1}$),

$$h_i = x_i - x_{i-1} \quad (i = 1, 2, \ldots, n),$$

$$d_i = h_{i+1}/(h_i + h_{i+1}), \quad e_i = 1 - d_i \quad (i = 1, 2, \ldots, n-1).$$

```
procedure interp2cf(n,x,y,a,b,c);
 value n;
 integer n;
 array x,y,a,b,c;
 begin
  integer i,i1,n2;
  real hi,hi1,s,s1,xi,xi1,yi,yi1;
  array h[1:n];
  xi1:=x[1];
  h[1]:=hi:=xi1-x[0];
  xi:=xi1;
  xi1:=x[2];
  h[2]:=hi1:=xi1-xi;
  s:=c[1]:=hi/(hi+hi1);
  s1:=1.0-s;
  hi:=hi1;
  xi:=xi1;
  yi:=y[2];
  yi1:=c[0]:=y[0];
  b[1]:=4.0×(s1×y[1]+s×yi)-s1×yi1;
  n2:=n-2;
  for i:=2 step 1 until n2 do
   begin
    i1:=i+1;
    xi1:=x[i1];
    yi1:=y[i1];
    h[i1]:=hi1:=xi1-xi;
    s:=c[i]:=hi/(hi+hi1);
    s1:=a[i]:=1.0-s;
    b[i]:=4.0×(s1×yi+s×yi1);
```

```
    xi:=xi1;
    yi:=yi1;
    hi:=hi1
   end i;
  n2:=n2+1;
  h[n]:=hi1:=x[n]-xi;
  s1:=a[n2]:=hi1/(hi+hi1);
  s:=1.0-s1;
  b[n2]:=4.0×(s1×yi+s×y[n])-s×y[n+1];
  xi:=c[1]:=-.3333333333×c[1];
  yi:=b[1]:=.3333333333×b[1];
  for i:=2 step 1 until n2 do
   begin
    hi:=a[i];
    s:=hi×xi+3.0;
    xi:=c[i]:=-c[i]/s;
    yi:=b[i]:=(b[i]-hi×yi)/s
   end i;
  c[n2]:=yi;
  n2:=n2-1;
  for i:=n2 step -1 until 1 do
   yi:=c[i]:=c[i]×yi+b[i];
  hi:=h[1];
  yi:=y[0];
  yi1:=y[1];
  s1:=c[1];
  b[0]:=(-3.0×yi-s1+4.0×yi1)/hi;
  a[0]:=2.0×(yi+s1-2.0×yi1)/(hi×hi);
  n2:=n2+1;
  for i:=2 step 1 until n2 do
```

```
begin
  i1:=i-1;
  hi:=h[i];
  xi:=c[i];
  yi:=y[i];
  b[i1]:=(-3.0×s1-xi+4.0×yi)/hi;
  a[i1]:=2.0×(s1+xi-2.0×yi)/(hi×hi);
  s1:=xi
 end i;
hi:=h[n];
s1:=2.0×y[n];
s:=y[n+1];
b[n2]:=(-3.0×xi-s+2.0×s1)/hi;
a[n2]:=2.0×(xi+s-s1)/(hi×hi)
end interp2cf
```

The function $s$ must satisfy the following system of equations [2] with unknowns $s_i$:

$$d_i s_{i-1} + 3s_i + e_i s_{i+1} = 4d_i y_i + 4e_i y_{i+1} \qquad (i = 1, 2, \ldots, n-1). \tag{2}$$

In the body of the procedure the method described in [1] is used to find the solution of system (2). In fact, we have

$$s(x_i) = s_i, \quad s(x_{i+1}) = s_{i+1} \quad \text{and} \quad s(t_{i+1}) = y_{i+1} \quad (i = 0, 1, \ldots, n-1).$$

Thus, by (1), we obtain

$$c_i = s_i, \quad a_i h_{i+1}^2 + b_i h_{i+1} + c_i = s_{i+1}, \quad a_i h_{i+1}^2/4 + b_i h_{i+1}/2 + c_i = y_{i+1} \quad (i = 0, 1, \ldots, n-1).$$

Hence

$$a_i = 2(s_i + s_{i+1} - 2y_{i+1})h_{i+1}^2, \quad b_i = (-3s_i - s_{i+1} + 4y_{i+1})/h_{i+1}, \quad c_i = s_i \quad (i = 0, 1, \ldots, n-1).$$

**3. Certification.** Procedure *interp2cf* has been extensively tested on the Odra 1204 computer. The obtained results were correct.

Example. We have

$$n = 3, \quad x_i = i \ (i = 0, 1, 2, 3), \quad y_0 = y_2 = y_4 = 0, \quad y_1 = y_3 = 1.$$

The exact values of the coefficients $a_i$, $b_i$ and $c_i$ are the following:

$$a_0 = a_2 = -20/7, \quad a_1 = 16/7, \quad b_0 = 24/7, \quad b_1 = -b_2 = -16/7,$$
$$c_0 = 0, \quad c_1 = c_2 = 4/7.$$

The calculated values of the coefficients of the function $s$ are con tained in Table 1.

TABLE 1

| $i$ | $a_i$ | $b_i$ | $c_i$ |
|---|---|---|---|
| 0 | −2.857142857 | 3.428571429 | 0.0 |
| 1 | 2.285714286 | −2.285714286 | 0.5714285714 |
| 2 | −2.857142857 | 2.285714286 | 0.5714285714 |

The time of execution of the procedure *interp2cf* is approximately equal to $0.03n+0.99$ sec.

**References**

[1] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The theory of splines and their applications*, New York 1967.
[2] M. J. Marsden, *Quadratic spline interpolation*, Bull. Amer. Math. Soc. 80 (1974), p. 903-906.

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ALGORYTM 46

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**WYZNACZENIE INTERPOLUJĄCEJ FUNKCJI SKLEJANEJ STOPNIA DRUGIEGO**

STRESZCZENIE

Dla danej liczby $n > 0$, węzłów $x_i$ $(i = 0, 1, \ldots, n)$ i liczb $y_i$ $(i = 0, 1, \ldots, n+1)$ procedura *interp2cf* wyznacza współczynniki interpolującej funkcji sklejanej $s$ stopnia drugiego, takiej, że

(i) $s \in C^1_{\langle x_0, x_n \rangle}$,

(ii) w każdym podprzedziale $\langle x_i, x_{i+1})$ $(i = 0, 1, \ldots, n-1)$ $s$ jest wielomianem stopnia co najwyżej 2,

(iii) $s(x_0) = y_0$, $s(x_n) = y_{n+1}$,

(iv) dla $t_i = (x_{i-1}+x_i)/2$ mamy $s(t_i) = y_i$ $(i = 1, 2, \ldots, n)$.

Dane:

$n$ — liczba określająca liczbę węzłów funkcji $s$;

$x[0:n]$ — tablica węzłów funkcji $s$ (uporządkowanych w sposób rosnący);

$y[0:n+1]$ — tablica rzędnych funkcji $s$ w węzłach $x_0$ i $x_n$ oraz w punktach $t_i$ $(i = 1, 2, \ldots, n)$.

Wyniki:

$a, b, c[0:n-1]$ — tablice współczynników funkcji $s$.

W każdym podprzedziale $\langle x_i, x_{i+1})$ $(i = 0, 1, \ldots, n-1)$ funkcja $s$ ma postać

$$s(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i.$$

Przykłady kontrolne potwierdziły poprawność procedury.