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A REAL-TIME METHOD FOR FAST DETERMINATION OF THE FUNDAMENTAL FREQUENCY

1. Introduction and technical requirements.

1.1. The first attempts to determine the period of a complicated function were started in the 18th century, when Lagrange had published his initial works. The necessity of solving such a problem has been dictated by the development of both astronomy and geophysics. Today, the main field of application of period determination methods are vibrating phenomena in technics. On the other hand the development of the theory of statistical phenomena makes it possible to generalize the problem and to formulate it more clearly. Also the advances in telecommunication based upon the periodic transient as a carrier of information set to the methods of periodicity determination quite new tasks.

While previously the time needed for problem solving was not a most important parameter, now the interest in telecommunication stresses the importance of developing methods which make it possible to solve the period determination problem in "real time". On the other hand the realization of such a method in the form of electronic equipment should be comparatively inexpensive for the actual technology.

In this paper we propose a solution of the problem mentioned above. The concrete technical conclusions are drawn from the theory constructed for this purpose. Some information concerning the special electronic equipment and realizations may be also found in [1], [2], [3], and [4]. From the multitude of applications of our method we can mention for example:

a. Determination of the fundamental frequency of a human voice. It is an important parameter playing the role in analysis and synthesis of speach as a carrier of information concerning affection of the speach.

b. Telemetric methods of information transmission by pulse modulation systems. It may happen that the transmission channel introduces phase-shift and attenuation for some frequencies creating additional zeros inside of a pulse. On the receiver side the previous shape of the pulse must be reproduced, and this requires the elimination of extra zeros, too.

c. Automatic recording of frequency in musical research (see e.g. [3]).

1.2. Let us consider a periodical transient with unknown period. Without loss of generality we may assume that it is given in the form of electrical voltage. The value of period is of our interest and the time given for its determination is only its small multiple. At the same time it is desired to convert step-changes in period-length into analogous form of voltage without the distorsion caused by the time-constant. The measurement must take place in real time, i.e. without the conventional memory circuits from which the information could be taken for a multiplex analysis. Attention should be paid to the possibility of the bounded but high contents of harmonics in the transient. For this reason the application of conventional methods of frequency measurement based upon counting zeros or measuring time between consecutive zeros by analog or digital device is impossible. Up today only the method of autocorrelation function in its full form, or limited to some symptomatic feature of function, gives the correct solution of the problem (see e.g. [6], Chapter 6).

In the sequel we shall show a number of transformations defined on periodic functions and easy for electrical realization. By means of such a transformation the periods of complicated signals can also be designated. Our method seems to be easier in electronic realization and more reasonable for practical purposes than the methods used up today.

2. Mathematical model and description of the method.

2.1. As a model for our periodical transient we choose, for the present, a real Lebesgue-measurable function $f_0(t)$ having bounded variation on every finite interval, finite period T, and the mean value equal zero. Every such function may be represented in the form of Fourier series

$$\sum_{n=1}^{\infty}\big(a_n\cos[(2\pi nt)/T]+b_n\sin[(2\pi nt)/T]\big)$$

almost everywhere convergent to f_0. It follows in the straightforward manner from the Dirichlet-Jordan theorem (see e.g. [7], Chapter II, 8) and from the fact the set of discontinuity points of a function with bounded variation has measure zero. Technical requirements (dumping of high frequencies) tell us that we can cut our series up without loss of precision. Now we see that a trigonometric polynomial

$$(1)\qquad f(t)=\sum_{n=1}^{M}\big(a_n\cos[(2\pi nt)/T]+b_n\sin[(2\pi nt)/T]\big),$$

where M is sufficiently large (and equal to the highest admissible frequency in the signal), is the enough reasonable model for our purpose;

it is also the best $L^2[0, T]$-approximation of f_0 by a trigonometric polynomial of M-th order. The coefficients a_n and b_n are real and we assume $a_M^2 + b_M^2 \neq 0$. In the sequel we shall also deal with polynomials of the form (1), where coefficients are random variables.

On the real axis there is a countable set $\mathscr{D}_f$, say $\ldots < t_{-1} < t_0 < < t_1 < \ldots$, of points (further called *distinguished zeros of* f) such that *for every* α_i, β_i *such that*: $\alpha_i < t_i < \beta_i$ *and both* $|\alpha_i - t_i|$ *and* $|\beta_i - t_i|$ *are sufficiently small*; *holds*

$$f(\alpha_i) < 0, \quad f(\beta_i) > 0, \quad i = \ldots, -1, 0, 1, \ldots$$

For a differentiable function $x(t)$ (and such is $f(t)$) the distinguished zeros t_i are exactly these ordinary zeros of x ($x(t_i) = 0$) for which the additional condition $x'(t_i) > 0$ is satisfied. In every interval of lenght T the number(1) $N = |\mathscr{D}_f \cap [t, t+T)|$ of distinguished zeros is finite and does not exceed M (see e.g. [7], Chapter X, Theorem 1.7.). Of course $t_{k-N} - t_k = T$ for any integer k.

Now we shall outline a method for determination of the period of function f. First, we transform f into the function F^* by means of operator $\mathfrak{C}^*$. Its acting can be described by the formula(2)

$$F^*(t) = (\mathfrak{C}^* f)(t) = \{t_i - t_{i-1} \quad \text{for} \quad t \in [t_i, t_{i+1}), \quad i = \ldots, -1, 0, 1, \ldots\}.$$

We recall that $t_i \in \mathscr{D}_f$. By $\mathfrak{C}$ we denote the operator that is defined by means of formula

$$\mathfrak{C}f = F = \mathfrak{C}^* f - \overline{\mathfrak{C}^* f}$$

where

$$\overline{\mathfrak{C}^* f} = \frac{1}{T} \int_t^{t+T} (\mathfrak{C}^* f)(t)\, dt. \tag{2}$$

The mean value $\overline{\mathfrak{C}^* f}$ should be calculated in an approximate way since the period T is essentially unknown. It is another question and we do not discuss it in detail, but at once it is easy to see that the interval of inte-

(1) By $|A|$ we denote the number of elements of the set A.

(2) Instead of $\mathfrak{C}^*$ we may use some other operators, for example

$$(\mathfrak{C}_1^* f)(t) = \{ \int_{t_{i-1}}^{t_i} |f(t)|\, dt \quad \text{for} \quad t \in [t_i, t_{i+1}), \quad i = \ldots, -1, 0, 1, \ldots\},$$

or

$$(\mathfrak{C}_2^* f)(t) = \{ \max_{t \in [t_{i-1}, t_i)} |f(t)| \quad \text{for} \quad t \in [t_i, t_{i+1},) \quad i = \ldots, -1, 0, 1, \ldots\}.$$

Their realization in the form of electronic circuits is also possible and the rest of the method remains the same.

gration in (2) should be then taken as large as it is possible. $\mathfrak{C}f$ is constant over the intervals between two consecutive distinguished zeros and, as instantly follows, it also has the period T. Obviously $\mathscr{D}'_{\mathfrak{C}f} \subset \mathscr{D}_f$ and, moreover,

$$|\mathscr{D}_{\mathfrak{C}f} \cap [t, t+T)| \leqslant (N+1)/2 \leqslant (M+1)/2.$$

Thus in the sequence of (well defined) iterations of $\mathfrak{C}$: $\mathfrak{C}f, \mathfrak{C}^2 f, \ldots, \mathfrak{C}^n f, \ldots$ the number of distinguished zeros $|\mathscr{D}_{\mathfrak{C}^n f} \cap [t, t+T)|$ decreases at least in a geometric progression. If $|\mathscr{D}_{\mathfrak{C}^K f} \cap [t, t+T)| = 1$ we finish our procedure. Now for the function $\mathfrak{C}^K f$ the distance between two succesive distinguished zeros is the sought period T of f. We easily can measure this distance ($\mathfrak{C}^{K+1} f \equiv T$).

From the electronic point of view the number K of necessary iterations is of interest. Complicacy of a circuit to perform succesive transformations $\mathfrak{C}^n$ increases together with K. K is necessarily finite (except the case mentioned below) by virtue of assumptions on the function f and, as it is easy to check, its order can be estimated by $\log_2 M$. Therefore we see that the complicacy of the needed circuit (it will be described in Section 3) depends on the highest admissible frequency in the signal.

It could also happen that $|\mathscr{D}_{\mathfrak{C}^n f} \cap [t, t+T)| \neq 1$ for all integers n. This is possible only if $|\tau_i - \tau_{i-1}| = \text{const}$ (independently of i), where $\tau_i \in \mathscr{D}_{\mathfrak{C}^n f}$ for some n, but this constant is not equal to T. Then, obviously, T is a multiple of this constant and to determine it we may use other methods (for example other transformations indicated in the footnote 2).

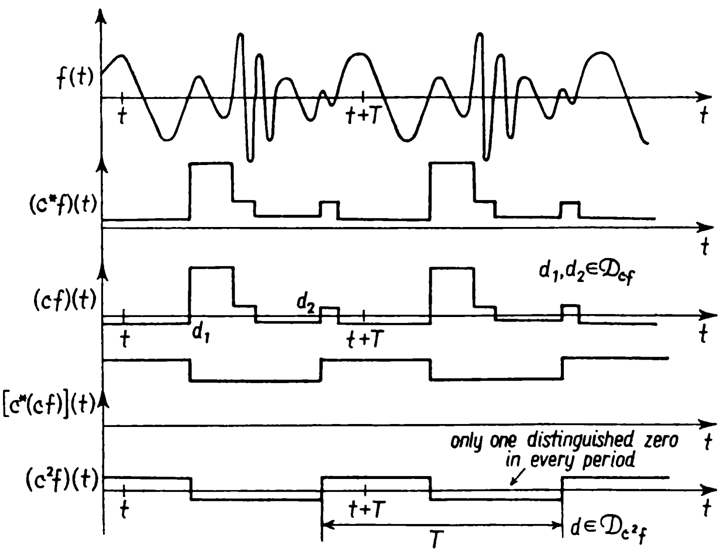

Fig. 1

2.2. An example; in Figure 1 we show a typical example of the functioning of the method which has been described in Section 2.1.

2.3. Let now

$$f(t, \omega) = \sum_{n=1}^{M} \big(a_n(\omega)\cos[(2\pi nt)/T] + b_n(\omega)\sin[(2\pi nt)/T]\big)$$

be a trigonometric polynomial with randomly distributed coefficients ($\omega \in [0, 1]$ is a random parameter). Suppose that the random variables $a_n(\omega)$ and $b_n(\omega)$ have finite first moments and that the distribution functions of the a_n's and b_n's are not degenerate i.e. for all n and for each real number a $Pr\{a_n(\omega) = a\} = Pr\{b_n(\omega) = a\} = 0$. Under these assumptions, with probability 1, the realization $f(t, \omega_0)$, $\omega_0 \in [0, 1]$, of the stochastic process $f(t, \omega)$ is a periodical function with period T and the number of distinguished zeros inside of the period is, with probability 1, less than M. Hence the whole precedent theory applies "almost surely". The special interest is paid to the case when the a_n's and b_n's are normally distributed mutually independent random variables. This corresponds to the existence of a white noise in our circuit. It would be interesting to estimate in this case more precisely, from the probabilistic point of view, the number of real zeros in the period. As far as we know, there are no general results in this field. We may quote only the theorem of Dunnage [5] concerning the random trigonometric polynomials of the form

$$\varphi(t, \omega) = \sum_{n=1}^{M} a_n(\omega)\cos[(2\pi nt)/T],$$

where $\{a_n(\omega)\}$ is a sequence of independent, normally distributed random variables, each having the distribution $N[0, 1]$. Using the Kac-Steinhaus method of independent functions Dunnage proved that when M is large the polynomial $\varphi(t, \omega)$ with probability $P = 1 - (\log M)^{-1}$ has $(2/\sqrt{3})M + O(M^{11/13}(\log M)^{3/13})$ real zeros in the interval $[t, t+T)$. Applying this result for our purposes we get that to find the period of φ the number of necessary iterations of $\mathfrak{C}$ is of the order $\log_2(M/\sqrt{3})$. Probability P increases quickly with M.

3. Realization.

The work performed by the circuit presented at the Fig. 2 could be described as follows.

The function $f(t)$ is converted in block A into a set of distinguished zeros, which synchronize block B creating the transformation $\mathfrak{C}$. The zeros from A are delivered to the frequency dividing circuit C and every "Z", zero synchronizes the integrating block D. The output voltage from D divided by "Z" is compared in the difference creating circuit F.

If the output from F is equal to zero, the information is delivered to gate G so that the path for the distinguished zeros from A should be opened. The fast response time-to-voltage converter converts the distance between the zeros into the voltage and displays it on the meter. If the output from block F is not equal to zero but is a signal with varying

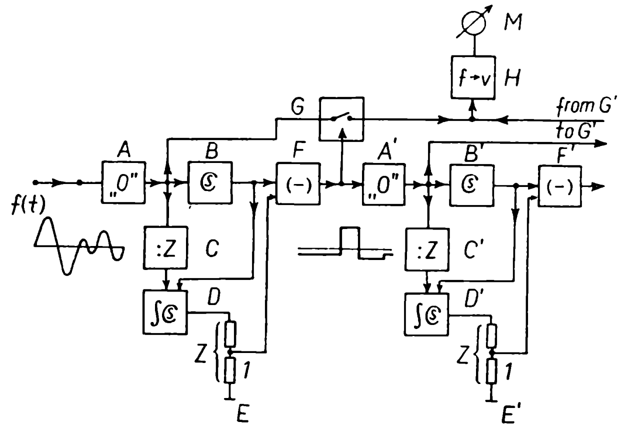

Fig. 2

polarity then the zeros of the new signal are distinguished by block A' and a new transformation $\mathfrak{C}$ is created by the circuit identical to the described one. In this case the converter H utilizes the zeros not from block A, but the zeros created by block A' (under the condition that the output from the corresponding block F' is equal to zero). On the principle of iterating the transformations an arbitrary number of desribed circuits can be connected. The limitation given in Section 2 can also be omitted on the expence of response time. Practical circuits have been built, proving the described theory. On the base of them some instruments have been constructed for different practical purposes.

References

[1] J. A. Czekajewski and P. A. Tove, *Pulse period meter with short response time applied to cardiotachometry*, Electronic Engineering (May 1964), pp. 1-5.

[2] J. A. Czekajewski, *Cardiotachometer with fast response*, Medical Electronics and Biological Engineering 3 (1965), pp. 57-63.

[3] J. A. Czekajewski, L. Isaksson, B. Norman and P. A. Tove, *A direct-recording frequency and amplitude meter for analysis of musical and other waveforms*, The Journal of the Acoustical Society of America 39 (1966), pp. 362-371.

[4] J. A. Czekajewski, L. Isaksson, B. Norman and P. A. Tove, *Frekvensmätare med kort inställningstid*, Elektronik (April 1966), pp. 64-73.

[5] J. E. A. Dunnage, *The number of real zeros of a random trigonometric polynomial*, Proc. London Math. Soc. 16 (1966), pp. 53-84.

[6] F. H. Lange, *Korrelationselektronik*; *Grundlagen und Anwendungen der Korrelationsanalyse in der modernen Nachrichten-, Mess-und Regelungstechnik*, 2-nd edition, VEB Verlag Technik, Berlin 1962.

[7] A. Zygmund, *Trigonometric series*, Vols. I and II, University Press, Cambridge 1959.

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METODA PRZYBLIŻONEGO WYZNACZANIA CZĘSTOTLIWOŚCI PODSTAWOWEJ W CZASIE REALNYM

STRESZCZENIE

W pracy podano pewną metodę szybkiego wyznaczania okresu periodycznych sygnałów o nieznanym okresie. Metoda ta pozwala mierzyć okres skomplikowanych nawet przebiegów bez opóźnień spowodowanych w zwykle w takich przypadkach stosowanych urządzeniach przez stałą czasową lub urządzenia pamięciowe. W celu przedyskutowania metody zbudowano pewien model matematyczny badanego sygnału (dopuszczający również stochastyczne szumy). Wreszcie podano opis i schemat blokowy elektronicznego urządzenia pracującego według podanej i przedyskutowanej metody. Wskazano również kilka zastosowań.

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МЕТОД БЫСТРОГО ОПРЕДЕЛЕНИЯ ОСНОВОЙ ЧАСТОТЫ В РЕАЛЬНОМ ВРЕМЕНИ

РЕЗЮМЕ

В статьи указан метод быстрого определения периода периодического сыгнала. Представленный метод позволяет измерять период даже сложных процессов без запаздывания вызванного влиянием временной постоянной и свойствами запоминающих устройств, которые обычно применяются для этой цели.

Для иллюстрации метода построена математическая модель исследованного сигнала (здесь допускается возможность стохастического шума). В заключении представлено описание и блок схема электронного устройства, работающего по изложенному принципу. Кроме того указаны некоторые применения предлагаемого метода.
