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PURSUIT GAMES WITH BOUNDED ACCELERATIONS

1. H. Steinhaus drew in [12] attention to the mathematical problem of pursuit and evasion as early as in 1925. It appeared that the theory of this game is similar to a certain kind of mechanics of a system of material points into which some antagonistic elements characterizing the games have been introduced. An example of such a game can be given as a pursuit of one ship by another ship or of one plane by another plane, provided that the adversaries do not see each other in the course of pursuit, and that their maximal velocities are bounded; one of the adversaries — the pursuer — tries to capture the evader as soon as possible, while the latter has a quite opposite tendency and makes all attempts to make his evasion as much as possible delayed. It is easy to imagine various generalizations of such games. They can be realized either by introducing a greater number of partners divided into two fighting parties, or by assuming more general conditions for maximal velocities or for spaces in which the pursuit takes place, etc. These problems have been dealt with among others in the papers [3], [7], [11], [13]-[17]. In particular we should like to draw the reader's attention to the paper [17] to which our paper bears many relations.

In the theory of pursuit games considered in this paper we use differential equations; the games described in this way are called differential games. The way in which the problems are formulated here resembles the theory of dynamic processes ([1], [5], [9]). On the other hand, the control process may be considered as a particular case of differential games. Namely, if we simplify the game by assuming that one of the partners knows the strategy of his adversary (the adversary is deciphered) then such a differential game reduces to a problem of the theory of control processes, being called there the problem of the theory of games, though, strictly speaking it loses then its character of game. All, so far used, applications of the theory of control processes to a more general theory, i.e. to the theory of differential games, refer exclusively to such simplified "one-person" games ([2], [4], [10]). This situation can be explained by mathematical difficulties encountered in the theory of control processes which become exceedingly greater if this theory is to be applied to differential games ([5], p. 163). For this reason, although the notions and mathematical tools of the theory of control processes are adjusted to the theory of differential games, no deeper results have been obtained. Therefore concrete problems solved directly are still of interest in the theory of differential games. Since, however, non-trivial problems are usually very difficult, there are not many positive results which could be quoted. Even in the problems which are easy to formulate the authors must take different and non-natural assumptions to obtain at least partial results.

In the present paper we are concerned with a pursuit game with one evader and one pursuer. The pursuit takes place in an n-dimensional Euclidean space. The evader and the pursuer know at each moment the positions and velocity vectors both of their own and of their adversary, their accelerations being regulated by them to a certain extent arbitrarily. Speaking more exactly, according to the physical nature of the problem, we assume that the acceleration of the elements taking part in the game are bounded.

The class of evader-strategies consists of continuous functions with values belonging to an n-dimensional Euclidean space, of the form $f_0(x_0, x_1, \dot{x}_0, \dot{x}_1)$, where $x_0, \dot{x}_0$ denote the position and the velocity of the evader, while $x_1, \dot{x}_1$ denote the respective bectors of the pursuer. A similar class of functions $f_1(x_0, x_1, \dot{x}_0, \dot{x}_1)$ will form the class of pursuer-strategies. We assume that these functions satisfy the following conditions:

$$(1.1) \qquad |f_0(x_0, x_1, \dot{x}_0, \dot{x}_1)| \leqslant a_0, \qquad |f_1(x_0, x_1, \dot{x}_0, \dot{x}_1)| \leqslant a_1$$

where a_0 and a_1 are constants subject to the restriction $a_0 < a_1$. If both parties of the game, their initial positions x_0^0, x_1^0 and velocities $\dot{x}_0^0, \dot{x}_1^0$ being given, choose the strategies f_0 and f_1 then the trajectories of pursuit and evasion are described by the differential equations:

$$(1.2) \qquad \ddot{x}_0 = f_0(x_0, x_1, \dot{x}_0, \dot{x}_1), \qquad \ddot{x}_1 = f_1(x_0, x_1, \dot{x}_0, \dot{x}_1).$$

(In the present paper we assume the following rules of game: $f_0 \epsilon F_0$ and $f_1 \epsilon F_1$, where F_0 and F_1 are classes of functions which guarantee the existence and uniqueness of solutions of the system (1.2).) Let us further remark that the assumptions (1.1) are fulfilled for real pursuit games in which the antagonists really do not have unbounded accelerations at their disposal.

Let $x_0(t)$ and $x_1(t)$ denote the solutions of equations (1.2) satisfying the given initial conditions. By the *time of pursuit* we mean the earliest moment $T > 0$ at which

$$\lim_{t \to T-0} [x_0(t) - x_1(t)] = 0.$$

If we put, in addition, $T = +\infty$ in the case when there is no finite T satisfying the relation given above, T will be uniquely determined by the choice of pursuer and evader strategies

$$T = T(f_0, f_1).$$

Assuming T to be the pay-off function, our game takes a standard form $\langle F_0, F_1, T(f_0, f_1)\rangle$. Clearly, this game depends on the initial positions and velocities. (By admitting different initial conditions we speak of the whole class of related pursuit games.) If a game is determined then its value is called the *optimal pursuit time.*

In the present paper we describe the construction of a certain function $\tau(x_0^0, x_1^0, \dot{x}_0^0, \dot{x}_1^0)$ and prove that this function is an optimal pursuit time within the region in which τ is continuous. At the same time we prove that the game is strictly determined within the region of initial values for which τ is continuous. We give also a construction by which optimal strategies of a game can be determined for given τ.

2. Let us consider a two-person pursuit game in an n-dimensional Euclidean space E_n. The players will be called the pursuer and the evader. The position, velocity and acceleration of the evader at any arbitrary moment t will be denoted by $x_0(t), \dot{x}_0(t), \ddot{x}_0(t)$, respectively, or in coordinates, by $(\xi_0^1(t), \xi_0^2(t), \ldots, \xi_0^n(t))$, $(\dot{\xi}_0^1(t), \dot{\xi}_0^2(t), \ldots, \dot{\xi}_0^n(t))$ and by $(\ddot{\xi}_0^1(t), \ddot{\xi}_0^2(t), \ldots, \ddot{\xi}_0^n(t))$. Similarly, the position, velocity and acceleration of the pursuer at the moment t will be denoted by $x_1(t), \dot{x}_1(t), \ddot{x}_1(t)$ or by $(\xi_1^1(t), \xi_1^2(t), \ldots, \xi_1^n(t))$, $(\dot{\xi}_1^1(t), \dot{\xi}_1^2(t), \ldots, \dot{\xi}_1^n(t))$ and by $(\ddot{\xi}_1^1(t), \ddot{\xi}_1^2(t), \ldots, \ddot{\xi}_1^n(t))$, respectively.

A pursuit game will be in general considered in a 4-dimensional phase space with points $X = (x_0, x_1, \dot{x}_0, \dot{x}_1)$, where x_i and $\dot{x}_i$ represent the position and the velocity of the i-th player ($i = 0, 1$), respectively. In the sequel we shall confine ourselves to a certain domain Ω of the phase space.

We assume the following initial conditions satisfied at moment $t = 0$

$$x_i(0) = x_i^0, \quad \dot{x}_i(0) = \dot{x}_i^0 \quad (i = 0, 1) \tag{2.1}$$

where $X^0 = (x_0^0, x_1^0, \dot{x}_0^0, \dot{x}_1^0) \in \Omega$.

Now we shall describe the strategies of pursuit and evasion. We assume that each player determines his acceleration according to both his own and his opponent's positions and velocities which are assumed to be known to him. In this way the course of a separate pursuit game will be determined by the system of differential equations

$$\ddot{x}_0 = f_0(x_0, x_1, \dot{x}_0, \dot{x}_1), \quad \ddot{x}_1 = f_1(x_0, x_1, \dot{x}_0, \dot{x}_1) \tag{2.2}$$

with initial conditions (2.1).

We shall state now the conditions which are to be fulfilled by the functions $f_0(X)$ and $f_1(X)$.

Let $\tau(X) = \tau(x_0, x_1, \dot{x}_0, \dot{x}_1)$ be a certain fixed function defined and continuous on Ω and having the following properties:

1° $\tau(x_0, x_1, \dot{x}_0, \dot{x}_1) > 0$,

2° for each trajectory(1) $(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t))$ lying within the domain Ω and for each number $T > 0$, if

$$\lim_{t \to T-0} \tau\big(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t)\big) = 0$$

then

$$\lim_{t \to T-0} [x_0(t) - x_1(t)] = 0.$$

Also let F_i $(i = 0, 1)$ be fixed non-empty sets of functions $f_i(X)$ defined and continuous on the domain Ω and such that for any function $f_0 \in F_0$ and for any function $f_1 \in F_1$ there exists $T > 0$ such that for $0 \leqslant t < T$ there exists one and only one solution $x_0(t), x_1(t)$ of the differential equations (2.2) satisfying the initial conditions (2.1), and such that if $T < +\infty$, then

$$\lim_{t \to T-0} \tau\big(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t)\big) = 0. \tag{2.3}$$

Instead of the uniqueness of the solutions of the equations (2.2) with initial conditions (2.1), the authors assume in the papers [2], [4], [10] a stronger condition, namely that the partial derivatites of first order of the functions $f_0(X)$ and $f_1(X)$ are continuous.

We shall assume, moreover, that for each function $f_0 \in F_0$ and for each function $f_1 \in F_1$ and for all $X \in \Omega$ the following conditions are satisfied:

$$|f_0(X)| \leqslant a_0, \qquad |f_1(X)| \leqslant a_1 \tag{2.4}$$

where a_0 and a_1 are constants satisfying the inequality

$$0 \leqslant a_0 < a_1. \tag{2.5}$$

Functions f_0 belonging to the set F_0 will be called *strategies of evasion*, while the functions f_1 belonging to the set F_1 will be called *strategies of pursuit*.

To each pair f_0, f_1 of the evasion and pursuit strategies there corresponds uniquely a number $T > 0$ occurring in the definitions of F_0 and F_1. This number will be called the *time of pursuit*. If the time of pursuit is finite then the boundary point of the domain Ω in which the condition (2.3) holds will be called the *point of capture* (since then $\lim_{t \to T-0} [x_0(t) - x_1(t)] = 0$).

(1) By a trajectory we mean a curve in the phase space $(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t))$, each of the components being of class C^1.

The conditions for F_0, F_1 and Ω described above guarantee that we consider such pursuits and evasions for which the trajectory in the phase space lies either constantly within the domain Ω or if it reaches the boundary point at a certain moment then at this point the capture will take place.

The evader aims to choose such a strategy of evasion that the time of pursuit be as long as possible, whereas the pursuer tries to find out a strategy of pursuit with time as short as possible.

The pursuit time T is a functional defined on the Cartesian product of the sets F_0 and F_1:

$$T = T(f_0, f_1),$$

where $f_0 \in F_0, f_1 \in F_1$. The system $\langle F_0, F_1, T\rangle$ will be called a *pursuit game*. In this definition F_0 is the set of strategies of the first player, F_1 is the set of strategies of the second player, and T is the function of the pay-off [6].

3. Let $\langle F_0, F_1, T\rangle$ be a fixed pursuit game.

LEMMA. *If there exists a function $\varepsilon(X)$ continuous and positive within the domain Ω and*

1^0 *if there exists a strategy of pursuit $f_1^* \in F_1$ such that for any arbitrary strategy of evasion $f_0 \in F_0$ and for every $X \in \Omega$ and for every number δ such that $0 \leqslant \delta < \varepsilon(X)$ we have*

$$(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0\delta^2,\ x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1^*\delta^2,\ \dot{x}_0 + f_0\delta,\ \dot{x}_1 + f_1^*\delta) \in \Omega \tag{3.1}$$

and

$$\tau(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0\delta^2,\ x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1^*\delta^2,\ \dot{x}_0 + f_0\delta,\ \dot{x}_1 + f_1^*\delta) - \tau(x_0, x_1, \dot{x}_0, \dot{x}_1) \leqslant -\delta; \tag{3.2}$$

2^0 *if there exists a strategy of evasion $f_0^* \in F_0$ such that for any arbitrary strategy of pursuit $f_1 \in F_1$ and for every $X \in \Omega$ and for every number δ such that $0 \leqslant \delta < \varepsilon(X)$ we have*

$$(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0^*\delta^2,\ x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1\delta^2,\ \dot{x}_0 + f_0^*\delta,\ \dot{x}_1 + f_1\delta) \in \Omega \tag{3.3}$$

and

$$\tau(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0^*\delta^2,\ x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1\delta^2,\ \dot{x}_0 + f_0^*\delta,\ \dot{x}_1 + f_1\delta) - \tau(x_0, x_1, \dot{x}_0, \dot{x}_1) \geqslant -\delta, \tag{3.4}$$

then the pursuit game $\langle F_0, F_1, T\rangle$ is determined, f_0^* and f_1^* are optimal strategies of the game, and $\tau(X^0)$ is the value of game (optimal time of pursuit).

Proof. Let the pursuer apply a strategy f_1^* and the evader any arbitrary strategy $f_0 \in F_0$. Let us denote by $\bar{x}_0(t), \bar{x}_1(t)$ the solution of the system of differential equations

$$\ddot{x}_0 = f_0(X), \quad \ddot{x}_1 = f_1^*(X) \tag{3.5}$$

with initial conditions (2.1). Let this solution be determined in the interval $0 \leqslant t < T$.

We are taking now an arbitrary closed interval $0 \leqslant t \leqslant t'$ contained in the interval $0 \leqslant t < T$. We construct in this interval a sequence of continuous functions $\bar{x}_0^k(t), \bar{x}_1^k(t)$ which for the equations (3.5) are the Euler broken lines [8]. We divide the interval $0 \leqslant t \leqslant t'$ by means of the points

$$t_0^k = 0 < t_1^k < t_2^k < \ldots < t_l^k < t_{l+1}^k < \ldots < t_{m_k}^k = t'$$

into a finite number of subintervals; we assume

$$\bar{x}_0^k(t) = x_0^0 + \dot{x}_0^0 t + \tfrac{1}{2} f_0(X^0) t^2, \qquad \bar{x}_1^k(t) = x_1^0 + \dot{x}_1^0 t + \tfrac{1}{2} f_1^*(X^0) t^2$$

for $0 \leqslant t \leqslant t_1^k$, and

$$\bar{x}_0^k(t) = \bar{x}_0^k(t_l^k) + \dot{\bar{x}}_0^k(t_l^k)(t - t_l^k) + \tfrac{1}{2} f_0\big(\bar{X}^k(t_l^k)\big)(t - t_l^k)^2,$$

$$\bar{x}_1^k(t) = \bar{x}_1^k(t_l^k) + \dot{\bar{x}}_1^k(t_l^k)(t - t_l^k) + \tfrac{1}{2} f_1^*\big(\bar{X}^k(t_l^k)\big)(t - t_l^k)^2$$

for $t_l^k < t \leqslant t_{l+1}^k$ $(l = 1, 2, \ldots, m_k - 1)$; where $\bar{X}^k(t) = \big(\bar{x}_0^k(t), \bar{x}_1^k(t), \dot{\bar{x}}_0^k(t), \dot{\bar{x}}_1^k(t)\big)$.

Due to our assumption the trajectory $\bar{X}(t) = \big(\bar{x}_0(t), \bar{x}_1(t), \dot{\bar{x}}_0(t), \dot{\bar{x}}_1(t)\big)$, $0 \leqslant t \leqslant t'$ lies in the domain Ω, therefore a continuous and positive function $\varepsilon(X)$ considered on this trajectory attains its smallest positive value. Let ε' be this value. It follows from the uniqueness of the solutions of the equations (3.5) that if

$$\lim_{k\to\infty} \max_{0 \leqslant l \leqslant m_k - 1} |t_{l+1}^k - t_l^k| = 0,$$

then the sequence of functions $\bar{X}^k(t)$ is in the interval $0 \leqslant t \leqslant t'$ uniformly convergent with the function $\bar{X}(t)$. Hence, for sufficiently large k

$$|\bar{X}(t) - \bar{X}^k(t)| < \varepsilon'$$

for $0 \leqslant t \leqslant t'$. Therefore, in virtue of (3.1) the trajectory $\bar{X}^k(t)$, $0 \leqslant t \leqslant t'$, lies within the domain Ω. If we now choose k so that

$$\max_{0 \leqslant l \leqslant m_k - 1} |t_{l+1}^k - t_l^k| < \varepsilon',$$

then, due to (3.2), we obtain

$$\tau\big(\bar{X}^k(t)\big) - \tau(X^0) \leqslant -t$$

for $0 \leqslant t \leqslant t'$. The function $\tau(X)$ is continuous in Ω, therefore from the above inequality it follows that

$$\tau\big(\bar{X}(t)\big) - \tau(X^0) \leqslant -t$$

for $0 \leqslant t \leqslant t'$. Since this inequality holds in every closed interval $0 \leqslant t \leqslant t'$ contained in the interval $0 \leqslant t < T$, then it is also satisfied in the interval $0 \leqslant t < T$, i.e.

$$t \leqslant \tau(X^0) - \tau(\bar{X}(t))$$

for $0 \leqslant t < T$. Hence, it follows that the choice of strategy f_1^* made by the pursuer, guarantees him the capture of the evader in a finite time, i.e. that $T < +\infty$, since $\tau(X) > 0$ for $X \in \Omega$. Finally, with $t \to T-0$ we obtain

$$T \leqslant \tau(X^0).$$

But T is the time of pursuit corresponding to the assumed strategies of pursuit and evasion, thus

$$T(f_0, f_1^*) \leqslant \tau(X^0) \tag{3.6}$$

for each strategy of evasion $f_0 \in F_0$.

Let now the evader use a strategy f_0^* and the pursuer any arbitrary strategy $f_1 \in F_1$. Let us denote by $\tilde{X}(t) = (\tilde{x}_0(t), \tilde{x}_1(t), \dot{\tilde{x}}_0(t), \dot{\tilde{x}}_1(t))$ $(0 \leqslant t < T)$ the solution of the system of differential equations

$$\ddot{x}_0 = f_0^*(X), \qquad \ddot{x}_1 = f_1(X)$$

with initial conditions (2.1). Like in the former case we use the assumption 2° of the Lemma and construct a sequence of functions $\tilde{X}^k(t)$ uniformly convergent to $\tilde{X}(t)$ in an arbitrary subinterval $0 \leqslant t \leqslant t''$ of the interval $0 \leqslant t < T$, and satisfying the condition

$$\tau(\tilde{X}^k(t)) - \tau(X^0) \geqslant -t.$$

Hence, for $0 \leqslant t < T$ we obtain

$$\tau(X^0) - \tau(\tilde{X}(t)) \leqslant t.$$

Clearly, in this case the time of pursuit cannot be finite. If $T < +\infty$, then from the definition of the evasion and pursuit strategies follows that

$$\lim_{t \to T-0} \tau(\tilde{X}(t)) = 0,$$

hence

$$\tau(X^0) \leqslant T.$$

If $T = +\infty$, then the above inequality is obvious. But T is the pursuit time corresponding to the assumed strategies of pursuit and evasion, thus

$$\tau(X^0) \leqslant T(f_0^*, f_1) \tag{3.7}$$

for an arbitrary strategy $f_1 \in F_1$.

The inequalities (3.6) and (3.7) tell us that the pursuit game $\langle F_0, F_1, T\rangle$ is determined, $\tau(X^0)$ is the value of the game (optimal pursuit time), and f_0^*, f_1^* are optimal strategies of evasion and pursuit, respectively [6].

4. Let us now construct the function $\tau(X)$ and the domain Ω satisfying all properties mentioned in section 2. At the same time we shall define some new notions used in the sequel.

Let at the initial moment $t = 0$ the positions and velocities of both the evader and pursuer be equal to x_i and $\dot{x}_i (i = 0, 1)$, respectively. Let is denote by $K(x_i, \dot{x}_i; a_i, t)$ a closed ball lying in the space E_n with the centre at the point $x_i + \dot{x}_i t$ and radius equal to $\frac{1}{2}a_i t^2$:

$$(4.1) \qquad K(x_i, \dot{x}_i; a_i, t) = \{x \in E_n: |x - (x_i + \dot{x}_i t)| \leqslant \tfrac{1}{2}a_i t^2\}.$$

Let us denote by $\mathscr{T}$ the set of moments $t \geqslant 0$, for which the ball $K(x_1, \dot{x}_1; a_1, t)$ contains a ball $K(x_0, \dot{x}_0; a_0, t)$, i.e.

$$\mathscr{T} = \{t \geqslant 0: K(x_0, \dot{x}_0; a_0, t) \subseteq K(x_1, \dot{x}_1; a_1, t)\}.$$

The set $\mathscr{T}$ is non-empty due to (2.5) and being closed it contains the smallest element. Let τ be this element. The moment τ is uniquely determined by the initial positions and velocities of the evader and pursuer, i.e. $\tau = \tau(X) = \tau(x_0, x_1, \dot{x}_0, \dot{x}_1)$.

From the definition of $\tau(X)$ it follows immediately that this function is the smallest non-negative solution of the equation

$$(4.2) \qquad [(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)\tau]^2 - \tfrac{1}{4}(a_1 - a_0)^2 = 0.$$

From the form of this equation it follows that such a solution exists for arbitrary $(x_0, x_1, \dot{x}_0, \dot{x}_1)$ being equal to zero if and only if $x_0 - x_1 = 0$.

The continuity of the function $\tau(X)$ is very essential in our considerations. The domain Ω of the phase space in which our problem will be considered must be chosen so as to fulfil this condition. We should be able to determine directly this domain where the function $\tau(X)$ is continuous if we could determine effectively the solution of the equation (4.2), as for example for $n = 1$ (we are, however, mainly interested in cases where $n \geqslant 2$). The simplest way case when the required condition is satisfied will be this if we confine ourselves to the set Ω of such points of phase space that

$$(4.3) \qquad 2[(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)\tau(X)](\dot{x}_1 - \dot{x}_0) - (a_1 - a_0)^2\tau^3(X) \neq 0,$$

i.e. to the points in which $\tau(X)$ is a onefold solution of the equation (4.2). In this way we have excluded the points in which $x_0 = x_1$.

Thus due to the choice of the set Ω the function $\tau(X)$ is continuous and positive in Ω. Moreover, if $\big(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t)\big)$ $(0 \leqslant t < T)$ is an arbitrary trajectory lying within the set Ω and such that

$$\lim_{t \to T-0} \tau\big(x_0(t), x_1(t), \dot{x}_0(t), \dot{x}_1(t)\big) = 0,$$

then from the form of the equation (4.2) it follows immediately that

$$\lim_{t \to T-0} [x_0(t) - x_1(t)] = 0.$$

Thus in Ω the function $\tau(X)$ has all properties required in section 2.

We shall show now that the set Ω is a domain of a $4n$-dimensional phase space.

The function $\tau(X)$ for $X \in \Omega$ is defined by (4.2) and by the condition

$$[(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)t]^2 - \frac{1}{4}(a_1 - a_0)^2 t^4 > 0 \tag{4.4}$$

for $0 \leqslant t < \tau(X)$. If for a fixed point $X \in \Omega$ we denote by τ the function $\tau(X)$ and by $F(t)$ the left-hand side of the inequality (4.4) then because of (4.2) we have

$$\frac{F(\tau) - F(t)}{\tau - t} < 0$$

for $0 \leqslant t < \tau$, whence we obtain $\frac{dF(\tau)}{dt} \leqslant 0$. But $X \in \Omega$, thus by virtue of (4.3) $\frac{dF(\tau)}{dt} \neq 0$. The condition (4.3) is thus equivalent to the condition

$$2[(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)\tau](\dot{x}_1 - \dot{x}_0) - (a_1 - a_0)^2 \tau^3 < 0. \tag{4.5}$$

We shall transform the relation (4.4) by introducing instead of the variable t a dimensionless parameter θ:

$$t = \tau(1 - \theta), \qquad (0 < \theta \leqslant 1).$$

Then (4.4) will take the form

$$[(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)(1 - \theta)\tau]^2 - \frac{1}{4}(a_1 - a_0)^2 (1 - \theta)^4 \tau^4 > 0 \tag{4.6}$$

for $0 < \theta \leqslant 1$. The set Ω can be now defined as the set of all points $(x_0, x_1, \dot{x}_0, \dot{x}_1)$ of the phase space, such that they all satisfy the conditions (4.2), (4.6) and (4.5).

Let us map the set Ω onto the set Ω_1 of a $2n$-dimensional space by means of the linear transformation

$$z = x_1 - x_0, \qquad \dot{z} = \dot{x}_1 - \dot{x}_0. \tag{4.7}$$

The set Ω_1 is defined by the conditions

$$(4.8)\qquad (z+\dot{z}\tau)^2-\frac{1}{4}(a_1-a_0)^2\tau^4=0$$

$$(4.9)\qquad [z+\dot{z}\tau(1-\theta)]^2-\frac{1}{4}(a_1-a_0)^2\tau^4(1-\theta)^4>0$$

for $0<\theta\leqslant 1$, and

$$(4.10)\qquad 2(z+\dot{z}\tau)\dot{z}-(a_1-a_0)^2\tau^3<0.$$

If the set Ω_1 is a domain of a $2n$-dimensional space of variables $(z, \dot{z})$, then, due to (4.7), the set Ω will be also the domain of a $4n$-dimensional space of variables $(x_0, x_1, \dot{x}_0, \dot{x}_1)$.

Let us transform now the set Ω_1 into the set Ω_2 by means of the transformation

$$(4.11)\qquad u=\frac{2}{a_1-a_0}(z+\dot{z}\tau),\qquad v=\frac{2}{a_1-a_0}\dot{z}\tau$$

where $\tau=\tau(z,\dot{z})$ is a function continuous and positive for $(z,\dot{z})\in\Omega_1$, satisfying (4.8), (4.9) and (4.10). This transformation is continuous. In virtue of (4.8) and (4.11) we have

$$\tau=(u^2)^{1/4}$$

where $u^2>0$ for $(u,v)\in\Omega_2$. It follows from the above inequality that the transformation

$$(4.12)\qquad z=\frac{a_1-a_0}{2}(u-v),\qquad \dot{z}=\frac{a_1-a_0}{2}(u^2)^{-1/4}v,$$

is an inverse transformation of (4.11), and its form shows that this transformation is continuous. Thus the transformation (4.11) is a homeomorphism.

In order to show that the set Ω is a domain it suffices to state, due to preceding considerations, that the set Ω_2 is a domain. The set Ω_2 is defined by conditions

$$[1-(1-\theta)^4]u^2-2\theta uv+\theta^2v^2>0\qquad (0<\theta\leqslant 1),$$

$$uv-2u^2<0.$$

Those conditions are equivalent to the inequality

$$(4u^2-2uv)+(v^2-6u^2)\theta+u^2\theta^2(4-\theta)>0$$

for $0 \leqslant \theta \leqslant 1$, and since the left-hand side of this inequality is a continuous function of the variable θ, then the inequality is equivalent to the condition

$$\min_{0\leqslant\theta\leqslant 1} \{(4u^2-2uv)+(v^2-6u^2)\theta+u^2\theta^2(4-\theta)\} > 0. \tag{4.13}$$

Consequently, if a point (u_0, v_0) belongs to the set Ω_2, then a neighbourhood of this point belongs to this set too, as the left-hand side of the inequality (4.13) is a continuous function of the variables (u, v). Thus Ω_2 is an open set. We are to show that Ω_2 is connected.

If $(u, v)\in\Omega_2$ then from (4.13) it follows that $(\varkappa u, \varkappa v)\in\Omega_2$, where $\varkappa$ is an arbitrary real number different from zero. To this end we shall transform the set Ω_2 into the set Ω_3 by means of the formulae

$$\bar{u} = (u^2)^{-1/2}u, \quad \bar{v} = (u^2)^{-1/2}v. \tag{4.14}$$

Then the set Ω_3 is defined by the condition

$$\min_{0\leqslant\theta\leqslant 1} \{(4-2\bar{u}\bar{v})+(\bar{v}^2-6)\theta+\theta^2(4-\theta)\} > 0 \tag{4.15}$$

since $\bar{u}^2 = 1$. From (4.15) it follows that the set Ω_3 has been characterized by two scalar parameters $\bar{u}\bar{v}$ and $\bar{v}^2$. Let us map the set Ω_3 into the set Ω_4 of a two-dimensional space, by means of the transformation

$$\lambda = \bar{u}\bar{v}, \quad \mu = \bar{v}^2. \tag{4.16}$$

Then, by virtue of (4.15) and (4.16), the set Ω_4 is defined by the condition

$$\min_{0\leqslant\theta\leqslant 1} \{(4-2\lambda)+(\mu-6)\theta+\theta^2(4-\theta)\} > 0. \tag{4.17}$$

We shall show now that the set Ω_4 is connected whence, because of (4.16) and (4.14), it follows that the set Ω_2 is connected.

Let (λ_1, μ_1) and (λ_2, μ_2) be arbitrary points belonging to the set Ω_4, and let μ_3 be any arbitrary number greater than $\max(\mu_1, \mu_2, 6)$. Then, from (4.17) it follows immediately that the broken line

$$\lambda = \begin{cases} \lambda_1 & \text{for} \quad 0 \leqslant s \leqslant 1, \\ (\lambda_2-\lambda_1)s+2\lambda_1-\lambda_2 & \text{for} \quad 1 < s \leqslant 2, \\ \lambda_2 & \text{for} \quad 2 < s \leqslant 3, \end{cases}$$

$$\mu = \begin{cases} (\mu_3-\mu_1)s+\mu_1 & \text{for} \quad 0 \leqslant s \leqslant 1, \\ \mu_3 & \text{for} \quad 1 < s \leqslant 2, \\ (\mu_2-\mu_3)s+3\mu_3-2\mu_2 & \text{for} \quad 2 < s \leqslant 3 \end{cases}$$

connecting the points (λ_1, μ_1) and (λ_2, μ_2) lies within Ω_4. Thus Ω_4 is a connected set.

In this way we have shown that the set Ω_2 is a domain, and at the same time that the set Ω is a domain.

Let us introduce now certain new notations and notions which will be used in the sequel. The balls $K(x_i, \dot{x}_i; a_i, \tau)$ $(i = 0, 1)$, defined by the formula (4.1) and corresponding to the moment $\tau = \tau(X)$ will be denoted by K_i; their centres by s_i, where $s_i = s_i(X)$. A point of the ball K_0 that lies on the boundary of the ball K_1 will be called *Apollonian point*. It is a function of a point of the phase space and will be denoted by $a = a(X)$.

5. Now we shall define and examine a fixed trajectory lying within the domain Ω, which will serve us to define certain strategies of evasion and pursuit.

Let $X^0 = (x_0^0, x_1^0, \dot{x}_0^0, \dot{x}_1^0)$ be a fixed point of the phase space belonging to the domain Ω, and let $\tau^0 = \tau(X^0)$. Then the balls K_0 and K_1 defined in section 4 will be of the form

$$K_0 = \{x \in E_n \colon |x-(x_0^0+\dot{x}_0^0\tau^0)| \leqslant \tfrac{1}{2}a_0\tau^{02}\},$$

$$K_1 = \{x \in E_n \colon |x-(x_1^0+\dot{x}_1^0\tau^0)| \leqslant \tfrac{1}{2}a_1\tau^{02}\}$$

hence

$$\begin{aligned} & s_0^0 = s_0(X^0) = x_0^0+\dot{x}_0^0\tau^0, \\ (5.1) \qquad & s_1^0 = s_1(X^0) = x_1^0+\dot{x}_1^1\tau^0, \\ & a^0 = a(X^0) = \frac{a_1}{a_1-a_0}(x_0^0+\dot{x}_0^0\tau^0)-\frac{a_0}{a_1-a_0}(x_1^0+\dot{x}_1^0\tau^0), \end{aligned}$$

and

$$(5.2) \qquad |a^0-s_0^0| = \tfrac{1}{2}a_0\tau^{02}, \qquad |a^0-s_1^0| = \tfrac{1}{2}a_1\tau^{02}.$$

Consider the following trajectories

$$(5.3) \qquad \begin{aligned} & x_0^*(t) = x_0^0+\dot{x}_0^0 t+\tfrac{1}{2}a_0\frac{a^0-s_0^0}{|a^0-s_0^0|}t^2, \qquad x_1^*(t) = x_1^0+\dot{x}_1^0 t+\tfrac{1}{2}a_1\frac{a^0-s_1^0}{|a^0-s_1^0|}t^2, \\ & \dot{x}_0^*(t) = \dot{x}_0^0+a_0\frac{a^0-s_0^0}{|a^0-s_0^0|}t, \qquad \dot{x}_1^*(t) = \dot{x}_1^0+a_1\frac{a^0-s_1^0}{|a^0-s_1^0|}t \end{aligned}$$

defined for $0 \leqslant t \leqslant \tau^0$. From the definition of the trajectories (5.3) we immediately obtain that

$$x_0^*(0) = x_0^0,\ x_1^*(0) = x_1^0,\ \dot{x}_0^*(0) = \dot{x}_0^0,\ \dot{x}_1^*(0) = \dot{x}_1^0$$

and due to (5.1) and (5.2)

$$x_0^*(\tau^0) = x_1^*(\tau^0) = a^0.$$

It will be proved later that along the trajectories (5.3) we have

$$\tau\big(x_0^*(t),\, x_1^*(t),\, \dot{x}_0^*(t),\, \dot{x}_1^*(t)\big) = \tau^0 - t. \tag{5.4}$$

Hence, by simple calculations we get

$$a\big(x_0^*(t),\, x_1^*(t),\, \dot{x}_0^*(t),\, \dot{x}_1^*(t)\big) = a^0$$

and

$$\frac{a\big(X^*(t)\big)-s_0\big(X^*(t)\big)}{\big|a\big(X^*(t)\big)-s_0\big(X^*(t)\big)\big|} = \frac{a^0-s_0^0}{|a^0-s_0^0|}$$

and

$$\frac{a\big(X^*(t)\big)-s_1\big(X^*(t)\big)}{\big|a\big(X^*(t)\big)-s_1\big(X^*(t)\big)\big|} = \frac{a^0-s_1^0}{|a^0-s_1^0|},$$

where $X^*(t) = \big(x_0^*(t),\, x_1^*(t),\, \dot{x}_0^*(t),\, \dot{x}_1^*(t)\big)$. Similarly we get from (5.4)

$$2\Big[\big(x_1^*(t)-x_0^*(t)\big)+\big(\dot{x}_1^*(t)-\dot{x}_0^*(t)\big)\tau\big(X^*(t)\big)\Big]\big(\dot{x}_1^*(t)-\dot{x}_0^*(t)\big)-(a_1-a_0)^2\times$$
$$\times\,\tau^3\big(X^*(t)\big) = \{2\,[(x_1^0-x_0^0)+(\dot{x}_1^0-\dot{x}_0^0)\tau^0](\dot{x}_1^0-\dot{x}_0^0)-(a_1-a_0)^2\tau^{03}\}\left(\frac{\tau^0-t}{\tau^0}\right)^2,$$

whence, by virtue of the definition of the domain Ω, it follows that if the initial point lies within the domain Ω then the trajectory (5.3) lies within this domain for $0 \leqslant t < \tau^0$.

Proof of (5.4): By definition, $\tau\big(x_0^*(t),\, x_1^*(t),\, \dot{x}_0^*(t),\, \dot{x}_1^*(t)\big)$ is the smallest positive root of the equation

$$\Big[\big(x_1^*(t)-x_0^*(t)\big)+\big(\dot{x}_1^*(t)-\dot{x}_0^*(t)\big)\tau\Big]^2-\tfrac{1}{4}(a_1-a_0)^2\tau^4 = 0.$$

By a direct checking we state that $\tau^0 - t$ is a root of this equation. We are to show that this equation has no smaller root. To this end we shall examine the function

$$\Phi(t,\tau) = [z^*(t)+\dot{z}^*(t)\tau]^2-\frac{1}{4}\,a^2\tau^4, \tag{5.5}$$

for $t \geqslant 0$, $\tau \geqslant 0$ and $t+\tau \leqslant \tau^0$, where in accordance with (5.1), (5.2) and (5.3)

$$\begin{aligned} z^*(t) &= x_1^*(t)-x_0^*(t) = \frac{\tau^0-t}{\tau^{02}}\,[z^0\tau^0+(z^0+\dot{z}^0\tau^0)t],\\ \dot{z}^*(t) &= \dot{x}_1^*(t)-\dot{x}_0^*(t) = \frac{1}{\tau^{02}}\,[\dot{z}^0\tau^0-2(z^0+\dot{z}^0\tau^0)t], \end{aligned} \tag{5.6}$$

and, moreover, $z^0 = x_1^0-x_0^0$, $\dot{z}^0 = \dot{x}_1^0-\dot{x}_0^0$ and $a = a_1-a_0$. From the form of the function $\Phi(t,\tau)$ as well as from the former considerations it follows immediately that

$$\Phi(t,\tau^0-t) = 0 \quad \text{for} \quad 0 \leqslant t \leqslant \tau^0 \tag{5.7}$$

and

$$(5.8)\qquad \begin{aligned} \Phi(t,0) > 0 \quad &\text{for} \quad 0 \leqslant t < \tau^0 \\ \Phi(0,\tau) > 0 \quad &\text{for} \quad 0 \leqslant \tau < \tau^0. \end{aligned}$$

To prove (5.4) it suffices therefore to show that $\Phi(t,\tau) > 0$ for $t \geqslant 0$, $\tau \geqslant 0$ and $t+\tau < \tau^0$. This will be proved by showing that if the derivative of the function $\Phi(t,\tau)$ along the straight line $t+\tau = \text{const}$ vanishes in an interior point of the domain $t \geqslant 0$, $\tau \geqslant 0$, $t+\tau \leqslant \tau^0$ then the function $\Phi(t,\tau)$ is positive in this point.

Let us consider the expression $\Phi_t(t,\tau)-\Phi_\tau(t,\tau)$ for $t \geqslant 0$, $\tau \geqslant 0$, $t+\tau \leqslant \tau^0$. Using (5.5) we obtain

$$(5.9)\qquad \Phi_t(t,\tau)-\Phi_\tau(t,\tau) = \tau[a^2\tau^2+2\dot{z}^*(t)\ddot{z}^*(t)\tau+2z^*(t)\ddot{z}^*(t)].$$

It follows from (5.7) that

$$(5.10)\qquad \Phi_t(t,\tau^0-t)-\Phi_\tau(t,\tau^0-t) = 0$$

for $0 \leqslant t \leqslant \tau^0$. t being fixed the expression (5.9) is a polynomial of the variable τ. From (5.10) it follows that

$$\tau_1 = \tau^0 - t$$

is a root of this polynomial. Hence,

$$\tau_2 = -\tau^0+t-\frac{2z^*(t)\ddot{z}^*(t)}{a^2}$$

is also the root of this polynomial.

By using (5.6) and performing some simple transformations we obtain

$$\tau_2 = \tau^0-t+\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}$$

Hence

$$(5.11)\quad \Phi_t(t,\tau)-\Phi_\tau(t,\tau) = a^2\tau[\tau-(\tau^0-t)]\left[\tau-\left(\tau^0-t+\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}\right)\right]$$

for $t \geqslant 0$, $\tau \geqslant 0$, and $t+\tau \leqslant \tau^0$.

Now, we shall define the function $\Phi(t,\gamma-t)$ for $0 \leqslant t \leqslant \gamma$ where $0 \leqslant \gamma \leqslant \tau^0$. We have

$$\frac{d}{dt}\Phi(t,\gamma-t) = \Phi_t(t,\gamma-t)-\Phi_\tau(t,\gamma-t)$$

and because of (5.11)

$$\frac{d}{dt}\Phi(t,\gamma-t) = a^2(\gamma-t)(\gamma-\tau^0)\left(\gamma-\tau^0-\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}\right)$$

whence, by integration we obtain

$$\Phi(t,\gamma-t) = -\tfrac{1}{2}a^2(\gamma-t)^2(\gamma-\tau^0)\left(\gamma-\tau^0-\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}\right)+\Phi(\gamma,0) \tag{5.12}$$

for $0 \leqslant t \leqslant \gamma$.

Suppose now that for same point (t,τ), where $t \geqslant 0$, $\tau \geqslant 0$ and $t+\tau < \tau^0$ the function $\Phi(t,\tau)$ assumes a non-positive value. Then from (5.7) and (5.8) and from the continuity of this function it follows that at a point $(\bar{t},\bar{\tau})$, where $\bar{t} \geqslant 0$, $\bar{\tau} \geqslant 0$ and $\bar{t}+\bar{\tau} < \tau^0$, $\Phi(t,\tau)$ attains its minimum value which is non-positive,

$$\Phi(\bar{t},\bar{\tau}) \leqslant 0. \tag{5.13}$$

By virtue of (5.11) we have

$$a^2\bar{\tau}(\bar{t}+\bar{\tau}-\tau^0)\left(\bar{t}+\bar{\tau}-\tau^0-\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}\right)=0$$

whence

$$\bar{t}+\bar{\tau}-\tau^0-\frac{2\Phi_\tau(0,\tau^0)}{a^2\tau^{02}}=0.$$

From (5.12) with $\gamma=\bar{t}+\bar{\tau}$ and $t=\bar{t}$ and from (5.8) we obtain

$$\Phi(\bar{t},\bar{\tau})=\Phi(\bar{t}+\bar{\tau},0)>0$$

which contradicts (5.13). This ends the proof of (5.4).

6. We shall now define some strategies of pursuit and evasion. Let us consider the following system of differential equations

$$\ddot{x}_0=\frac{a(X)-s_0(X)}{|a(X)-s_0(X)|}a_0, \qquad \ddot{x}_1=\frac{a(X)-s_1(X)}{|a(X)-s_1(X)|}a_1$$

subject to the initial conditions

$$X(0)=X^0.$$

From the considerations contained in section 5 it follows that the functions $x_0^*(t)$ and $x_1^*(t)$ defined by (5.3) form a solution of this system satisfying the initial conditions formulated above and which for $0 \leqslant t < \tau^0$ belongs to the domain Ω and for which

$$\lim_{t\to\tau^0-0}\tau\big(x_0^*(t),x_1^*(t),\dot{x}_0^*(t),\dot{x}_1^*(t)\big)=0.$$

Hence, the functions

$$f_0^*(X)=\frac{a(X)-s_0(X)}{|a(X)-s_0(X)|}a_0, \qquad f_1^*(X)=\frac{a(X)-s_1(X)}{|a(X)-s_1(X)|}a_1 \tag{6.1}$$

are the strategies of pursuit and evasion in the domain Ω.

If the pursuer and evader use the strategies (6.1) then during the pursuit the vectors of accelerations are constant and their value is maximum. The time of pursuit is then equal to τ^0 and the capture takes place in a^0. In the course of pursuit the actual Apollonian point $a(X)$ coincides with the initial Apollonian point a^0.

7. We shall show now that the pursuit of one evader by one pursuer is a determined game in the domain Ω.

Let $\langle F_0, F_1, T\rangle$ be a pursuit game which has been discussed in section 2, and let the functions $f_0^*(X)$ and $f_1^*(X)$, defined in the former section, belong to the sets F_0 and F_1, respectively. The sets F_0 and F_1 contain also some other strategies. Let for instance, Σ_0 and Σ_1 be arbitrary closed regions of a phase space which are contained in the domain Ω. Let the functions $f_0(X)$ and $f_1(X)$ be continuous in Ω, and let $f_0(X)$ be in Σ_0 an arbitrary function with continuous first order derivatives and such that $|f_0(X)| \leqslant a_0$, and moreover, $f_0(X) = f_0^*(X)$ in $\Omega \setminus \Sigma_0$; $f_1(X)$ is an arbitrary function with continuous first-order derivatives in Σ_1 and such that $|f_1(X)| \leqslant a_1$, moreover, $f_1(X) = f_1^*(X)$ in $\Omega \setminus \Sigma_1$. The functions $f_0(X)$ and $f_1(X)$ thus defined are strategies.

The following theorem holds:

THEOREM. *A pursuit game $\langle F_0, F_1, T\rangle$ is a determined game for which f_0^* is the optimal strategy of evasion, f_1^* is the optimal strategy of pursuit, and $\tau(X^0)$ is the optimal time of pursuit.*

Proof. To prove our theorem we apply the Lemma formulated and proved in section 3. We begin with defining an auxiliary function $\varepsilon(X)$. We do it as follows:

Let $\varepsilon_1(X)$ be a positive and continuous function in the domain Ω, such that for every $X \in \Omega$ and for arbitrary f_0 and f_1, where $|f_0| \leqslant a_0$ and $|f_1| \leqslant a_1$, and for every number $0 \leqslant \delta < \varepsilon_1(X)$ the point $(x_0 + \dot{x}_0\delta + \frac{1}{2}f_0\delta^2, x_1 + \dot{x}_1\delta + \frac{1}{2}f_1\delta^2, \dot{x}_0 + f_0\delta, \dot{x}_1 + f_1\delta)$ belongs to the domain Ω. Such a function exists because Ω is open. Let $\varepsilon_2(X)$ be an upper bound of the numbers ε such that

$$|(x_1 - x_0) + (\dot{x}_1 - \dot{x}_0)\delta| > \tfrac{1}{2}(a_0 + a_1)\delta^2$$

holds for every $0 \leqslant \delta < \varepsilon$ and $X \in \Omega$. The function $\varepsilon_2(X)$ is continuous in the domain Ω. Let now

$$\varepsilon(X) = \min\left\{\varepsilon_1(X),\ \varepsilon_2(X),\ \frac{\sqrt{a_1}}{\sqrt{a_0}+\sqrt{a_1}}\tau(X)\right\}. \tag{7.1}$$

The function $\varepsilon(X)$ is defined in the domain Ω and is continuous and positive there.

To prove the theorem we shall show that the inequalities (3.2) and (3.4) hold, f_0^* and f_1^* being determined by (6.1) and δ an arbitrary number satisfying the condition $0 \leqslant \delta < \varepsilon(X)$.

Proof of (3.2). Fix a point $X \in \Omega$ and let $f_0(X) \in F_0$. For the sake of brevity the argument X of the considered functions will be omitted, and the values of those functions in the point

$$(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0\delta^2,\ x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1^*\delta^2,\ \dot{x}_0 + f_0\delta,\ \dot{x}_1 + f_1^*\delta)$$

belonging to Ω will be denoted by adding an upper index δ, when $0 \leqslant \delta < \varepsilon$. We are to prove that if $0 \leqslant \delta < \varepsilon$, then

$$\tau^\delta \leqslant \tau - \delta.$$

By virtue of the definition of τ it sufficies to show that

$$K_0^\delta \subset K_1^\delta, \tag{7.2}$$

where

$$K_0^\delta = K(x_0 + \dot{x}_0\delta + \tfrac{1}{2}f_0\delta^2,\ \dot{x}_0 + f_0\delta;\ a_0,\ \tau - \delta),$$
$$K_1^\delta = K(x_1 + \dot{x}_1\delta + \tfrac{1}{2}f_1^*\delta^2,\ \dot{x}_1 + f_1^*\delta;\ a_1,\ \tau - \delta).$$

Let us introduce in the space E_n a rectangular system of coordinates $(\xi^1, \xi^2, \ldots, \xi^n)$ assuming the centre s_1 of the ball K_1 to lie at the origin of this system, and the axis ξ^1 be directed toward the centre s_0 of the ball K_0. Then

$$s_0 = (\tfrac{1}{2}(a_1 - a_0)\tau^2, 0, \ldots, 0), \qquad s_1 = (0, 0, \ldots, 0),$$

and because of (6.1)

$$f_1^* = (a_1, 0, \ldots, 0).$$

Let us denote by $(\cos\varphi_1, \cos\varphi_2, \ldots, \cos\varphi_n)$ the direction cosines of the vector f_0 and let $|f_0| = \beta_0$. Then

$$f_0 = (\beta_0\cos\varphi_1, \beta_0\cos\varphi_2, \ldots, \beta_0\cos\varphi_n).$$

Due to (5.1) we obtain

$$\begin{aligned} s_0^\delta &= \big(\tfrac{1}{2}(a_1 - a_0)\tau^2 + (\tau\delta - \tfrac{1}{2}\delta^2)\beta_0\cos\varphi_1, \\ &\qquad (\tau\delta - \tfrac{1}{2}\delta^2)\beta_0\cos\varphi_2, \ldots, (\tau\delta - \tfrac{1}{2}\delta^2)\beta_0\cdot\cos\varphi_n\big) \\ s_1^\delta &= ((\tau\delta - \tfrac{1}{2}\delta^2)a_1, 0, \ldots, 0), \end{aligned} \tag{7.3}$$

where $s_0^\delta \neq s_1^\delta$, which results from the definition of ε. The relation (7.2) is equivalent to the inequality

$$|s_1^\delta - s_0^\delta| \leqslant \tfrac{1}{2}(a_1 - a_0)(\tau - \delta)^2. \tag{7.4}$$

Let us consider now the expression $|s_1^\delta - s_0^\delta|$. From (7.3) we have

$$|s_1^\delta - s_0^\delta| = \{[\tfrac{1}{2}(a_1-a_0)(\tau-\delta)^2-(\tau\delta-\tfrac{1}{2}\delta^2)(a_0-\beta_0\cos\varphi_1)]^2+ \\ +[(\tau\delta-\tfrac{1}{2}\delta^2)\beta_0\cos\varphi_2]^2+\ldots+[(\tau\delta-\tfrac{1}{2}\delta^2)\beta_0\cos\varphi_n]^2\}^{1/2}.$$

Since $\delta < \varepsilon$, then because of (7.1) we have

$$\delta < \frac{\sqrt{a_1}}{\sqrt{a_0}+\sqrt{a_1}}\tau,$$

thus the expression $|s_1^\delta - s_0^\delta|$ attains for $\varphi_1 = 0$, $\varphi_2 = \ldots = \varphi_n = \pi/2$ its maximum value equal to

$$\tfrac{1}{2}(a_1-a_0)(\tau-\delta)^2-(a_0-\beta_0)(\tau\delta-\tfrac{1}{2}\delta^2).$$

Hence, we immediately obtain (7.4), since $\beta_0 \leqslant a_0$ and $\delta < \tau$.

Proof of the inequality (3.4): Let again $X \in \Omega$ and $f_1 \in F_1$. We introduce notations similar to those given above. Thus we have to show that, if $0 \leqslant \delta < \varepsilon$, then

$$\tau^\delta \geqslant \tau - \delta$$

where the index δ denotes the value at the point

$$(x_0+\dot{x}_0\delta+\tfrac{1}{2}f_0^*\delta^2,\ x_1+\dot{x}_1\delta+\tfrac{1}{2}f_1\delta^2,\ \dot{x}_0+f_0^*\delta,\ \dot{x}_1+f_1\delta)$$

belonging to the domain Ω. To do this it suffices to show that there exists at least one point of the ball

$$K_0^\delta = K(x_0+\dot{x}_0\delta+\tfrac{1}{2}f_0^*\delta^2,\ \dot{x}_0+f_0^*\delta;\ a_0,\ \tau-\delta)$$

lying outside or on the boundary of the ball

$$K_1^\delta = K(x_1+\dot{x}_1\delta+\tfrac{1}{2}f_1\delta^2,\ \dot{x}_1+f_1\delta;\ a_1,\ \tau-\delta).$$

The Apollonian point is just such a point. From the definition it lies on the boundary of the balls K_0 and K_1, and because of (6.1) it lies also on the boundary of the ball K_0^δ. In order to show that it does not lie inside the ball K_1^δ we shall show that

$$K_1^\delta \subseteq K_1.$$

Let $x \in K_1^\delta$, i.e.,

$$|x-s_1^\delta| \leqslant \tfrac{1}{2}a_1(\tau-\delta)^2.$$

Then we have

$$|x-s_1| \leqslant |x-s_1-f_1(\tau\delta-\tfrac{1}{2}\delta^2)|+|f_1(\tau\delta-\tfrac{1}{2}\delta^2)|$$

and since $s_1+f_1(\tau\delta-\tfrac{1}{2}\delta^2) = s_1^\delta$ and $\delta < \tau$ and $|f_1| \leqslant a_1$, then

$$|x-s_1| \leqslant \tfrac{1}{2}a_1(\tau-\delta)^2+|f_1|(\tau\delta-\tfrac{1}{2}\delta^2) \leqslant \tfrac{1}{2}a_1\tau^2,$$

whence it follows that $x \in K_1$. In this way we have proved the inequality (3.4) which finishes the proof of our theorem.

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