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R. MAGIERA (Wrocław)

ESTIMATION WITH DELAYED OBSERVATIONS

1. Introduction. Let $X_1, \ldots, X_n$ be independent random variables with the same probability distribution depending on an unknown parameter ϑ. Suppose that X_i is observed at time t_i, where $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$, and $t_1, \ldots, t_n$ are independent of $X_1, \ldots, X_n$. We will, in fact, suppose that $t_1, \ldots, t_n$ are the order statistics of positive exchangeable random variables $U_1, \ldots, U_n$ which are independent of $X_1, \ldots, X_n$. In the paper we shall be interested in the problem of estimating the parameter ϑ when the information is accessible, as described above, at random moments of time. We assume that the loss incurred by the statistician in estimating the parameter ϑ is not only due to the error of estimation but also to the cost of observation.

The decision of the statistician is determined by a Markov stopping time τ, which is the moment when the statistician decides to stop the observation, and by an estimator f of ϑ chosen by him when he does stop. The estimator f is a function of observations and of the number of observations made up to time τ. We assume that at least one observation is taken. The problem is to find sequential decisions which minimize the expected value of the over-all loss due to the estimation and due to observation costs.

We will adopt a Bayesian approach by placing a prior distribution over ϑ and find a class of optimal decisions for the statistician when the loss due to the estimation error is the squared error loss. Moreover, we assume that the observation cost equals c units for unit time. For independent normally distributed random variables $X_1, \ldots, X_n$ with unknown mean and known variance the underlying problem was considered by Starr et al. [4].

In the present paper we treat the estimation problem for a large class of distributions belonging to a family $\mathscr{E}(\vartheta, a)$ from the exponential family of distributions. We show that for a prior distribution Φ from the family $\mathscr{E}_0(a_0, \gamma)$, which is conjugate with $\mathscr{E}(\vartheta, a)$, the Bayes estimator of the parameter ϑ is of the form (14). Further, we infer that the posterior risk corresponding to this estimator is equal to $[a_0+\beta+ak(\tau)]^{-1}$. Therefore, the sequential estimation problem could be reduced to an optimal stopping problem. Under assumptions on $\mathscr{E}(\vartheta, a)$ and on the distribution function with a failure rate ϱ for the random variables $U_1, \ldots, U_n$ we show that for a weighted quadratic loss function the sequential plan $\delta^0 = (\tau_0, \hat{f}^0_{\tau_0})$, where τ_0 and $\hat{f}^0_{\tau_0}$ are defined by (16) (with $a_0 = 0$) and (17), respectively, is minimax.

The problem considered here arises when we are interested in data which, no matter how we choose to manipulate our environment, are forthcoming only at random moments of time. Following [4] let us quote a few examples of the described situation. In studying the effectiveness of safety devices in mobile objects the relevant data can only be obtained as the results of a failure or an accident. For instance, medical data concerning the effectiveness of a medicine can be obtained at random moments of time when patients seek help or are examined in one or another way. One can supply more examples of the described situation when observing mail services or archeological discoveries.

2. Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Denote by $(\mathscr{X}, \mathscr{B})$ the measurable space, where $\mathscr{X} \subseteq R$ (R denotes the real line) and $\mathscr{B}$ is a σ-algebra of Borel subsets of $\mathscr{X}$. Consider the random variables $X_1, \ldots, X_n$ defined on $(\Omega, \mathscr{F}, P)$, with values in $(\mathscr{X}, \mathscr{B})$ and with the same distribution $\mathscr{P}_\vartheta$ on $(\mathscr{X}, \mathscr{B})$ depending on a parameter $\vartheta \in D$. We assume that D is an open interval (possibly infinite or semi-infinite) of the real line. Further, we suppose that all distributions $\mathscr{P}_\vartheta$, $\vartheta \in D$, are absolutely continuous with respect to a σ-finite measure ν on $(\mathscr{X}, \mathscr{B})$. The probability measure P is, in fact, interpreted as an element of the family of probability measures P_ϑ, $\vartheta \in D$, on $(\Omega, \mathscr{F})$. By $\mathrm{E}_\vartheta(\cdot)$ and $\mathrm{D}_\vartheta(\cdot)$ we denote the expected value and the variance, respectively, evaluated with respect to the measure P_ϑ. We assume that $\mathrm{E}_\vartheta(X_i^2) < \infty$ for all $\vartheta \in D$.

We suppose that the distributions $\mathscr{P}_\vartheta$, $\vartheta \in D$, belong to some family $\mathscr{E}(\vartheta, a)$ from the exponential family of distributions, defined as follows:

By $\mathscr{E}(\vartheta, a)$ we mean the family of distributions $\mathscr{P}_\vartheta$, $\vartheta \in D$, for which the densities with respect to the measure ν are of the form

$$(1)\qquad \frac{d\mathscr{P}_\vartheta}{d\nu}(x) = p(x; \vartheta, a) = s(x, a)\exp[aw_1(\vartheta) + xw_2(\vartheta)],$$

where a is a positive constant, $s(x, a)$ denotes a (non-negative) $\mathscr{B}$-measurable function independent of ϑ, $w_1(\vartheta)$ and $w_2(\vartheta)$ are functions defined on D, twice continuously differentiable in D and with the first derivatives $w_1'(\vartheta)$ and $w_2'(\vartheta)$ such that $w_2'(\vartheta) > 0$ and $w_1'(\vartheta)/w_2'(\vartheta)$ is strictly decreasing in the whole interval D.

The expected value and the variance of the considered random variables X_i $(i = 1, \ldots, n)$ are given by

$$\mathrm{E}_\vartheta(X_i) = -a\frac{w_1'(\vartheta)}{w_2'(\vartheta)} \tag{2}$$

and

$$\mathrm{D}_\vartheta(X_i) = -\frac{a}{w_2'(\vartheta)}\frac{d}{d\vartheta}\left[\frac{w_1'(\vartheta)}{w_2'(\vartheta)}\right]. \tag{3}$$

Let us remark that the most known distributions relevant to theory and applications belong to the family $\mathscr{E}(\vartheta, a)$, namely: normal $\mathscr{N}(a\vartheta, a)$ with $\vartheta \in (-\infty, \infty)$, gamma $\mathscr{G}(\vartheta^{-1}, a)$, Poisson $\mathscr{P}(a\vartheta)$ and negative-binomial $n\mathscr{B}(\vartheta(1+\vartheta)^{-1}, a)$ with $\vartheta \in (0, \infty)$.

Let $X_1, \ldots, X_n$ be independent random variables with the same distribution $\mathscr{P}_\vartheta$ belonging to $\mathscr{E}(\vartheta, a)$ with unknown parameter ϑ and known value of a. We shall consider the problem of estimating ϑ when observations become available at random moments of time. We suppose that X_i is observed at time t_i $(i = 1, \ldots, n)$, where $t_1, \ldots, t_n$ are the order statistics of positive exchangeable random variables $U_1, \ldots, U_n$. We assume that $U_1, \ldots, U_n$ are independent of $X_1, \ldots, X_n$. Let

$$k(t) = \sum_{i=1}^{n} I_{[0,t]}(U_i)$$

be the number of observations made during time $t \geqslant 0$, and let

$$\mathscr{F}_t = \sigma\{k(s), s \leqslant t, X_1, \ldots, X_{k(t)}\}.$$

Thus $\mathscr{F}_t$ denotes the information available to the statistician at time t.

An $\mathscr{F}_t$-measurable random variable f will be called an *estimator* of ϑ. We suppose that the loss due to the estimation error is determined by a weighted quadratic loss function $L(\vartheta, f)$ and that the cost of observing the process for unit time is a given constant $c > 0$. Thus, if the statistician decides to stop at time t, then the loss incurred by him is determined by

$$L_t(\vartheta, f) = L(\vartheta, f) + ct,$$

where ϑ is the true value of the parameter and f is the chosen estimator.

By a *stopping time* we shall mean an extended random variable τ for which $P_\vartheta(0 \leqslant \tau < \infty) = 1$ for all $\vartheta \in D$ and $\{\tau > t\} \in \mathscr{F}_t$ for every $t \geqslant 0$, and by a *sequential plan* we shall understand any pair $\delta = (\tau, f)$.

The statistician decides when observing the process should be stopped and what estimator should be taken when he does stop. He is interested in making such a choice of τ and f that the expected value of the over-all loss function $L_\tau(\vartheta, f) = L(\vartheta, f) + c\tau$ be small.

If $\delta = (\tau, f)$ is a chosen sequential plan and ϑ is the true value of the parameter, then the *risk function* is defined by

$$R(\vartheta, \delta) = \mathrm{E}_\vartheta[L_\tau(\vartheta, f)].$$

We consider only such sequential plans δ for which $R(\vartheta, \delta) < \infty$ for all $\vartheta \in D$.

We use a Bayesian approach considering a prior probability distribution of the parameter ϑ. Let us formulate this more formally. We introduce a random variable Θ with values $\vartheta \in D$. Let $\mathscr{M}$ be a σ-algebra of Borel subsets of D, and let Φ be a prior probability distribution of Θ on $(D, \mathscr{M})$. We suppose that, given $\Theta = \vartheta$, the random variables $X_1, \ldots, X_n$ are independent and have a common distribution $\mathscr{P}_\vartheta \in \mathscr{E}(\vartheta, a)$ and that $\Theta, X_1, \ldots, X_n$ are independent of $U_1, \ldots, U_n$. We denote unconditional probability measure by P_Φ, and we suppose that $\mathrm{E}_\Phi(\Theta^2) < \infty$. For a given sequential plan δ the *expected risk* with respect to Φ is then defined by

$$r(\Phi, \delta) = \mathrm{E}_\Phi(L_\tau) = \int_D R(\vartheta, \delta)\Phi(d\vartheta).$$

Suppose that for the prior distribution Φ, the posterior distribution $\Phi^{\mathscr{F}_t}$, given $\mathscr{F}_t$, is well defined. Then the conditional expected loss, given $\mathscr{F}_t$, corresponding to Φ and f is defined by

$$r^{\mathscr{F}_t}(\Phi, f) = \int_D L(\vartheta, f)\Phi^{\mathscr{F}_t}(d\vartheta).$$

$r^{\mathscr{F}_t}(\Phi, f)$ will be called the *posterior risk* corresponding to Φ and f.

It is clear that for any stopping time τ the functional $r^{\mathscr{F}_\tau}(\Phi, f)$ is minimized by $f = \hat{f}$, being a Bayes estimator with respect to Φ. Thus the problem of finding Bayes sequential plans may be reduced to an optimal stopping problem.

In sequential estimation problems without delaying the observations it turns out that for some processes with a proper loss function the only minimax sequential plans are the fixed-time ones. For example, it is known (see [1] and [3]) that for processes with independent increments (most frequently involved in mathematical statistics) and for a weighted quadratic loss function, the minimax (sequential) plan reduces to a fixed-time plan. The solution of the problem considered in this paper leads to the plans which are essentially sequential.

3. Bayes and minimax sequential plans. Let D be an open interval (a, b). We will find a class of optimal sequential plans $\delta = (\tau, f)$ for $\vartheta \in D$ based on a sequence $X_1, \ldots, X_n$ of independent random variables with a common distribution $\mathscr{P}_\vartheta \in \mathscr{E}(\vartheta, a)$ satisfying the following conditions:

(i) for each $\vartheta \in D$,

$$\vartheta = -\frac{w_1'(\vartheta)}{w_2'(\vartheta)}; \tag{4}$$

(ii) there exists a constant $\beta \geqslant 0$ such that the relation

$$\int_D \exp[aw_1(\vartheta)+xw_2(\vartheta)]d\vartheta = \frac{1}{(a-\beta)s(x, a)} \tag{5}$$

is valid for all $a > \beta$ and $x \in \mathscr{X}$ for which $s(x, a) > 0$;

(iii) for every $a > \beta$ and for each $x \in \mathscr{X}$ except perhaps $x = \inf \mathscr{X}$,

$$\lim_{\vartheta \to a^+} \exp[aw_1(\vartheta)+xw_2(\vartheta)] = \lim_{\vartheta \to b^-} \exp[aw_1(\vartheta)+xw_2(\vartheta)]$$

and

$$\lim_{\vartheta \to a^+} \vartheta \exp[aw_1(\vartheta)+xw_2(\vartheta)] = \lim_{\vartheta \to b^-} \vartheta \exp[aw_1(\vartheta)+xw_2(\vartheta)].$$

It is easily verified that conditions (i)-(iii) are fulfilled for all above-mentioned well-known distributions belonging to $\mathscr{E}(\vartheta, a)$, namely: for $\mathscr{N}(a\vartheta, a)$ with $a = -\infty$, $b = \infty$, $\beta = 0$, for $\mathscr{G}(\vartheta^{-1}, a)$ and $n\mathscr{B}(\vartheta(1+\vartheta)^{-1}, a)$ with $a = 0$, $b = \infty$, $\beta = 1$, and for $\mathscr{P}(a\vartheta)$ with $a = 0$, $b = \infty$, $\beta = 0$.

Let $\varphi(\vartheta)$ be the density (with respect to the Lebesgue measure) of the probability distribution Φ on $(D, \mathscr{M})$, and let $\varphi^{\mathscr{F}_t}(\vartheta)$ denote the density of the conditional probability distribution $\Phi^{\mathscr{F}_t}$.

We assume that $\varphi(\vartheta)$ is of the form

$$\begin{aligned} \varphi(\vartheta) &= a_0 p(\gamma; \vartheta, a_0+\beta) \\ &= a_0 s(\gamma, a_0+\beta)\exp[(a_0+\beta)w_1(\vartheta)+\gamma w_2(\vartheta)], \end{aligned} \tag{6}$$

where $a_0 > 0$ and γ are constants, and the function s is positive. Note that $\varphi(\vartheta)$ is a density of a probability distribution on D, since

$$\int_D \varphi(\vartheta)d\vartheta = 1.$$

This follows from (5), because in view of (6) we have

$$\int_D \varphi(\vartheta)d\vartheta = a_0 s(\gamma, a_0+\beta)\int_D \exp[(a_0+\beta)w_1(\vartheta)+\gamma w_2(\vartheta)]d\vartheta.$$

Let $\mathscr{E}_0(a_0, \gamma)$ denote the family of all probability distributions on D with densities defined by (6). We have the following

LEMMA 1. *Let* $\mathscr{P}_\vartheta \in \mathscr{E}(\vartheta, a)$ *and let* (ii) *be valid. If* $\Phi \in \mathscr{E}_0(a_0, \gamma)$, *then*

$$\Phi^{\mathscr{F}_t} \in \mathscr{E}_0\Big(a_0+ak(t), \gamma+\sum_{i=1}^{k(t)} X_i\Big).$$

Proof. By the Bayes theorem, since the considered random variables are independent, we have

$$\varphi^{\mathscr{F}_t}(\vartheta) = \frac{\varphi(\vartheta)\sum\limits_{i=1}^{k(t)} p(X_i;\vartheta, a)}{\int\limits_D \varphi(\vartheta)\prod\limits_{i=1}^{k(t)} p(X_i;\vartheta, a)\,d\vartheta}.$$

Substituting (6) into this formula and using (1) we obtain

$$\varphi^{\mathscr{F}_t}(\vartheta) = \frac{\exp\left[[a_0+\beta+ak(t)]w_1(\vartheta)+\left(\gamma+\sum\limits_{i=1}^{k(t)} X_i\right)w_2(\vartheta)\right]}{\int\limits_D \exp\left[[a_0+\beta+ak(t)]w_1(\vartheta)+\left(\gamma+\sum\limits_{i=1}^{k(t)} X_i\right)w_2(\vartheta)\right]d\vartheta}.$$

Now, taking into account (5) we get

$$\varphi^{\mathscr{F}_t}(\vartheta) = [a_0+ak(t)]s\left(\gamma+\sum_{i=1}^{k(t)} X_i,\ a_0+\beta+ak(t)\right)\exp\Big[[a_0+\beta+ak(t)]w_1(\vartheta)+ \\ +\left(\gamma+\sum_{i=1}^{k(t)} X_i\right)w_2(\vartheta)\Big],$$

which completes the proof of the lemma.

In other words, the family $\mathscr{E}_0(a_0, \gamma)$ of distributions Φ is conjugate with the family $\mathscr{E}(\vartheta, a)$ of distributions $\mathscr{P}_\vartheta$ satisfying (ii).

By Lemma 1 and by the strong Markov property we have the following

Corollary. *For any stopping time* τ

$$\Phi^{\mathscr{F}_\tau} \in \mathscr{E}_0\left(a_0+ak(\tau),\ \gamma+\sum_{i=1}^{k(\tau)} X_i\right). \tag{7}$$

Conditions (i) and (iii) imply the following relations which are useful in further considerations:

$$a\int\limits_D \vartheta w_2'(\vartheta)\exp[aw_1(\vartheta)+xw_2(\vartheta)]\,d\vartheta = x\int\limits_D w_2'(\vartheta)\exp[aw_1(\vartheta)+xw_2(\vartheta)]\,d\vartheta, \tag{8}$$

$$\int\limits_D \vartheta(x-a\vartheta)w_2'(\vartheta)\exp[aw_1(\vartheta)+xw_2(\vartheta)]\,d\vartheta = -\int\limits_D \exp[aw_1(\vartheta)+xw_2(\vartheta)]\,d\vartheta. \tag{9}$$

Using (5), (8) and (9) we have

$$(10)\qquad \int_D (x-a\vartheta)^2 w_2'(\vartheta)\exp[aw_1(\vartheta)+xw_2(\vartheta)]\,d\vartheta = \frac{a}{(a-\beta)s(x,a)}.$$

In view of (2)-(4) the expected value and the variance of X_i ($i = 1, \ldots, n$) are given by

$$(11)\qquad \mathrm{E}_\vartheta(X_i) = a\vartheta$$

and

$$(12)\qquad \mathrm{D}_\vartheta(X_i) = \frac{a}{w_2'(\vartheta)}.$$

We take the weighted quadratic loss function

$$(13)\qquad L(\vartheta, f) = w_2'(\vartheta)(f-\vartheta)^2,$$

i.e. the squared error measured in terms of the variance.

LEMMA 2. *Let* $\mathscr{P}_\vartheta \in \mathscr{E}(\vartheta, a)$ *and let conditions* (i)-(iii) *be valid. Then for the loss function* (13) *and for any stopping time* τ *the Bayes estimator of* ϑ *with respect to* $\Phi \in \mathscr{E}_0(a_0, \gamma)$ *is of the form*

$$(14)\qquad \hat{f}_\tau = \frac{\gamma + \sum_{i=1}^{k(\tau)} X_i}{a_0+\beta+ak(\tau)}.$$

Proof. In view of (13) the posterior risk $r^{\mathscr{F}_\tau}(\Phi, f)$ takes the form

$$(15)\qquad r^{\mathscr{F}_\tau}(\Phi, f) = \int_D w_2'(\vartheta)(f-\vartheta)^2 \varphi^{\mathscr{F}_\tau}(\vartheta)\,d\vartheta.$$

It is easily seen that this risk is minimized by

$$f = \hat{f}_\tau = \frac{\int_D \vartheta w_2'(\vartheta)\varphi^{\mathscr{F}_\tau}(\vartheta)\,d\vartheta}{\int_D w_2'(\vartheta)\varphi^{\mathscr{F}_\tau}(\vartheta)\,d\vartheta}.$$

By (7) we obtain

$$\hat{f}_\tau = \frac{\int_D \vartheta w_2'(\vartheta)\exp\left[[a_0+\beta+ak(\tau)]w_1(\vartheta)+\left(\gamma+\sum_{i=1}^{k(\tau)} X_i\right) w_2(\vartheta)\right]d\vartheta}{\int_D w_2'(\vartheta)\exp\left[[a_0+\beta+ak(\tau)]w_1(\vartheta)+\left(\gamma+\sum_{i=1}^{k(\tau)} X_i\right) w_2(\vartheta)\right]d\vartheta},$$

and the application of (8) leads to formula (14).

Let G be the common distribution function of the independent random variables $U_1, \ldots, U_n$. We suppose that $G(0) = 0$, $G(t) > 0$ for $t > 0$, G is absolutely continuous with density g, and g is the right-hand derivative of G on $(0, \infty)$. The class of such distribution functions G is denoted by $\mathscr{G}$. Let $\zeta = \sup\{t : G(t) < 1\}$ and let $\varrho(z) = g(z)[1-G(z)]^{-1}$, $0 \leqslant z < \zeta$, denote the failure rate.

We have the following

THEOREM 1. *Suppose that $G \in \mathscr{G}$ and G has a non-increasing failure rate ϱ. If $\Phi \in \mathscr{E}_0(a_0, \gamma)$, $0 < a_0 < \infty$, then the Bayes sequential plan is $\delta_{a_0} = (\tau_{a_0}, \hat{f}_{\tau_{a_0}})$, where*

$$\tau_{a_0} = \inf\{t \geqslant 0: [n-k(t)]\varrho(t) \leqslant ca^{-1}\{a_0+\beta+a[k(t)+1]\}[a_0+\beta+ak(t)]\}. \tag{16}$$

Proof. Let us evaluate the posterior risk $r^{\mathscr{F}_\tau}(\Phi, f)$ corresponding to estimator (14). Substituting (14) into (15) and taking into account (7) we get

$$r^{\mathscr{F}_\tau}(\Phi, \hat{f}_\tau) = [a_0+ak(\tau)]s\Big(\gamma+\sum_{i=1}^{k(\tau)} X_i,\ a_0+\beta+ak(\tau)\Big)\times$$
$$\times\int_D w_2'(\vartheta)\left(\frac{\gamma+\sum_{i=1}^{k(\tau)} X_i}{a_0+\beta+ak(\tau)}-\vartheta\right)^2 \exp\Big[[a_0+\beta+ak(\tau)]w_1(\vartheta)+$$
$$+\Big(\gamma+\sum_{i=1}^{k(\tau)} X_i\Big)w_2(\vartheta)\Big]d\vartheta = \frac{[a_0+ak(\tau)]s\big(\gamma+\sum_{i=1}^{k(\tau)} X_i,\ a_0+\beta+ak(\tau)\big)}{[a_0+\beta+ak(\tau)]^2}\times$$
$$\times\int_D w_2'(\vartheta)\Big\{\gamma+\sum_{i=1}^{k(\tau)} X_i-\vartheta[a_0+\beta+ak(\tau)]\Big\}^2\exp\Big[[a_0+\beta+ak(\tau)]w_1(\vartheta)+$$
$$+\Big(\gamma+\sum_{i=1}^{k(\tau)} X_i\Big)w_2(\vartheta)\Big]d\vartheta.$$

By (10) we have

$$r^{\mathscr{F}_\tau}(\Phi, \hat{f}_\tau) = \frac{1}{a_0+\beta+ak(\tau)}.$$

Thus, if $\Phi \in \mathscr{E}_0(a_0, \gamma)$, then the problem of finding Bayes sequential plans reduces to the problem of minimizing

$$V_\Phi(a_0, \tau) = \mathrm{E}_\Phi\{[a_0+\beta+ak(\tau)]^{-1}+c\tau\}$$

with respect to τ.

If $G \in \mathscr{G}$ and G has a non-increasing failure rate, then it follows from Theorem 2.1 in [4] that for any prior distribution Φ (not only for

$\Phi \in \mathscr{E}_0(a_0, \gamma)) \ V_\Phi(a_0, \tau)$ is minimized by (16). Thus, taking into account Lemma 2 we obtain the theorem.

It follows also from Theorem 2.1 in [4] that $V_\Phi(a_0, \tau_{a_0})$ is independent of Φ.

The minimax sequential plans are determined in the following

THEOREM 2. *Let* $X_1, \ldots, X_n$ *be independent random variables with a common distribution* $\mathscr{P}_\vartheta \in \mathscr{E}(\vartheta, a)$ *satisfying conditions* (i)-(iii), *and let, moreover, for* $\beta > 0$

(iv) $\sup\limits_{\vartheta \in D} \vartheta^2 w_2'(\vartheta) = \beta^{-1}$.

If $\mathrm{E}(t_1) < \infty$, *then for the loss function* (13) *the sequential plan* $\delta^0 = (\tau_0, \hat{f}^0_{\tau_0})$ *with*

$$\hat{f}^0_{\tau_0} = \frac{\sum\limits_{i=1}^{k(\tau_0)} X_i}{ak(\tau_0)+\beta} \tag{17}$$

is minimax.

Proof. Let us write V instead of V_Φ when the distribution Φ is degenerate at $a_0 = 0$. Then, as remarked above, $r(\Phi, \delta_{a_0}) = V(a_0, \tau_{a_0})$. Let us now take into consideration the estimator (17). After simple computations, in view of (iv), we see that

$$\sup_{\vartheta \in D} R(\vartheta, \delta^0) = V(0, \tau_0)$$

is a finite constant. Thus, according to the well-known method of finding the minimax rules in decision theory (see, e.g., [2], p. 90), it suffices to show that $V(a_0, \tau_{a_0}) \to V(0, \tau_0)$ as $a_0 \to 0$. To show this one can apply an argument used in the proof of Theorem 4.2 in [4].

Condition (iv) is fulfilled, e.g., for the negative-binomial and gamma distributions ($\beta = 1$).

One may also consider, adopting the methods used in [4], certain modification of the discussed model, assuming that the distribution of the random variables $U_1, \ldots, U_n$ belongs to some exponential family of distributions with an unknown parameter and taking a conjugate prior distribution of this parameter.

In the analogous way as in [4] one may propose an adaptive plan which requires knowledge of neither n nor G and performs nearly as well as possible when n is large for a wide class of G.

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW

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R. MAGIERA (Wrocław)

ESTYMACJA PRZY OPÓŹNIONYCH OBSERWACJACH

STRESZCZENIE

W pracy rozważa się pewien problem estymacji nieznanego parametru rozkładów, gdy obserwacje stają się dostępne dla statystyka w chwilach losowych.

Niech $X_1, \ldots, X_n$ będą niezależnymi zmiennymi losowymi o tym samym rozkładzie, należącym do pewnej rodziny $\mathscr{E}(\vartheta, a)$ z wykładniczej rodziny rozkładów. Przypuśćmy, że X_i jest obserwacją w chwili t_i $(i = 1, \ldots, n)$, gdzie $t_1, \ldots, t_n$ są statystykami pozycyjnymi dodatnich zmiennych losowych $U_1, \ldots, U_n$ niezależnych od $X_1, \ldots, X_n$.

W pracy wyznaczono klasę optymalnych planów sekwencyjnych nieznanego parametru ϑ dla zmiennych losowych $X_1, \ldots, X_n$, zakładając, że strata związana z błędem estymacji jest określona przez ważoną kwadratową funkcję straty, a związany z obserwacją koszt c $(c > 0)$, przypadający na jednostkę czasu, jest stały. Przy pewnych założeniach dotyczących $\mathscr{E}(\vartheta, a)$ i dystrybuanty G zmiennych losowych $U_1, \ldots, U_n$ wyznaczono bayesowski plan sekwencyjny parametru ϑ względem rozkładu a priori z rodziny skoniugowanych rozkładów dla $\mathscr{E}(\vartheta, a)$ i pokazano, że plan sekwencyjny $\delta^0 = (\tau_0, \hat{f}^0_{\tau_0})$, gdzie τ_0 i $\hat{f}^0_{\tau_0}$ są określone odpowiednio przez (16) (dla $a_0 = 0$) i (17), jest minimaksowy.
