ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVII, 1 (1980)

TERESA LEDWINA (Wroclaw)

ON THE ADMISSIBILITY OF TESTS FOR EXTENDED HYPOTHESES OF FIT

1. Introduction. We are concerned with a multinomial distribution with, say, k classes and with the hypothesis that the point $p = (p_1, \ldots, p_k)$, where p_i is the probability of the i-th class, ranges over some set U_q.

The hypothesis of fit for a fixed distribution, say F, is usually formulated as

$$H_0 : p = q,$$

where $q = (q_1, \ldots, q_k)$ and $q_1, \ldots, q_k$ are generated by F. But in a large sample this hypothesis is nearly always rejected, since no natural population has ever the exact distribution F. In fact, we would want to reject H_0 only when the departure of the real distribution from F were great. Consequently, Hodges and Lehmann [3] have proposed to test instead of H_0 the hypothesis

$$H^* : d(p, q) \leqslant c^*,$$

where $d(p, q)$ is a measure of distance.

We are concerned with the problem of testing $H: p \in U_q$ against $K: p \notin U_q$, where U_q is a neighbourhood of q having the property: $p \in U_q$ implies $p_i > c > 0$ for $i = 1, \ldots, k$.

For the simplest case of testing H_0, Birnbaum [1] has proved that every test with convex acceptance region is admissible. We extend this result to the case of testing H against K.

The usage of customary tests of fit for testing H depends on the possibility of control of the level of significance. An example of such an extended hypothesis is given in Section 4.

2. Notation. Let $X = (X_1, \ldots, X_k)$ stand for a multinomially distributed random vector (with parameters n and $p = (p_1, \ldots, p_k)$), where X_i take non-negative integral values and

$$n = \sum_{i=1}^{k} X_i.$$

Let $\mathscr{X}$ denote the set of values of X. For every $x \in \mathscr{X}$ we have

$$\mathrm{P}(X_1 = x_1, \ldots, X_k = x_k) = l_n(x) \prod_{i=1}^{k} p_i^{x_i},$$

where

$$l_n(x) = n! \Big(\prod_{i=1}^{k} x_i!\Big)^{-1}.$$

Throughout this paper we assume that $p_i > 0$ for $i = 1, \ldots, k$ and use the notation W_p for such a multinomial distribution.

Finally, let us note that a subset of $\mathscr{X}$ is called *convex* if it is a common part of $\mathscr{X}$ and of a convex set in R^k. We say that x is an *extremal point* of the subset A of $\mathscr{X}$ if x is an extremal point of the convex hull of A.

3. Admissibility of tests with convex acceptance regions.

THEOREM. *Let φ be a test of $H: p \in U_q$ against the unrestricted alternative $K: p \notin U_q$, where each $p_i > c > 0$ for every $p \in U_q$. If the acceptance region of φ is convex and randomization is done at most at the extremal points of this region, then φ is admissible.*

Proof. Let us suppose to the contrary that φ satisfies the assumptions of the theorem and that there exists a test ψ better than φ. So we have

$$\sum_{x \in \mathscr{X}} \big(\psi(x) - \varphi(x)\big) W_p(x) \leqslant 0 \quad \text{for } p \in U_q, \tag{1}$$

$$\sum_{x \in \mathscr{X}} \big(\psi(x) - \varphi(x)\big) W_p(x) \geqslant 0 \quad \text{for } p \notin U_q \tag{2}$$

and at least one of these inequalities is sharp for some p.

Now we can write $W_p(x)$ in the form

$$W_p(x) = l_n(x) \Big(1 + \sum_{i=1}^{k-1} \exp\{\theta_i\}\Big)^{-n} \exp\Big\{\sum_{i=1}^{k-1} \theta_i x_i\Big\},$$

where

$$\theta_i = \ln \frac{p_i}{p_k} \quad \text{for } i = 1, \ldots, k-1.$$

Consequently, (2) implies

$$\sum_{x \in \mathscr{X}} l_n(x) \big(\psi(x) - \varphi(x)\big) \exp\Big\{\sum_{i=1}^{k-1} \theta_i x_i\Big\} \geqslant 0 \quad \text{for } p \notin U_q. \tag{3}$$

Moreover, from (1) and (2) it follows that there exists a point $x^0 \in \mathscr{X}$ such that $\varphi(x^0) > \psi(x^0)$.

Now, let us denote by A the acceptance region of φ. Then

$$A = \{x \in \mathscr{X}\colon \varphi(x) < 1\}$$

and by the conditions imposed on φ we see that $x^0 \notin A$ or x^0 is an extremal point of A. In both cases there exists a vector $a \neq 0$, having not all components equal, such that

$$\langle a, x-x^0\rangle < 0 \quad \text{for } x \in A, x \neq x^0,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on R^k. Moreover, in view of the equalities

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} x_i^0 = n$$

the vector $a^* = (a_1 - a_k, \ldots, a_{k-1} - a_k, 0)$ has the property

$$\langle a^*, x-x^0\rangle = \langle a, x-x^0\rangle.$$

Since the vector $\theta = (\theta_1, \ldots, \theta_{k-1})$ ranges over R^{k-1}, we can put $\theta_i = Na_i^*$ for $i = 1, \ldots, k-1$, where N is a natural number.

Now we show that there exists a subsequence of $\{(Na_1^*, \ldots, Na_{k-1}^*)\}$ such that the solutions

$$p(N) = \big(p_1(N), \ldots, p_k(N)\big)$$

of $Na_i^* = \ln p_i - \ln p_k$ $(i = 1, \ldots, k-1)$ belong to the alternative K.

Let now $\theta_i^0 = \ln q_i - \ln q_k$ $(i = 1, \ldots, k-1)$. The condition imposed on U_q guarantees that

$$\sum_{i=1}^{k-1} (\theta_i - \theta_i^0)^2 = \sum_{i=1}^{k-1} \left(\ln \frac{p_i q_k}{p_k q_i}\right)^2 < C$$

for some C and every $p \in U_q$. Thus, for every sequence of numbers, say $\{d_j\}$, tending to infinity and such that $C < d_1 < d_2 < \ldots$, the solutions of

$$\sum_{i=1}^{k-1} (\theta_i - \theta_i^0)^2 = d_j$$

lead to p's which belong to K. So for a sufficiently large N the vectors $(Na_1^*, \ldots, Na_{k-1}^*)$ belong to K and

$$N\langle a^*, x-x^0\rangle < 0 \quad \text{for } x \in A \setminus \{x_0\}.$$

Multiplying now both sides of (3) by $\exp\{-N\langle a^*, x^0\rangle\}$ we obtain

(4)
$$\sum_{x \in \mathscr{X} \setminus \{x^0\}} l_n(x)\big(\psi(x) - \varphi(x)\big)\exp\{N\langle a^*, x-x^0\rangle\} + l_n(x^0)\big(\psi(x^0) - \varphi(x^0)\big) \geqslant 0.$$

Since for $x \in A$ we have $\varphi(x) = 1$ and, consequently, $\psi(x) - \varphi(x) \leqslant 0$, inequality (4) implies

(5)
$$\sum_{x \in A \setminus \{x^0\}} l_n(x)\big(\psi(x) - \varphi(x)\big)\exp\{N\langle a^*, x-x^0\rangle\} + l_n(x^0)\big(\psi(x^0) - \varphi(x^0)\big) \geqslant 0.$$

Thus letting $N \to +\infty$ in (5) we obtain

$$\psi(x^0) - \varphi(x^0) \geqslant 0,$$

which contradicts $\varphi(x^0) > \psi(x^0)$ and the proof is completed.

4. Applications. The acceptance regions based on the χ^2-statistics

$$\sum_{i=1}^{k} \frac{(x_i - nq_i)^2}{nq_i}$$

and on the likelihood ratio

$$2\sum x_i \ln\left(\frac{x_i}{nq_i}\right)$$

are convex and, therefore, are admissible for testing the hypothesis of fit $H_0: p = q$ and the extended hypothesis $H: p \in U_q$.

However, the usage of those tests for testing H depends on the possibility of control of the level of significance. We show that this can be done for

$$U_q = \{p : |p_i - q_i| \leqslant \varepsilon\}, \quad \text{where } \varepsilon < \min(q_1, \ldots, q_k).$$

Since $W_p(x)$ is a continuous function of p for every fixed x, we infer that for every $\varepsilon_1 > 0$ $(\alpha > \varepsilon_1)$ there exists an $\varepsilon > 0$ such that

$$\Big|\sum_{x \in A} W_p(x) - \sum_{x \in A} W_q(x)\Big| \leqslant \varepsilon_1 \quad \text{for all } A \in 2^{\mathfrak{X}} \text{ and } p \in U_q.$$

For such ε_1 and ε we have

$$\sum_{x \in A} W_p(x) \leqslant \sum_{x \in A} W_q(x) + \varepsilon_1 \quad \text{for } p \in U_q.$$

Consequently, every test of level $\alpha - \varepsilon_1$ for testing H_0 is that of level α for H.

Remark. As we mentioned in Section 1, Hodges and Lehmann [3] considered the following extended hypothesis of fit:

$$H^* : d(p, q) \leqslant c^*.$$

They proposed to test H^* with nice asymptotic properties. Their result was generalized by Bjørnstad [2].

After investigating those test statistics one can see, however, that they are not convex functions of x for the distances d of the form

$$d(p, q) = \sum_{i=1}^{k} s_i(p_i - q_i)^2,$$

where the weights s_i can be functions of p and q. So the examination of admissibility of those tests by our theorem is rather difficult.

References

[1] A. Birnbaum, *Characterizations of complete classes of tests of some multiparametric hypotheses with applications to likelihood ratio tests*, Ann. Math. Statist. 26 (1955), p. 21-36.

[2] J. F. Bjørnstad, *Inference theory in contingency tables*, Statistical Research Report No. 2, University of Oslo, 1975.

[3] J. L. Hodges, Jr., and E. Lehmann, *Testing of approximate validity of statistical hypotheses*, J. R. Statist. Soc., B, 16 (1954), p. 261-268.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY WROCŁAW
50-370 WROCŁAW

Received on 30. 11. 1977

TERESA LEDWINA (Wrocław)

O DOPUSZCZALNOŚCI TESTÓW DLA ROZSZERZONEJ HIPOTEZY ZGODNOŚCI

STRESZCZENIE

W pracy rozważa się problem testowania rozszerzonej hipotezy zgodności. Udowodniono, że testy o wypukłych obszarach przyjęć są dopuszczalne.
