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APPLICATIONES MATHEMATICAE
XVIII, 2 (1984), p. 319-336

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HYPERBOLIC SPLINES WITH GIVEN DERIVATIVES AT THE KNOTS

Let τ be a hyperbolic spline function with knots x_i $(0 = x_0 < x_1 < \ldots < x_n = 1)$. In this paper, theorems on the existence and uniqueness of the solution of certain interpolation problems are given. We assume that the interpolant τ has the first or second derivative at the knots x_i $(i = 0, 1, \ldots, n)$ and satisfies the appropriate boundary conditions. The upper bounds for the error $\|f^{(k)} - \tau^{(k)}\|$ $(k = 0, 1, 2, 3)$ are given, where $f \in C^4[0, 1]$ or $f \in C^5[0, 1]$.

1. Introduction. Let $\Delta_n = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ be an arbitrary (but fixed) partition of the interval $[0, 1]$. Further, let $\alpha = \{\alpha_i > 0\}_{i=1}^{n}$ be a set of tension parameters. By $\mathrm{Sph}(\Delta_n, \alpha)$ we denote a space of hyperbolic splines in tension with the above knots x_i and tension parameters α_i, i.e. $\tau \in \mathrm{Sph}(\Delta_n, \alpha)$ if and only if

(i) in each interval $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, $\tau(x)$ is a linear combination of the functions $1, x, \sinh(\alpha_i x)$, and $\cosh(\alpha_i x)$;

(ii) $\tau \in C^2[0, 1]$.

For simplicity of notation, let $h_i = x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$, $y_i = \tau(x_i)$, $m_i = \tau'(x_i)$, $M_i = \tau''(x_i)$ $(i = 0, 1, \ldots, n)$.

This paper is concerned with some questions of the existence and uniqueness of a solution of certain interpolation problems in the space $\mathrm{Sph}(\Delta_n, \alpha)$. The first one is formulated as follows: does there exist $\tau \in \mathrm{Sph}(\Delta_n, \alpha)$ such that

(1.1) $$\tau'(x_i) = m_i \quad (i = 0, 1, \ldots, n)$$

with boundary conditions

(1.2) $$\tau(x_0) = y_0, \quad \tau(x_n) = y_n$$

or

(1.3) $$\tau''(x_0) = M_0, \quad \tau''(x_n) = M_n,$$

or initial conditions

$$(1.4) \qquad \tau(x_0) = y_0, \quad \tau(x_1) = y_1,$$

where $m_i, y_0, y_1, y_n, M_0, M_n$ are given real numbers.

The second interpolation scheme is of the form

$$(1.5) \qquad \tau''(x_i) = M_i \quad (i = 0, 1, \ldots, n)$$

with the boundary conditions

$$(1.6) \qquad \tau(x_0) = y_0, \quad \tau(x_n) = y_n$$

or

$$(1.7) \qquad \tau'(x_0) = m_0, \quad \tau'(x_n) = m_n,$$

or

$$(1.8) \qquad \tau(x_0) = y_0, \quad \tau(x_1) = y_1,$$

or

$$(1.9) \qquad \tau(x_0) = y_0, \quad \tau'(x_0) = m_0,$$

where, as above, $M_i, y_0, y_1, y_n, m_0, m_n$ are given real numbers.

The results of this paper are in some sense a generalization of the results due to Neuman [3], [4], and Carlson and Hall [1]. In those papers, the authors consider analogous interpolation problems but in the space of cubic splines. Namely, if $a_i \to 0$, then our results are the same as in [3], [4], and [1]. In Section 2, some conditions which guarantee the existence and, in some cases, also the uniqueness of a solution of the above interpolation problems are given. In Section 3, upper bounds for the error $\|f^{(k)} - \tau^{(k)}\|$ $(k = 0, 1, 2, 3)$ are presented, where $\|\cdot\|$ stands for the max-norm over the interval $[0, 1]$.

2. The existence and uniqueness theorems. We begin this section with some elementary facts concerning tridiagonal matrices $A_n = (r_{ij})$ $(i, j = 1, 2, \ldots, n)$, where

$$(2.1) \qquad r_{ij} = \begin{cases} a_i & (i = 2, 3, \ldots, n;\ i > j), \\ b_i & (i = 1, 2, \ldots, n;\ i = j), \\ c_i & (i = 1, 2, \ldots, n-1;\ i < j), \\ 0 & (|i-j| > 1;\ i, j = 1, 2, \ldots, n). \end{cases}$$

If d_k denotes the k-th main minor of A_n, then it is well known that

$$(2.2) \qquad d_k = b_k d_{k-1} - a_k c_{k-1} d_{k-2} \quad (k = 2, 3, \ldots, n;\ d_0 = 1, d_1 = b_1).$$

This recurrence formula will be used in the proofs of theorems given below.

Let $B_n = (q_{ij})$ $(i, j = 1, 2, \ldots, n)$ be an inverse matrix to A_n (we assume that B_n exists). If all main minors d_k $(k = 1, 2, \ldots, n)$ of A_n are different from zero, then from formulae (3.3)-(3.4) or (3.7) in [2] it follows that

$$(2.3)\qquad q_{nj} = (-1)^{n-j}\frac{d_{i-1}}{d_j}\prod_{l=j+1}^{n} a_l \quad (j = 1, 2, \ldots, n).$$

We now consider the interpolation problem (1.1), (1.2). Let $y_i = \tau(x_i)$ and $m_i = \tau'(x_i)$ $(i = 0, 1, \ldots, n)$. Then the hyperbolic spline $\tau(x)$ for $x \in [x_{i-1}, x_i]$ may be written in the form

$$(2.4)\qquad \tau(x) = y_{i-1}F_i(h_i - t) + y_iF_i(t) + m_{i-1}G_i(h_i - t) - m_iG_i(t),$$

where

$$(2.5)\qquad \begin{gathered} t = x - x_{i-1}, \quad F_i(t) = \frac{1}{h_i}\left(t + \frac{\sigma_i(h_i - t) - \sigma_i(t)}{\sigma_i'(0) + \sigma_i'(h_i)}\right), \\ G_i(t) = \frac{\sigma_i'(0)\sigma_i(h_i - t) + \sigma_i'(h_i)\sigma_i(t)}{\sigma_i'^2(0) - \sigma_i'^2(h_i),}, \quad \sigma_i(t) = \frac{1}{a_i^2}\left(\frac{\sinh(a_it)}{\sinh(a_ih_i)} - \frac{t}{h_i}\right). \end{gathered}$$

The continuity of $\tau''(x)$ at the knots x_i $(i = 1, 2, \ldots, n-1)$ implies the consistency relations

$$(2.6)\qquad \begin{aligned} -\frac{h_{i+1}}{h_i}s_iy_{i-1} + \left(\frac{h_{i+1}}{h_i}s_i - \frac{h_i}{h_{i+1}}s_{i+1}\right)y_i + \frac{h_i}{h_{i+1}}s_{i+1}y_{i+1} \\ = h_{i+1}t_im_{i-1} + (h_{i+1}t_iv_i + h_it_{i+1}v_{i+1})m_i + h_it_{i+1}m_{i+1} \\ (i = 1, 2, \ldots, n-1) \end{aligned}$$

(see [6], equation (1.15)), where

$$(2.7)\qquad s_i = t_i(v_i + 1), \quad v_i = \frac{-\sigma_i'(h_i)}{\sigma_i'(0)}, \quad t_i = \frac{h_i\sigma_i'(0)}{\sigma_i'^2(0) - \sigma_i'^2(h_i)}.$$

The matrix A_{n-1} of the linear system (2.6) with unknowns $y_1, y_2, \ldots$ $\ldots, y_{n-1}$ (y_0, y_n are given) is tridiagonal. By (2.2) we have

$$d_0 = 1, \quad d_1 = \frac{h_2}{h_1}s_1 - \frac{h_1}{h_2}s_2,$$

$$d_k = \left(\frac{h_{k+1}}{h_k}s_k - \frac{h_k}{h_{k+1}}s_{k+1}\right)d_{k-1} + \frac{h_{k+1}}{h_k}\frac{h_{k-1}}{h_k}s_k^2d_{k-2} \qquad (k = 2, 3, \ldots, n-1).$$

We can prove by induction that

$$d_k = \frac{(-1)^{k+1}\prod_{j=1}^{k+1} s_j}{h_1 h_{k+1}} \sum_{j=1}^{k+1} (-1)^j \frac{h_j^2}{s_j} \qquad (k = 0, 1, \ldots, n-1).$$

Hence $d_{n-1} = \det A_{n-1} \neq 0$ if and only if

$$\sum_{j=1}^{n} (-1)^j \frac{h_j^2}{s_j} \neq 0 \qquad (s_j > 0 \text{ for all } j).$$

This proves the following

THEOREM 2.1. *The interpolation problem* (1.1), (1.2) *has exactly one solution if and only if*

$$\sum_{j=1}^{n} (-1)^j \frac{h_j^2}{s_j} \neq 0. \tag{2.8}$$

By (2.8) and from the fact that s_j is a positive function in variables h_j and a_j, we have immediately

COROLLARY 2.1. *If the knots x_i are equidistant and all tension parameters are equal to one another, then the interpolation problem* (1.1), (1.2) *has exactly one solution if and only if n is odd.*

In the next theorem, the conditions which guarantee the existence of a solution of the problem (1.1), (1.3) are given. These conditions depend on the partition Δ_n, tension parameters a_j, and also on the interpolated values m_j $(j = 0, 1, \ldots, n)$, M_0, M_n.

THEOREM 2.2. *The interpolation problem* (1.1), (1.3) *has a solution if*

$$\sum_{j=1}^{n} (-1)^j \frac{m_j - m_{j-1}}{h_j} t_j(v_j - 1) = (-1)^n M_n - M_0. \tag{2.9}$$

The interpolant τ is nonunique.

Proof. In this case we must add to the system (2.6) two equations which follow from conditions (1.3). Thus we obtain the following system of linear equations with unknowns $y_0, y_1, \ldots, y_n$:

$$-y_0 + y_1 = \frac{h_1}{s_1}(t_1 v_1 m_0 + t_1 m_1 + h_1 M_0) - \tag{2.10}$$

$$-\frac{h_{i+1}}{h_i} s_i y_{i-1} + \left(\frac{h_{i+1}}{h_i} s_i - \frac{h_i}{h_{i+1}} s_{i+1}\right) y_i + \frac{h_i}{h_{i+1}} s_{i+1} y_{i+1}$$

$$= h_{i+1} t_i m_{i-1} + (h_{i+1} t_i v_i + h_i t_{i+1} v_{i+1}) m_i + h_i t_{i+1} m_{i+1}$$

$$(i = 1, 2, \ldots, n-1),$$

$$-y_{n-1}+y_n = \frac{h_n}{s_n}(t_n m_{n-1}+t_n v_n m_n - h_n M_n).$$

The matrix A_{n+1} of this system is tridiagonal. We shall show that $\det A_{n+1} = 0$ and $\operatorname{rank} A_{n+1} = n$. Let $h_0 = h_1/s_1$. By (2.2) we have $d_0 = 1$, $d_1 = -1$, and

$$d_k = \left(\frac{h_k}{h_{k-1}}s_{k-1} - \frac{h_{k-1}}{h_k}s_k\right)d_{k-1} + \frac{h_k h_{k-2}}{h_{k-1}^2}s_{k-1}^2 d_{k-2} \quad (k = 2, 3, \ldots, n), \tag{2.11}$$

$$d_{n+1} = d_n + \frac{h_{n-1}}{h_n}s_n d_{n-1}. \tag{2.12}$$

We can prove by induction that the numbers d_k satisfying (2.11) are equal to

$$d_k = (-1)^k \frac{h_1}{h_k s_1}\prod_{j=1}^{k} s_j \quad (k = 1, 2, \ldots, n). \tag{2.13}$$

From (2.12) and (2.13) we obtain

$$d_{n+1} = \det A_{n+1} = (-1)^n \frac{h_1}{h_n s_1}\prod_{j=1}^{n} s_j + \frac{h_{n-1}}{h_n}s_n\cdot(-1)^{n-1}\frac{h_1}{h_{n-1}s_1}\prod_{j=1}^{n-1} s_j = 0.$$

Consequently, by (2.13), we also have $\operatorname{rank} A_{n+1} = n$.

Let A_n be a submatrix formed with n first rows and n first columns of the matrix A_{n+1}. By (2.13) we have $\det A_n \neq 0$. Hence, all main minors of A_n are different from zero. We now put

$$y^{\mathrm{T}} = (y_0, y_1, \ldots, y_{n-1}), \quad \Phi^{\mathrm{T}} = (\varphi_1, \varphi_2, \ldots, \varphi_n),$$

where

$$\begin{aligned}
\varphi_1 &= \frac{h_1}{s_1}(t_1 v_1 m_0 + t_1 m_1 + h_1 M_0),\\
\varphi_i &= h_i t_{i-1} m_{i-2} + (h_i t_{i-1} v_{i-1} + h_{i-1} t_i v_i) m_{i-1} + h_{i-1} t_i m_i \\
&\qquad\qquad (i = 2, 3, \ldots, n-1),\\
\varphi_n &= h_n t_{n-1} m_{n-2} + (h_n t_{n-1} v_{n-1} + h_{n-1} t_n v_n) m_{n-1} + h_{n-1} t_n m_n - \\
&\qquad - \frac{h_{n-1}}{h_n} s_n y_n.
\end{aligned} \tag{2.14}$$

We now consider the system $A_n y = \varphi$. If q_{ij} $(i, j = 1, 2, \ldots, n)$ are entries of the inverse matrix to the matrix A_n, then from (2.3) it follows that

$$(2.15)\quad q_{nj} = (-1)^{n-j-1}\frac{h_n^2}{h_{j-1}h_j s_n} \qquad (j = 1, 2, \ldots, n;\ h_0 = h_1/s_1).$$

Hence

$$(2.16)\qquad y_{n-1} = \sum_{j=1}^{n} q_{nj}\varphi_j.$$

Putting (2.14) and (2.15) into (2.16) we obtain

$$(2.17)\quad -y_{n-1}+y_n = (-1)^n \frac{h_n^2}{s_n}\Big\{-\frac{1}{h_1}(t_1v_1m_0+t_1m_1+h_1M_0)+ \\ +\sum_{j=2}^{n}\frac{(-1)^j}{h_{j-1}h_j}[h_jt_{j-1}m_{j-2}+(h_jt_{j-1}v_{j-1}+h_{j-1}t_jv_j)m_{j-1}+h_{j-1}t_jm_j]\Big\}.$$

Comparing the right-hand side of (2.17) with the right-hand side of the last equation in (2.10), we obtain finally (2.9).

The matrix of the system (2.6) with unknowns $y_2, y_3, \ldots, y_n$ (y_0, y_1 are given) is lower triangular with entries $(h_i/h_{i+1})s_{i+1}$ ($i = 1, 2, \ldots, n-1$) on the main diagonal. Hence we obtain

THEOREM 2.3. *The interpolation problem* (1.1), (1.4) *has exactly one solution.*

Now we consider the interpolation scheme (1.5) with conditions (1.6)-(1.9). Let $M_i = \tau''(x_i)$ ($i = 0, 1, \ldots, n$). In each interval $[x_{i-1}, x_i]$ the function $\tau(x)$ from $\mathrm{Sph}(\Delta_n, a)$ can be written in the form

$$(2.18)\qquad \tau(x) = y_{i-1}\frac{h_i-t}{h_i}+y_i\frac{t}{h}+M_{i-1}\sigma_i(h_i-t)+M_i\sigma_i(t),$$

where t and $\sigma_i(t)$ are the same as in (2.5). By the continuity of $\tau'(x)$ at the points x_i we obtain the following system of equations:

$$(2.19)\quad h_{i+1}y_{i-1}-(h_i+h_{i+1})y_i+h_iy_{i+1} \\ = h_ih_{i+1}\{-M_{i-1}\sigma_i'(0)+[\sigma_i'(h_i)+\sigma_{i+1}'(h_{i+1})]M_i-M_{i+1}\sigma_{i+1}'(0)\} \\ (i = 1, 2, \ldots, n-1)$$

(see [5], equation (1.4)).

THEOREM 2.4. *The interpolation problem* (1.5), (1.6) *has exactly one solution.*

Proof. The proof of this theorem is similar to that of Theorem 2.1. In this case the matrix of the system (2.19) with unknowns $y_1, y_2, \ldots, y_{n-1}$

(y_0, y_n are given) is also tridiagonal and its determinant is equal to $(-1)^{n-1}\prod_{j=2}^{n-1} h_j$.

According to Theorem 2.2 we can prove the following

THEOREM 2.5. *The interpolation problem* (1.5), (1.7) *has a solution if*

$$\frac{1}{2}\sum_{j=1}^{n} h_j(M_{j-1}+M_j)p_j = m_n - m_0, \tag{2.20}$$

where

$$p_j = 2\,\frac{\sigma_j'(0)+\sigma_j'(h_j)}{h_j} \qquad (j = 1, 2, \ldots, n).$$

The interpolant τ is nonunique.

Proof. An idea of the proof is similar to that of Theorem 2.2. The system of equations (2.19) and two equations obtained from the boundary conditions (1.7) give the following system of linear equations with unknowns $y_0, y_1, \ldots, y_n$:

$$\begin{aligned} -y_0+y_1 &= h_1[M_0\sigma_1'(h_1)-M_1\sigma_1'(0)+m_0], \\ h_{i+1}y_{i-1}-(h_i+h_{i+1})y_i+h_iy_{i+1} &= h_ih_{i+1}\{-M_{i-1}\sigma_i'(0)+ \\ &\quad +[\sigma_i'(h_i)+\sigma_{i+1}'(h_{i+1})]M_i-M_{i+1}\sigma_{i+1}'(0)\} \quad (i = 1, 2, \ldots, n-1), \\ y_{n-1}-y_n &= h_n[-M_{n-1}\sigma_n'(0)+M_n\sigma_n'(h_n)-m_n]. \end{aligned} \tag{2.21}$$

The matrix A_{n+1} of this system is tridiagonal and the k-th main minor d_k is

$$d_k = (-1)^k\prod_{j=1}^{k-1} h_j \qquad (k = 1, 2, \ldots, n).$$

Hence, $d_{n+1} = \det A_{n+1} = 0$ and $\operatorname{rank} A_{n+1} = n$. Let A_n denote the submatrix of A_{n+1} formed in the same way as in the proof of Theorem 2.2. For the entries q_{nj} of the inverse matrix to the matrix A_n, by (2.3) we have

$$q_{nj} = -\frac{h_n}{h_{j-1}h_j} \qquad (j = 1, 2, \ldots, n;\ h_0 = 1). \tag{2.22}$$

Further, let $y^{\mathrm{T}} = (y_0, y_1, \ldots, y_{n-1})$ and $\varphi^{\mathrm{T}} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, where

$$\begin{aligned} \varphi_1 &= h_1[M_0\sigma_1'(h_1)-M_1\sigma_1'(0)+m_0], \\ \varphi_j &= h_{j-1}h_j\{-M_{j-2}\sigma_{j-1}'(0)+[\sigma_{j-1}'(h_{j-1})+\sigma_j'(h_j)]M_{j-1}-M_j\sigma_j'(0)\} \\ &\qquad (j = 2, 3, \ldots, n-1), \\ \varphi_n &= h_{n-1}h_n\{-M_{n-2}\sigma_{n-1}'(0)+[\sigma_{n-1}'(h_{n-1})+\sigma_n'(h_n)]M_{n-1}- \\ &\qquad -M_n\sigma_n'(0)\}-h_{n-1}y_n. \end{aligned} \tag{2.23}$$

For a linear system $A_n y = \varphi$ we have

$$y_{n-1} = \sum_{j=1}^{n} q_{nj}\varphi_j.$$

Putting (2.22) and (2.23) into the right-hand side of the last equality and performing some simple calculations, we obtain

$$y_{n-1}-y_n = h_n\Big\{\sum_{j=2}^{n}[-M_{j-2}\sigma'_{j-1}(0)+[\sigma'_{j-1}(h_{j-1})+\sigma'_j(h_j)]M_{j-1}-\sigma'_j(0)M_j]+$$
$$+M_0\sigma'_1(h_1)-M_1\sigma'_1(0)+m_0\Big\}.$$

Comparing the right-hand side of the above equality with the right-hand side of the last equation in (2.21) we obtain the desired result (2.20).

The system (2.19) with unknowns $y_2, y_3, \ldots, y_n$ (y_0, y_1 are given) has a lower triangular matrix with elements h_i ($i = 1, 2, \ldots, n-1$) on the main diagonal. Hence we get

THEOREM 2.6. *The interpolation problem* (1.5), (1.8) *has exactly one solution.*

We also have

THEOREM 2.7. *The interpolation problem* (1.5), (1.9) *has exactly one solution.*

Proof. Condition (1.9) yields the equality

$$\frac{y_1-y_0}{h_1} = M_0\sigma'_1(h_1)-M_1\sigma'_1(0)+m_0. \tag{2.24}$$

Equations (2.19) and (2.24) form the following system of linear equations with unknowns $y_1, y_2, \ldots, y_n$ (y_0, m_0 are given):

$$\begin{aligned} &\frac{y_1-y_0}{h_1} = M_0\sigma'_1(h_1)-M_1\sigma'_1(0)+m_0,\\ &-\frac{y_i-y_{i-1}}{h_i}+\frac{y_{i+1}-y_i}{h_{i+1}} = -M_{i-1}\sigma'_i(0)+[\sigma'_i(h_i)+\sigma'_{i+1}(h_{i+1})]M_i-\\ &\qquad -M_{i+1}\sigma'_{i+1}(0) \qquad (i = 1, 2, \ldots, n-1). \end{aligned} \tag{2.25}$$

The matrix A_n of this system is lower triangular with elements on the main diagonal equal to $1/h_i$ ($i = 1, 2, \ldots, n$). Hence $\det A_n \neq 0$.

3. Error bounds. For the sake of brevity we put

$$f_i^{(k)} = f^{(k)}(x_i), \quad y_i^{(k)} = \tau^{(k)}(x_i), \quad e_i^{(k)} = y_i^{(k)} - f_i^{(k)} \quad (i = 0, 1, \dots, n),$$

$$h = \max_{1 \leqslant i \leqslant n} h_i, \quad a = \max_{1 \leqslant i \leqslant n} a_i,$$

$$a_i = \frac{1}{3} - \frac{\sigma_i'(h_i)}{h_i}, \quad b_i = \frac{1}{6} + \frac{\sigma_i'(0)}{h_i}, \quad c_i = \frac{1}{6} h_i^2 \left(\frac{f_i'' - f_{i-1}''}{h_i} - f_{i-1}''' \right)$$
$$(i = 1, 2, \dots, n).$$

We denote by $\omega(f, h)$ the modulus of continuity of the function f with parameter h. Let $\|\cdot\|_\infty$ denote the matrix of the ∞-norm and let $\|\cdot\|$ stand for the max-norm over $[0, 1]$.

According to the definition of the function $\sigma(t)$ we obtain easily the following

LEMMA 3.1. *If $t \in [0, h_i]$, then*

(3.1) $$\sigma_i(t) \leqslant 0,$$

(3.2) $$\sigma_i'''(t) = a_i^2 \sigma_i'(t) + \frac{1}{h_i},$$

(3.3) $$0 < \frac{1}{3} - \frac{1}{v_i + 1} \leqslant \frac{1}{90} a_i^2 h_i^2,$$

(3.4) $$|\sigma_i'(t)| \leqslant \frac{1}{3} h_i,$$

(3.5) $$0 < a_i \leqslant \frac{1}{45} a_i^2 h_i^2,$$

(3.6) $$0 < b_i \leqslant \frac{7}{360} a_i^2 h_i^2,$$

(3.7) $$|G_i(h_i - t)| + |G_i(t)| \leqslant \frac{3}{4} h_i,$$

(3.8) $$F_i(t) \geqslant 0, \quad F_i(h_i - t) + F_i(t) \equiv 1$$

for $i = 1, 2, \dots, n$.

We can now prove the following

THEOREM 3.1. *Let $n \geqslant 1$ (n odd), $f \in C^5[0, 1]$, and let $\tau \in \mathrm{Sph}(\Delta_n, a)$ be a solution of the interpolation problem* (1.1), (1.2) *with tension parameters $a_j = a$ ($j = 1, 2, \dots, n$) and equidistant knots $x_i = i/n$ ($i = 0, 1, \dots, n$).*

If $m_i = f_i'$, $y_0 = f_0$, *and* $y_n = f_n$, *then*

$$\|\tau - f\| \leqslant \omega(f, h) + \frac{n-1}{45}\,(h^5\|f^{(5)}\| + a^2h^3\|m\|_\infty) + \frac{3}{4}\,h\|m\|_\infty, \tag{3.9}$$

where $h = 1/n$.

Proof. We use the obvious equalities

$$f_{i+1} - f_{i-1} = 2hf_i' + \frac{1}{3}h^3 f_i''' + \frac{1}{60}h^5 f^{(5)}(\delta_i), \tag{3.10}$$

$$f_{i-1}' - 2f_i' + f_{i+1}' = f_i''' h^2 + \frac{1}{12}h^4 f^{(5)}(\eta_i) \tag{3.11}$$

$$(x_{i-1} \leqslant \delta_i, \eta_i \leqslant x_{i+1};\ i = 1, 2, \ldots, n-1).$$

Under the above assumptions, the system of equations (2.6) takes the form

$$-y_{i-1} + y_{i+1} = h\left(\frac{1}{v+1}f_{i-1}' + 2\,\frac{v}{v+1}f_i' + \frac{1}{v+1}f_{i+1}'\right) \tag{3.12}$$

$$(i = 1, 2, \ldots, n-1),$$

where v is defined in (2.7). Subtracting (3.10) from the above equality and using (3.11) we obtain

$$-e_{i-1} + e_{i+1} = \frac{1}{36}h^5 f^{(5)}(\eta_i) - \frac{1}{60}h^5 f^{(5)}(\delta_i) - \tag{3.13}$$

$$-h\left(\frac{1}{3} - \frac{1}{v+1}\right)(f_{i-1}' - 2f_i' + f_{i+1}') \quad (i = 1, 2, \ldots, n-1;\ e_0 = e_n = 0).$$

Let A_{n-1} denote the matrix of the system (3.13). It is known (see [1], equation (2.10)) that $\|A_{n-1}^{-1}\|_\infty \leqslant (n-1)/2$. Consequently, by (3.3) we obtain

$$\|e\|_\infty \leqslant \frac{n-1}{45}\,(h^5\|f^{(5)}\| + a^2h^3\|m\|_\infty). \tag{3.14}$$

Let $u(x)$ be an auxiliary function defined in the following way:

(i) $u(x)$ belongs to $\operatorname{lin}\{1, x, \sinh(ax), \cosh(ax)\}$ for $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$;

(ii) $u(x_i) = f_i$, $u'(x_i) = f_i'$ $(i = 0, 1, \ldots, n)$.

It is obvious that such a function $u(x)$ always exists and is unique. For $x \in [x_{i-1}, x_i]$, $u(x)$ may be written in the form (see (2.4))

$$u(x) = f_{i-1}F_i(h-t) + f_iF_i(t) + f_{i-1}'G_i(h-t) - f_i'G_i(t), \tag{3.15}$$

where $t = x - x_{i-1}$, $F_i(t)$ and $G_i(t)$ are given in (2.5). Using (3.15) and (3.8), we obtain

$$f(x)-u(x) = [f(x)-f_{i-1}]F_i(h-t)+[f(x)-f_i]F_i(t)-f'_{i-1}G_i(h-t)+f'_iG_i(t).$$

Hence and from (3.7), (3.8) we obtain the estimation

$$\|f-u\| \leqslant \omega(f,h)+\frac{3}{4}h\|m\|_\infty. \tag{3.16}$$

By (ii), (3.15), and (2.4) we get

$$\tau(x)-u(x) = e_{i-1}F_i(h-t)+e_iF_i(t) \qquad (x \in [x_{i-1}, x_i]).$$

Consequently, by (3.8) we have

$$\|\tau-u\| \leqslant \|e\|_\infty. \tag{3.17}$$

From (3.14), (3.16), (3.17), and from the triangle inequality we obtain the thesis of our theorem.

THEOREM 3.2. *Let $n \geqslant 1$, $f \in C^{(5)}[0,1]$, and let $\tau \in \mathrm{Sph}(\Delta_n, a)$ be a solution of the problem (1.1), (1.4) with tension parameters $a_j = a$ ($j = 1, 2, \ldots, n$) and equidistant knots $x_i = i/n$ ($i = 0, 1, \ldots, n$). If $m_i = f'_i$, $y_0 = f_0$, and $y_1 = f_1$, then*

$$\|f-\tau\| \leqslant \omega(f,h)+\frac{1}{45}p(h^5\|f^{(5)}\|+a^2h^3\|m\|_\infty)+\frac{3}{4}h\|m\|_\infty,$$

where

$$p = \begin{cases} n-1 & \text{for } n \text{ odd}, \\ n & \text{for } n \text{ even}, \end{cases}$$

and $h = 1/n$.

Proof. The initial conditions (1.4) ($e_0 = e_1 = 0$) imply that the matrix A_{n-1} of the system (3.13) is of the form

$$\begin{vmatrix} 1 & & & & & \\ 0 & 1 & & & 0 & \\ 1 & 0 & 1 & & & \\ & & & \cdot & & \\ & 0 & & \cdot & \cdot & \\ & & & 1 & 0 & 1 \end{vmatrix}.$$

Hence

$$\|A_{n-1}^{-1}\|_\infty = \begin{cases} (n-1)/2 & \text{for } n \text{ odd}, \\ n/2 & \text{for } n \text{ even}. \end{cases}$$

From now on the proof goes along the same lines as the proof of Theorem 3.1.

Below we shall give the estimation of the errors for the interpolation problem (1.5) with conditions (1.6)-(1.9). The function $\tau(x)$, given by (2.18), can be written in the form

$$\tau(x) = y_{i-1}+ty'_{i-1}+\frac{1}{2}t^2M_i+\frac{1}{6}t^3\frac{M_i-M_{i-1}}{h_i}+ \\ +\frac{1}{6}a_i^2t^3[M_i\sigma_i'(\xi_i)-M_{i-1}\sigma_i'(\nu_i)], \tag{3.18}$$

where $t = x-x_{i-1}$, $x_{i-1}<\xi_i$, $\nu_i<x_i$ $(i=1,2,\ldots,n)$. Using the Taylor series expansion of the function $f\in C^4[0,1]$, we have

$$\frac{f_i-f_{i-1}}{h_i} = f'_{i-1}+\frac{1}{2}h_if''_{i-1}+\frac{1}{6}h_i^2f'''_{i-1}+\frac{1}{24}h_i^3f^{\mathrm{IV}}(a_i), \tag{3.19}$$

$$f'_i-f'_{i-1} = h_if''_{i-1}+\frac{1}{2}h_i^2f'''_{i-1}+\frac{1}{6}h_i^3f^{\mathrm{IV}}(\beta_i), \tag{3.20}$$

$$\frac{f''_i-f''_{i-1}}{h_i} = f'''_{i-1}+\frac{1}{2}h_if^{\mathrm{IV}}(\gamma_i), \tag{3.21}$$

where $x_{i-1}<a_i,\beta_i,\gamma_i<x_i$, and $i=1,2,\ldots,n$.

THEOREM 3.3. *Let $f\in C^4[0,1]$ and let $\tau(x)\in \mathrm{Sph}(\Delta_n, a)$ be a solution of problem* (1.5), (1.6) *with equidistant knots $x_i = i/n$. If $M_i = f''_i$, $y_0=f_0$, and $y_n=f_n$, then*

$$\|\tau^{(k)}-f^{(k)}\| \leqslant \begin{cases} \frac{h^2}{32}\|f^{\mathrm{IV}}\|(3+8h^2)+\frac{h^2}{32}a^2\|M\|_\infty\left(1+4\frac{8}{9}h^2\right) & \textit{for } k=0,\\ \frac{h}{16}\|f^{\mathrm{IV}}\|\left(1+8\frac{2}{3}h^2\right)+\frac{h}{48}a^2\|M\|_\infty(1+18h^2) & \textit{for } k=1,\\ \left(\frac{h}{2}\right)^{4-k}\|f^{\mathrm{IV}}\|+\left(\frac{h}{2}\right)^{4-k}a^2\|M\|_\infty & \textit{for } k=2,3, \end{cases} \tag{3.22}$$

where $h=1/n$.

Proof. We shall use the obvious identities

$$f_{i-1}-2f_i+f_{i+1} = h^2f''_i+\frac{h^4}{12}f^{\mathrm{IV}}(\delta_i), \tag{3.23}$$

$$f''_{i-1}-2f''_i+f''_{i+1} = h^4f^{\mathrm{IV}}(\eta_i), \tag{3.24}$$

where $x_{i-1} < \delta_i, \eta_i < x_{i+1}$, and $i = 1, 2, \ldots, n-1$. Since the knots x_i are equidistant, the consistency relations (2.19) take the form

$$(3.25)\quad y_{i-1} - 2y_i + y_{i+1} = h\{-f''_{i-1}\sigma'_i(0) + [\sigma'_i(h) + \sigma'_{i+1}(h)]f''_i - f''_{i+1}\sigma'_{i+1}(0)\}$$
$$(i = 1, 2, \ldots, n-1).$$

Subtracting (3.23) from (3.25) and using then (3.24), we obtain

$$(3.26)\quad e_{i-1} - 2e_i + e_{i+1} = \frac{h^4}{12}[2f^{\mathrm{IV}}(\eta_i) - f^{\mathrm{IV}}(\delta_i)] - h^2[(a_i + a_{i+1})f''_i + $$
$$+ b_i f''_{i-1} + b_{i+1}f''_{i+1}] \quad (i = 1, 2, \ldots, n-1;\ e_0 = e_n = 0).$$

Let A_{n-1} denote the matrix of the system (3.26). It is known that $\|A_{n-1}^{-1}\|_\infty \leqslant n^2/8$, which together with (3.26), (3.5), and (3.6), implies

$$(3.27)\quad \|e\|_\infty \leqslant \frac{h^2}{32}\|f^{\mathrm{IV}}\| + \frac{h^2}{96}a^2\|M\|_\infty.$$

From (2.18) it follows that

$$y'_{i-1} = \frac{y_i - y_{i-1}}{h} - f''_{i-1}\sigma'_i(h) + f''_i\sigma_i(0) \quad (i = 1, 2, \ldots, n).$$

Adding to this equality the expansion (3.19) we obtain

$$e'_{i-1} = \frac{e_i - e_{i-1}}{h} + h(f''_{i-1}a_i + f''_i b_i) - c_i + \frac{h^3}{24}f^{\mathrm{IV}}(a_i) \quad (i = 1, 2, \ldots, n).$$

Similarly, for e'_n we have the formula

$$e'_n = \frac{e_n - e_{n-1}}{h} - h(f''_{n-1}b_{n-1} + f''_n a_n) + c'_n - \frac{h^3}{24}f^{\mathrm{IV}}(\theta_n),$$

where

$$c'_n = \frac{h^2}{6}\left(\frac{f''_{n-1} - f''_n}{h} + f'''_n\right), \quad x_{n-1} \leqslant \theta_n \leqslant x_n.$$

Applying the Taylor expansions to c_i $(i = 1, 2, \ldots, n)$ and c'_n, and then using (3.5) and (3.6) we have

$$(3.28)\quad \|e'\|_\infty \leqslant \frac{2}{h}\|e\|_\infty + \frac{h^3}{24}a^2\|M\|_\infty + \frac{h^3}{8}\|f^{\mathrm{IV}}\|.$$

Expanding the function $f(x)$ into the Taylor series and subtracting from

(3.18) we finally obtain

$$|\tau(x)-f(x)| \leqslant |e_{i-1}|+h|e'_{i-1}|+\frac{h^4}{24}[2|f^{\mathrm{IV}}(\gamma_i)|+|f^{\mathrm{IV}}(\eta_i)|]+$$

$$+\alpha_i^2\frac{h^2}{6}\|M\|_\infty[|\sigma'_i(\xi_i)|+|\sigma'_i(\nu_i)|] \quad (x_{i-1}<\eta_i<x_i).$$

Consequently, by (3.27), (3.28), and (3.4) we obtain the estimation (3.22) for $k=0$ and $k=1$. The proof for the cases $k=2$ and $k=3$ is similar to that of Theorem 2.4.3.1 in [7].

THEOREM 3.4. *Let $f\in C^4[0,1]$ and let $\tau\in \mathrm{Sph}(\Delta_n, \alpha)$ be a solution of the problem* (1.5), (1.8). *If $M_i=f''_i$ $(i=0,1,\ldots,n)$, $y_0=f_0$, and $y_1=f_1$, then*

(3.29)

$$\|\tau^{(k)}-f^{(k)}\| \leqslant \begin{cases} \dfrac{h^4}{24}\|f^{\mathrm{IV}}\|(n^2+15n-4)+\dfrac{h^4}{24}\alpha^2\|M\|_\infty\left(n^2+2\dfrac{13}{15}n-\dfrac{1}{5}\right) \\ \hfill \textit{for } k=0, \\ h^3\|f^{\mathrm{IV}}\|\left(\dfrac{5}{12}n+\dfrac{3}{8}\right)+h^3\alpha^2\|M\|_\infty\left(\dfrac{1}{12}n+\dfrac{1}{24}\right) \quad \textit{for } k=1, \\ \left(\dfrac{h}{2}\right)^{4-k}\|f^{\mathrm{IV}}\|+\left(\dfrac{h}{2}\right)^{4-k}\alpha^2\|M\|_\infty \quad \textit{for } k=2,3, \end{cases}$$

where

$$h=\max_{1\leqslant i\leqslant n} h_i.$$

Proof. Case $k=0$. The consistency relations (2.19) can be written in the form

$$-\frac{1}{h_i}(y_i-y_{i-1})+\frac{1}{h_{i+1}}(y_{i+1}-y_i)=\varphi_i, \tag{3.30}$$

where

$$\varphi_i=-f''_{i-1}\sigma'_i(0)+f''_i[\sigma'_i(h_i)+\sigma'_{i+1}(h_{i+1})]-f''_{i+1}\sigma'_{i+1}(0) \quad (i=1,2,\ldots,n-1).$$

Adding the first $l-1$ equations we have

$$\frac{y_l-y_{l-1}}{h_l}=\sum_{i=1}^{l-1}\varphi_i+\frac{f_1-f_0}{h_1} \quad (l=2,3,\ldots,n).$$

Subtracting (3.19) from the above equalities, after simple calculations we obtain

$$\frac{e_l-e_{l-1}}{h_l} = -\sum_{i=1}^{l-1} h_{i+1}(a_{i+1}+b_{i+1})f_i'' - \sum_{i=1}^{l-1} h_i(a_i+b_i)f_i'' + h_1b_1(f_1''-f_0'') -$$
$$-h_lb_l(f_l''-f_{l-1}'')-c_1+c_l+\frac{h_1^3}{24}f^{\mathrm{IV}}(a_1)-\frac{h_l^3}{24}f^{\mathrm{IV}}(a_l)+$$
$$+\left[\sum_{i=1}^{l-1}\frac{h_i}{2}(f_{i-1}''+f_i'')+f_0'-f_{l-1}'\right].$$

Now we apply the trapezoidal rule for the expression in the brackets. Further, making use of (3.21), (3.5), and (3.6), we obtain the following estimation for $\|e\|_\infty$:

(3.31) $$\|e\|_\infty \leqslant \frac{h^4}{24}\|f^{\mathrm{IV}}\|(n^2+5n-6)+\frac{h^4}{24}a^2\|M\|_\infty\left(n^2+\frac{13}{15}n-\frac{2}{5}\right).$$

Case $k=1$. By (2.18) we obtain

$$y_i'-y_{i-1}' = (f_{i-1}''+f_i'')[\sigma_i'(h_i)-\sigma_i'(0)].$$

Subtracting (3.20) from this equality we have

(3.32) $$e_i'-e_{i-1}' = -h_i(a_i+b_i)(f_i''+f_{i-1}'')+3c_i-\frac{h_i^3}{6}f^{\mathrm{IV}}(\beta_i)$$
$$(i=1,2,\ldots,n).$$

Taking into account (2.18) together with the initial conditions and (3.19) we get

$$e_0' = y_0'-f_0' = \frac{1}{2}f_0''h_1+\frac{1}{6}f_0'''h_1^2+\frac{1}{24}f^{\mathrm{IV}}(a_1)h_1^3.$$

Adding the first l equations of (3.32) and making use of the last equality, after simple calculations we obtain

$$e_l' = -\sum_{i=2}^{l} h_i(a_i+b_i)(f_i''+f_{i-1}'')+3\sum_{i=2}^{l}c_i-\frac{1}{6}\sum_{i=1}^{l}h_i^3f^{\mathrm{IV}}(\beta_i)+2c_1+$$
$$+\frac{h_1^3}{24}f^{\mathrm{IV}}(a_1)-h_1(f_0''b_1+f_1''a_1)\qquad (l=1,2,\ldots,n).$$

By (3.21) and (3.5), (3.6) we obtain

(3.33) $$\|e'\|_\infty \leqslant h^3\|f^{\mathrm{IV}}\|\left(\frac{5}{12}n-\frac{1}{24}\right)+h^3a^2\|M\|_\infty\left(\frac{1}{12}n-\frac{1}{24}\right).$$

Expanding the function $f(x)$ into the Taylor series and subtracting from (3.18), we obtain

$$\tau(x)-f(x) = e_{i-1}+te'_{i-1}+\frac{1}{6}t^3\left(\frac{f''_i-f''_{i-1}}{h_i}-f'''_{i-1}\right)+$$
$$+\frac{1}{6}t^3\alpha^2[f''_i\sigma'_i(\xi_i)-f''_{i-1}\sigma'_i(\nu_i)]-\frac{1}{24}t^4f^{\mathrm{IV}}(\theta_i)$$
$$(x_{i-1}<\theta_i<x_i;\ i=1,2,\ldots,n),$$

where $t = x-x_{i-1}$. Consequently, by (3.21), (3.31), and (3.33) we obtain inequality (3.29) for $k = 0$.

Similarly one can prove (3.29) for $k=1$ using (3.21) and (3.33). The proof in the case $k = 2, 3$ is quite similar to that of Theorem 2.4.3.1 in [7].

THEOREM 3.5. *Let $f\in C^4[0,1]$ and let $\tau\in \mathrm{Sph}(\Delta_n, \alpha)$ be a solution of the problem* (1.5), (1.9). *If $M_i = f''_i$ $(i = 0,1,\ldots,n)$, $y_0 = f_0$, and $m_0 = f'_0$, then*

$$\|\tau^{(k)}-f^{(k)}\| \leqslant \begin{cases} \dfrac{h^4}{24}(n^2+4n+3)\|f^{\mathrm{IV}}\|+\dfrac{h^4}{24}\left(n^2+3\cdot\dfrac{1}{15}n+2\cdot\dfrac{2}{3}\right)\alpha^2\|M\|_\infty & \text{for } k=0,\\ \dfrac{h^3}{12}(n+5)\|f^{\mathrm{IV}}\|+\dfrac{h^3}{12}(n+4)\alpha^2\|M\|_\infty & \text{for } k=1,\\ \left(\dfrac{h}{2}\right)^{4-k}\|f^{\mathrm{IV}}\|+\left(\dfrac{h}{2}\right)^{4-k}\alpha^2\|M\|_\infty & \text{for } k=2,3, \end{cases}$$

where

$$h = \max_{1\leqslant i\leqslant n} h_i.$$

Proof. Case $k = 0$. Adding the first l equations of (2.25) we obtain

$$\frac{y_l-y_{l-1}}{h_l} = f'_0+f''_0\sigma'_1(h_1)-f''_1\sigma'_1(0)+\sum_{i=1}^{l-1}\varphi_i, \tag{3.34}$$

where φ_i is such as defined in the proof of Theorem 3.4. Subtracting (3.19) from (3.34), after simple calculations we obtain

$$\frac{e_l-e_{l-1}}{h_l} = -h_1(a_1+b_1)f''_0-\sum_{i=1}^{l-1}h_{i+1}(a_{i+1}+b_{i+1})f''_i-\sum_{i=1}^{l-1}h_i(a_i+b_i)f''_i+$$
$$+h_lb_l(f''_l-f''_{l-1})+c_l-\frac{1}{24}h_l^3f^{\mathrm{IV}}(a_l)+$$
$$+\left[\sum_{i=1}^{l-1}\frac{h_i}{2}(f''_{i-1}+f''_i)+f'_0-f'_{l-1}\right].$$

As in the proof of Theorem 3.4 we use the trapezoidal rule for the expression in the brackets while for c_l we apply the expansion (3.21). From the initial conditions it follows that $e_0 = 0$. Consequently, by (3.5) and (3.6) we finally have

$$\|e\|_\infty \leqslant \frac{1}{24} h^4 (n^2+2n) \|f^{\mathrm{IV}}\| + \frac{1}{24} h^4 a^2 \|M\|_\infty \left(n^2 + \frac{14}{15} n\right).$$

Case $k = 1$. From (2.18) we have

$$y_l' = \frac{y_l - y_{l-1}}{h_l} - M_{l-1}\sigma_l'(0) + M_l \sigma_l'(h_l) \qquad (l = 1, 2, \ldots, n).$$

Subtracting f_l' from the both sides of the last equality, by (3.34) we obtain

$$e_l' = -\sum_{i=1}^{l} h_i(a_i+b_i)(f_{i-1}''+f_i'') + \left[\sum_{i=1}^{l} \frac{h_i}{2}(f_{i-1}''+f_i'') + f_0' - f_l'\right]$$
$$(l = 1, 2, \ldots, n).$$

For the expression in the brackets we use the trapezoidal rule. From the initial conditions it follows that $e_0' = 0$. Consequently, by (3.5) and (3.6) we have now

$$\|e'\|_\infty \leqslant \frac{1}{12} nh^3 \|f^{\mathrm{IV}}\| + \frac{1}{12} nh^3 a^2 \|M\|_\infty.$$

Further, the proof goes along the same lines as that of Theorem 3.4.

Acknowledgement. The author is grateful to Dr. E. Neuman for helpful discussion and careful reading of the manuscript.

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Received on 2. 12. 1981