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# *RECURRENT EVENTS IN QUEUES WITH INDEPENDENT ARRIVAL INTERVALS*

**1. Definitions.** We consider a single-channel first-come-first-served queue with a general independent input distribution and a general service time (a $GI/G/1$ service system). For $n \geqslant 1$ let $\tau_n$ be the arrival interval between the $n$th and $(n+1)$-st unit and $\gamma_n$ the service time of the $n$th unit. The $\{\tau_n\}$ is assumed to be a sequence of independent non-negative random variables with a common distribution. So is the sequence $\{\gamma_n\}$. Then the differences $\delta_n = \gamma_n - \tau_n$ are independent and identically distributed random variables with a distribution function, say, $F(x) = P\{\delta_n \leqslant x\}$. The sums of the differences $\delta_n$ will be denoted by

$$S_n = \delta_1 + \ldots + \delta_n, \quad n \geqslant 1,$$
$$S_0 \equiv 0.$$

Let $V_n$ stand for the waiting time (in the queue) of the $(n+1)$-st unit. As soon as the service of a unit is completed, the service of the next one begins, unless the waiting line is empty. If there are no waiting units, the server is idle up to the arrival of a new unit. The time interval between the moment when the $n$th unit leaves the system and the moment when the $(n+1)$-st unit enters the service will be denoted by $W_n$. The measurement of time begins with the arrival moment of the first unit. Therefore, $V_0 = W_0 \equiv 0$. Clearly $V_n \geqslant 0$ and $W_n \geqslant 0$, $V_n > 0$ implies $W_n = 0$ and $W_n > 0$ implies $V_n = 0$. Inversely, $V_n = 0$ implies $W_n \geqslant 0$ and $W_n = 0$ implies $V_n \geqslant 0$. Owing to this $V_n + W_n$ is equal either to $V_n$ or to $W_n$. It will be shown further that the random variables $V_n$ and $W_n$ are strictly determined by $\delta_n$ or $S_n$.

**2. Basic relations.** Consider two moments: $t_1$ when the $n$th unit leaves the system and $t_2$ when the $(n+1)$-st unit arrives. From the definitions mentioned previously we find

$$t_1 = \gamma_1 + W_1 + \ldots + \gamma_{n-1} + W_{n-1} + \gamma_n, \quad t_2 = \tau_1 + \ldots + \tau_n.$$

The difference $t_1 - t_2$, denoted by $\xi_n$, gives $V_n$ or $-W_n$ according as $t_2 < t_1$ or $t_1 < t_2$. If $t_1 = t_2$, both $V_n$ and $W_n$ are zeroes. Hence we have the following relations:

$$\xi_n = t_1 - t_2 = \sum_1^n (\gamma_j - \tau_j) + \sum_1^{n-1} W_j = \sum_1^n \delta_j + \sum_1^{n-1} W_j,$$

$$V_n = 0 \quad \text{and} \quad W_n = -\xi_n, \quad \text{if} \quad \xi_n \leqslant 0,$$

$$V_n = \xi_n \quad \text{and} \quad W_n = 0, \quad \text{if} \quad \xi_n > 0,$$

$$V_n - W_n = \xi_n.$$

The first and the last equations give furthermore a recurrent formula

$$\xi_{n+1} = \sum_1^{n+1} \delta_j + \sum_1^n W_j = \delta_{n+1} + W_n + \xi_n = \delta_{n+1} + V_n.$$

These relations can be written in the equivalent form

(1) $$\xi_{n+1} = \delta_{n+1} + \max(0, \xi_n), \qquad \xi_0 \equiv 0,$$

(2) $$V_n = \max(0, \xi_n),$$

(3) $$W_n = V_n - \xi_n = -\min(0, \xi_n).$$

We will now prove a lemma expressing the quantities $\xi_n$, $V_n$ and $W_n$ in terms of the sequence $\{S_j\}$. Since the elements of this sequence are sums of independent random variables $\delta_i$ having the common distribution $F(x)$, the distributions of $\xi_n$, $V_n$ and $W_n$ depend on $F(x)$ only.

LEMMA. *For $n \geqslant 1$ we have*

(4) $$\xi_n = S_n - \min(0, S_1, \ldots, S_{n-1}),$$

(5) $$V_n = S_n - \min(0, S_1, \ldots, S_n),$$

(6) $$W_n = \min(0, S_1, \ldots, S_{n-1}) - \min(0, S_1, \ldots, S_n).$$

We prove (4) by induction. For $n = 1$ the formula holds because $\xi_1 = \delta_1$. Assuming that it is true for arbitrary $n > 1$, we shall show its validity for $n+1$. Since $\xi_{n+1} = \delta_{n+1} + \max(0, \xi_n)$, it is sufficient to consider only the second term

$$\begin{aligned} \max(0, \xi_n) &= \max[0, S_n - \min(0, S_1, \ldots, S_{n-1})] \\ &= S_n + \max[-S_n, -\min(0, S_1, \ldots, S_{n-1})] \\ &= S_n - \min[S_n, \min(0, S_1, \ldots, S_{n-1})] \\ &= S_n - \min(0, S_1, \ldots, S_n). \end{aligned}$$

Thus $\xi_{n+1} = \delta_{n+1} + S_n - \min(0, S_1, \ldots, S_n) = S_{n+1} - \min(0, S_1, \ldots, S_n)$ as asserted.

Formulas (5) and (6) follow immediately from (2), (3) and (4).

**3. Recurrent events.** We shall now describe our service system in terms of the theory of recurrent events. For the necessary definitions and theorems see Feller [1].

DEFINITION. We say that an event $E$ *occurs at the $n$-th place in the sequence* $S_1, S_2, \ldots, S_n, \ldots$ if one of the equivalent events (7) or (8) is observed:

$$\xi_n \leqslant 0 \equiv S_n \leqslant \min(0, S_1, \ldots, S_{n+1})\text{(1)}, \tag{7}$$

$$V_n = 0 \equiv S_n = \min(0, S_1, \ldots, S_n). \tag{8}$$

In order to examine whether $E$ is a recurrent event, one needs to verify two points:

(a) $E$ occurs at the $m$th and $(m+n)$-th place in the sequence $S_1, \ldots, S_m, \ldots, S_{m+n}$ if and only if it occurs at the last place in each of the subsequences $S_1, \ldots, S_m$ and $S_{m+1}, \ldots, S_{m+n}$.

(b) Whenever this is the case, these two events are independent.

THEOREM. *$E$ is an recurrent event.*

Proof. Only the second point requires testing. Assuming that $E$ has occured at the $m$th and $(m+n)$-th place in the sequence $\{S_j\}$, we have

$$\begin{aligned}
&P\{V_m = 0, V_{m+n} = 0\} \\
&\quad = P\{S_m = \min(0, \ldots, S_m), S_{m+n} = \min(0, \ldots, S_{m+n})\} \\
&\quad = P\{S_m = \min(0, \ldots, S_m), S_{m+n} = \min(S_m, \ldots, S_{m+n})\} \\
&\quad = P\{S_m = \min(0, \ldots, S_m), S_{m+n}-S_m = \min(S_m-S_m, \ldots, S_{m+n}-S_m)\}.
\end{aligned}$$

For fixed $m$ let us write $S_{m+j}-S_m = S_j^*$, $S_j^*-\min(0, \ldots, S_j^*) = V_j^*$, $j = 0, \ldots, n$.

Then we get

$$P\{S_m = \min(0, \ldots, S_m), S_n^* = \min(0, \ldots, S_n^*)\} = P\{V_m = 0\}P\{V_n^* = 0\}$$

since the random variables $S_i = \delta_1+\ldots+\delta_i$ $(i \leqslant m)$ and $S_j^* = \delta_{m+1}+ +\ldots+\delta_{m+j}$ are independent.

With a recurrent event there are associated two probabilities: the probability $u_n$ that $E$ occurs at the $n$th place and the probability $a_n$ that it occurs for the first time at the $n$th place. For $n \geqslant 1$ we have

$$u_n = P\{\xi_n \leqslant 0\} = P\{V_n = 0\}, \tag{9}$$

$$\begin{aligned}
a_n &= P\{\xi_1 > 0, \ldots, \xi_{n-1} > 0, \xi_n \leqslant 0\} \\
&= P\{V_1 > 0, \ldots, V_{n-1} > 0, V_n = 0\}.
\end{aligned} \tag{10}$$

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(1) In this case the index $n$ is called a *ladder point*. On ladder points in Bernoulli trials, see [1], p. 280.

It will be convenient to define $u_0 = 1$ and $a_0 = 0$. Let $U(s)$ and $A(s)$ denote the generating functions of the sequences $\{u_n\}$ and $\{a_n\}$:

$$U(s) = \sum_0^\infty u_n s^n, \quad A(s) = \sum_1^\infty a_n s^n.$$

It has been shown ([1]) that

(11) $$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \ldots + a_{n-1} u_1 + a_n u_0, \quad n \geqslant 1,$$

(12) $$U(s) = \frac{1}{1-A(s)}.$$

Note that $\{u_n\}$ is not a probability distribution. The sequence $\{a_n\}$, however, is one, and therefore

$$a = A(1) = \sum_1^\infty a_n \leqslant 1.$$

An integral-valued random variable $T$ with the distribution

$$P\{T = n\} = a_n$$

is called the *waiting time for $E$* or the *recurrence time of $E$*. More exactly, $T$ is the waiting time between successive occurrences of $E$. If $a < 1$ then $1-a$ should be interpreted as the probability that $E$ will never occur. In this case one may conventionally write $P\{T = \infty\} = 1-a$.

Following Feller, a recurrent event is called *persistent* if $a = 1$ and *transient* if $a < 1$.

Further, note two important theorems ([1]):

1) *$E$ is transient if and only if the series $u = \sum_0^\infty u_n$ converges. The probability that $E$ will ever occur is $1-u^{-1}$.*

2) *If $E$ is persistent and not periodic, then $\lim_{n\to\infty} u_n = \mu^{-1}$ where $\mu = A'(1)$ is the mean of the recurrence time $T$. If $u_n \to 0$, $\mu$ is infinite.*

**4. Distributions of recurrence time and waiting times.** If $F(x)$ is the distribution function of the differences $\delta_k$, then the joint distribution of the $n$-dimensional random variable $(S_1, \ldots, S_n)$ is given by

$$P\{S_1 \leqslant x_1, \ldots, S_n \leqslant x_n\}$$
$$= \int_{-\infty}^{x_1} d_{t_1} F(t_1) \int_{-\infty}^{x_2} d_{t_2} F(t_2 - t_1) \ldots \int_{-\infty}^{x_{n-1}} d_{t_{n-1}} F(t_{n-1} - t_{n-2}) \int_{-\infty}^{x_n} d_{t_n} F(t_n - t_{n-1}).$$

Moreover, if $\delta_k$ is a continuous random variable with a density $f(x) = F'(x)$ then the density function of $(S_1, \ldots, S_n)$ is $f(x_1) f(x_2 - x_1) \ldots f(x_n - x_{n-1})$.

Consider first the probabilities $u_n$ and $a_n$. From (9) we have

$$\begin{aligned} u_n &= P\{V_n = 0\} = P\{S_n = \min(0, S_1, \dots, S_n)\} \\ &= P\{S_n \leqslant 0, S_n \leqslant S_1, \dots, S_n \leqslant S_{n-1}\} \\ &= P\{S_n \leqslant 0, S_n - S_1 \leqslant 0, \dots, S_n - S_{n-1} \leqslant 0\}. \end{aligned}$$

For fixed $n$ we now write $S'_j = S_n - S_{n-j} = \delta_{n-j+1} + \dots + \delta_n$, $j = 0, 1, \dots, n$. We have

$$\begin{aligned} u_n &= P\{S'_1 \leqslant 0, S'_2 \leqslant 0, \dots, S'_{n-1} \leqslant 0, S'_n \leqslant 0\} \\ &= \int_{-\infty}^{0} d_{t_1}F(t_1) \int_{-\infty}^{0} d_{t_2}F(t_2 - t_1) \dots \int_{-\infty}^{0} d_{t_n}F(t_n - t_{n-1}) \\ &= \int_{-\infty}^{0} f(x_1)\,dx_1 \int_{-\infty}^{0} f(x_2 - x_1)\,dx_2 \dots \int_{-\infty}^{0} f(x_n - x_{n-1})\,dx_n. \end{aligned} \tag{13}$$

From (10) we have

$$\begin{aligned} a_n &= P\{\xi_1 > 0, \dots, \xi_{n-1} > 0, \xi_n \leqslant 0\} \\ &= P\{S_1 > 0, \dots, S_{n-1} > 0, S_n \leqslant 0\} \\ &= \int_0^{\infty} d_{t_1}F(t_1) \dots \int_0^{\infty} d_{t_{n-1}}F(t_{n-1} - t_{n-2}) \int_{-\infty}^{0} d_{t_n}F(t_n - t_{n-1}) \\ &= \int_0^{\infty} f(x_1)\,dx_1 \dots \int_0^{\infty} f(x_{n-1} - x_{n-2})\,dx_{n-1} \int_{-\infty}^{0} f(x_n - x_{n-1})\,dx_n. \end{aligned} \tag{14}$$

Note that

$$P\{S_1 > 0, \dots, S_n > 0\} = 1 - a_1 - \dots - a_n. \tag{15}$$

We shall now find the distribution functions of the random variables $\xi_n$, $V_n$ and $W_n$. For $\xi_n$ we get

$$\begin{aligned} P\{\xi_n \leqslant x\} &= \sum_{j=0}^{n-1} P\{\xi_n \leqslant x, S_j = \min(0, S_1, \dots, S_{n-1})\} \\ &= \sum_{j=0}^{n-1} P\{S_n - S_j \leqslant x, S_j = \min(0, \dots, S_j), S_j = \min(S_j, \dots, S_{n-1})\} \\ &= \sum_{j=0}^{n-1} [P\{S_j = \min(0, \dots, S_j)\} \times \\ &\qquad \times P\{0 = \min(0, S_{j+1} - S_j, \dots, S_{n-1} - S_j), S_n - S_j \leqslant x\}] \\ &= \sum_{j=0}^{n-1} P\{V_j = 0\} P\{S'_1 > 0, \dots, S'_{n-j-1} > 0, S'_{n-j} \leqslant x\}. \end{aligned}$$

For brevity we put

$$P\{S_1' > 0, \ldots, S_{k-1}' > 0, S_k' \leqslant x\} = a_k(x).$$

Thus

$$(16) \qquad P\{\xi_n \leqslant x\} = \sum_{j=0}^{n-1} u_j a_{n-j}(x).$$

Hence in a particular case we obtain formula (11)

$$P\{\xi_n \leqslant 0\} = \sum_{j=0}^{n-1} u_j a_{n-j}(0) = \sum_{j=0}^{n-1} u_j a_{n-j} = u_n.$$

Using (2), (3) and (16) we get the distribution functions of $V_n$ and $W_n$

$$(17) \qquad P\{V_n \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ P\{\xi_n \leqslant x\} & \text{if} \quad x \geqslant 0; \end{cases}$$

$$(18) \qquad P\{W_n \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ P\{\xi_n \geqslant -x\} & \text{if} \quad x \geqslant 0. \end{cases}$$

**5. Busy period.** The service channel is busy up to the first occurence of $E$. The busy period is therefore the sum of a random number of service times $\gamma_k$. Namely it is

$$\gamma = \gamma_1 + \ldots + \gamma_T,$$

where $\gamma$ indicates the busy period and $T$ is the recurrence time with the distribution $\{a_n\}$. To obtain the busy period distribution we write

$$P\{\gamma \leqslant x\} = \sum_1^\infty P\{\gamma_1 + \ldots + \gamma_T \leqslant x/T = n\}P\{T = n\}$$
$$= \sum_1^\infty a_n P\{\gamma_1 + \ldots + \gamma_n \leqslant x\}.$$

Denote by $B_n(x)$ the distribution function of the sum $\gamma_1 + \ldots + \gamma_n$ and by $B(x)$ the distribution function of the busy period $\gamma$. Now

$$(19) \qquad B(x) = P\{\gamma \leqslant x\} = \sum_1^\infty a_n B_n(x).$$

Let $b$ stand for the mean busy period and $\beta$ for the mean service time. If $b$ and $\beta$ are finite then

$$b = \int_0^\infty x\,dB(x) = \int_0^\infty x \sum_1^\infty a_n dB_n(x) = \sum_1^\infty a_n \int_0^\infty x\,dB_n(x)$$
$$= \sum_1^\infty a_n n\beta = \beta \sum_1^\infty n a_n.$$

Since this series express the mean recurrence time, we have proved that the mean busy period is equal to the mean service time multiplied by the mean recurrence time:

$$b = \beta\mu. \tag{20}$$

**6. The special case of symmetrically distributed differences $\delta_k$.** We shall prove the following

THEOREM. *If the differences $\delta_k$ have a symmetric distribution function, i. e. $F(-x)+F(x) \equiv 1$ at all points of continuity of $F$, then $E$ is persistent and its mean recurrence time is infinite.*

Proof. Remark that in this case the relation

$$P\{S_1 \leqslant 0, \ldots, S_n \leqslant 0\} = P\{S_1 > 0, \ldots, S_n > 0\}$$

is true. Using (13) and (15) we may write

$$u_n = 1-a_1-\ldots-a_n, \qquad u_{n-1}-u_n = a_n, \qquad n \geqslant 1.$$

Multiplying the last equality by $s^n$ and summarizing it, we get after some calculations

$$(1-s)U(s) = 1-A(s).$$

Together with (12) we have two equations from which we may obtain $U(s)$ and $A(s)$. Considering the conditions $U(0) = u_0 = 1$ and $A(0) = a_0 = 0$, we get

$$U(s) = \frac{1}{\sqrt{1-s}}, \qquad A(s) = 1-\sqrt{1-s}.$$

Thus $a = A(1) = 1$ and $\mu = A'(1) = \infty$. The proof is completed.

The expansion of the generating functions into power series gives us the probabilities

$$u_n = \frac{(2n-1)!!}{2n!!}, \qquad u_0 = 1,$$

$$a_n = \frac{(2n-3)!!}{2n!!}, \qquad a_1 = \frac{1}{2}.$$

**7. Examples.**

(a) Arrival intervals and service times are exponentially distibuted with means $a$ and $\beta$ respectively.

$$P\{\tau_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1-e^{-x/a} & \text{if} \quad x \geqslant 0; \end{cases}$$

$$P\{\gamma_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1-e^{-x/\beta} & \text{if} \quad x \geqslant 0. \end{cases}$$

Then the distribution of the differences $\delta_k$ is given by

$$F(x) = \begin{cases} pe^{x/a} & \text{if} \quad x < 0, \\ 1-qe^{-x/\beta} & \text{if} \quad x \geqslant 0, \end{cases}$$

where $p = F(0) = a/(a+\beta)$ and $q = 1-p = \beta/(a+\beta)$. Using (13) and (14) we get after some calculation the generating functions

$$A(s) = \frac{1-\sqrt{1-4pqs}}{2q}, \qquad U(s) = \frac{2q}{2q-1+\sqrt{1-4pqs}}.$$

Thus

$$a = A(1) = \frac{1-|p-q|}{2q} = \begin{cases} 1 & \text{if} \quad p \geqslant 1/2, \\ p/q & \text{if} \quad p < 1/2. \end{cases}$$

We see that $E$ is persistent if $p \geqslant 1/2$ and transient if $p < 1/2$. In the second case the probability that $E$ will ever occur equals $p/q$. For the first case we obtain the mean recurrence time

$$\mu = A'(1) = \begin{cases} \dfrac{p}{p-q} & \text{if} \quad p > 1/2, \\ \infty & \text{if} \quad p = 1/2. \end{cases}$$

For $p > 1/2$ we can also write $\mu = a/(a-\beta)$.

From (20) we get the mean busy period $b = a\beta/(a-\beta)$.

(b) Exponentially distributed arrival intervals and constant service time.

$$P\{\tau_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1-e^{-x/a} & \text{if} \quad x \geqslant 0; \end{cases}$$

$$P\{\gamma_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < \beta, \\ 1 & \text{if} \quad x \geqslant \beta. \end{cases}$$

The distribution of the differences is given by

$$F(x) = \begin{cases} pe^{x/a} & \text{if} \quad x < \beta, \\ 1 & \text{if} \quad x \geqslant \beta, \end{cases}$$

where $p = F(0) = e^{-\beta/a}$. It is not easy to obtain here explicit expressions for the generating funcions. It may, however, be shown that

$$a_n = p(-p\log p)^{n-1}\frac{n^{n-1}}{n!}$$

and that $E$ is persistent if $p \geqslant e^{-1}$ and transient if $p < e^{-1}$. If $p = e^{-1}$, its mean recurrence time is infinite.

(c) Constant arrival intervals and exponential service time.

$$P\{\tau_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < a, \\ 1 & \text{if} \quad x \geqslant a; \end{cases}$$

$$P\{\gamma_k \leqslant x\} = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - e^{-x/\beta} & \text{if} \quad x \geqslant 0. \end{cases}$$

The distribution of the differences $\delta_k$ is given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x < -a, \\ 1 - qe^{-x/\beta} & \text{if} \quad x \geqslant -a, \end{cases}$$

where $q = e^{-a/\beta}$, $p = F(0) = 1-q$. It may be shown that

$$u_{n-1} - u_n = q(-q\log q)^{n-1}\frac{n^{n-1}}{n!}.$$

Now $E$ is persistent if $p \geqslant 1-e^{-1}$ and transient if $p < 1-e^{-1}$.

In these examples $p$ always indicates the probability

$$F(0) = P\{\delta_k \leqslant 0\} = P\{\gamma_k \leqslant \tau_k\}$$

that the service time will not exceed the arrival interval. The difference between those three cases are significant.

## Reference

[1] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1, 2nd ed., New York 1957.

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### *ZDARZENIA REKURENCYJNE W KOLEJKACH O NIEZALEŻNYCH ODSTĘPACH MIĘDZY WEJŚCIAMI*

STRESZCZENIE

Tematem niniejszej pracy jest próba zastosowania teorii zdarzeń rekurencyjnych w systemie obsługi masowej typu $GI/G/1$. W systemie takim czasy czekania kolejnych jednostek tworzą ciąg nieujemnych zmiennych losowych $\{V_n\}$. W ciągu tym zdarzenie $V_k = 0$ jest zdarzeniem rekurencyjnym. Rozkład czasu powrotu takiego zdarzenia w ciągu $\{V_n\}$ jest jednoznacznie określony rozkładem różnicy $\gamma-\tau$ czasu obsługi ($\gamma$) i odstępu między zgłoszeniami kolejnych jednostek ($\tau$). Wyznaczono rozkłady czasu powrotu dla czterech szczególnych przypadków.

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## *РЕКУРЕНТНЫЕ СОБЫТИЯ В СИСТЕМАХ МАССОВОГО ОБСЛУЖИВАНИЯ С НЕЗАВИСИМОСТЬЮ РАССТОЯНИЙ ЗАЯВОК*

**РЕЗЮМЕ**

Автор применяет теорию рекурентных событий к системам массового обслуживания типа $GI/G/1$. В такой системе отрезки времён ожидания последовательных единиц образуют последовательность неотрицательных случайных переменных $\{V_n\}$. В этой последовательности событие $V_k = 0$ является рекурентным событием. Распределение времени возврата таких событий в последовательности $\{V_n\}$ однозначно определяется распределением разности $\gamma - \tau$ времени обслуживания ($\gamma$) и промежутка времени между последовательными заявками ($\tau$). В работе вычислены распределения времени возврата для четырех частных случаев системы массового обслуживания.