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MINIMAX ESTIMATION OF A CLASS OF FUNCTIONS OF THE SCALE PARAMETER IN THE GAMMA AND OTHER DISTRIBUTIONS IN THE CASE OF TRUNCATED PARAMETER SPACE

Abstract. We consider the problem of minimax estimation of the scale parameter λ in the gamma distribution (1) with truncated parameter space. We prove some sufficient conditions for minimaxity in the classes of rational, analytical and other functions and give some examples of minimax estimators. The results of the paper can be applied to the estimation of the scale parameter for the normal, lognormal, Pareto, generalized gamma, generalized Laplace and other distributions.

1. INTRODUCTION

This paper deals with the problem of minimax estimation of the scale parameter λ in the gamma distribution

$$f(x, s, \lambda) = \frac{\lambda^s}{\Gamma(s)} x^{s-1} \exp(-\lambda x), \qquad x > 0, \tag{1}$$

where $\lambda \in \Lambda$, $\Lambda = (0, \lambda_0)$ or $\Lambda = (\lambda_0, \infty)$, $\lambda_0 \geqslant 0$, $s > 0$, λ_0 and s are given constants; the paper is a continuation of [6]. There are two methods of investigations of minimax estimators. The first of them uses the simple fact that if the estimator δ is admissible under the loss $L(\cdot, \cdot)$, then the same estimator is minimax under the new loss

$$\tilde{L}(\cdot, \cdot) = [\mathrm{E}_\lambda L(\lambda, \delta)]^{-1} L(\cdot, \cdot)$$

(see [16], Theorem 8.1.1). Thus from the results of [3], [7], [12], [15] and other authors one can obtain the admissible and minimax estimators for $g(\lambda) = a\lambda + b/c\lambda + d$ under the restriction $\lambda \geqslant \lambda_0$ or $\lambda \in (0, \lambda_0)$, where λ_0, a, b, c, d are given constants and $\lambda_0 \geqslant 0$. In the same way the minimax estimator for λ^r was found by Singh [13], where r is an integer and $\lambda \in (0, \infty)$. A different approach was presented in [4], [6] and [17]. Using the well-known theorem of Lehmann (see [9], p. 256), Zubrzycki [17] and Ghosh and Singh [4] obtained minimax estimators for λ^{-1} and λ. Their results were generalized in the paper [6] which provides a necessary and sufficient condition for the minimax estimation of λ^r, where r is any real number, $r < s/2$, and which gives a few examples of minimax estimators. The authors mentioned above did not consider the case of two-sided restrictions $\lambda \in (\lambda_0, \lambda_1)$, where $0 < \lambda_0 < \lambda_1 < \infty$. Such a problem requires different methods (see [1], [2], [8] – estimation of the mean in the normal distribution) and will be the subject of further investigations.

In this paper we give a few sufficient conditions for minimaxity in the case of restrictions imposed on the parameter. The Theorem in Section 2 provides a sufficient condition for minimax estimation of any measurable function $g(\cdot)$ of the scale parameter λ in the gamma distribution. In Section 3 there are given examples of minimax estimators in the gamma, normal, generalized gamma, generalized Laplace, lognormal, particular cases of the beta and other distributions. Section 4 contains some sufficient conditions for the weight function under which every estimator is minimax or, more precisely, every estimator has the unbounded maximal risk.

2. SUFFICIENT CONDITIONS FOR MINIMAXITY

What we need first are some preliminary lemmas.

LEMMA 1. *Let* $f(y) = \Gamma^2(y-r)/\Gamma(y-2r)\Gamma(y)$, *where* $y > \max(0, 2r)$ *and* $r \in \boldsymbol{R}^1$. *Then* $f(\cdot)$ *is strictly increasing.*

Proof. Since $f'(y) = f(y)[\ln f(y)]'$, it suffices to show that

$$[\ln f(y)]' > 0 \quad \text{for all } y > \max(0, 2r).$$

This is equivalent to

$$2\psi(y-r) - \psi(y) - \psi(y-2r) > 0, \qquad \text{where } \psi(y) = [\ln \Gamma(y)]'.$$

Note that from (8.363.3) of [5] we obtain

$$\psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left(\frac{1}{y+k} - \frac{1}{x+k} \right) = (x-y) \sum_{k=0}^{\infty} \frac{1}{(x+k)(y+k)}$$

for all $x > 0$ and $y > 0$. Consequently,

$$2\psi(y-r) - \psi(y-2r) - \psi(y) = \sum_{k=0}^{\infty} \frac{1}{(y+k)(y+k-r)(y+k-2r)} > 0.$$

This completes the proof.

Let us write

$$F(y, a) = \frac{\{\sum_{i=n}^{N} \sum_{k=m}^{M} b_i c_k [a(1+y)]^{N+M-i-k} \Gamma(u+i+k, a(1+y))\}^2}{\sum_{k=m}^{M} \sum_{i=m}^{M} c_i c_k [a(1+y)]^{2M-i-k} \Gamma(u+i+k, a(1+y))}, \tag{2}$$

where $y > -1$, $a > 0$, $c_M \neq 0$, $u > \max(2M, N+M)$, $m < M$, $n < N$, and c_i ($i = m, \ldots, M$) are chosen such that the denominator takes a positive value. Here

$$\Gamma(x, y) = \int_y^\infty w^{x-1} \exp(-w)\, dw.$$

LEMMA 2. *There exists a constant $C > 0$ such that $F(y, a) \leqslant C$ for each $y > -1$.*

Proof. Observe that $F(\cdot, a)$ is continuous on $[-1, \infty[$ and for all $y_1 > -1$ and $a_1 > 0$ there exist $y_2 > -1$ and $a_2 > 0$ such that

$$F(y_1, a_1) = F(y_2, a_2) \quad \text{and} \quad a_1(1+y_1) = a_2(1+y_2).$$

Then it is sufficient to show that

$$\lim_{y \to \infty} F(y, 1) < \infty.$$

We prove that

$$\lim_{y \to \infty} F(y, 1) = 0.$$

If we apply de L'Hôspital's rule $2(M-m)+1$ times, then the denominator of (2) takes the form

$$\{[1+y]^{2M+u-1} + o[(1+y)^{2M+u-1}]\} \exp[-(1+y)],$$

whereas the numerator is a sum of expressions of the form

$$G(1+y)^A \Gamma(u+B, 1+y)\, \Gamma(u+D, 1+y),$$

$$G(1+y)^A \Gamma(u+B, 1+y) \exp[-(1+y)]$$

and

$$G(1+y)^A \exp[-2(1+y)],$$

where A, B, G, D are some reals, $u+B > 0$, $u+D > 0$. Let us multiple the numerator and the denominator by $(1+y)^{1-u-2M} \exp(1+y)$. Now observe that if y tends to infinity, then the denominator tends to one, whereas the numerator, as a sum of expressions of the form

$$G(1+y)^A \Gamma(u+B, 1+y)\, \Gamma(u+D, 1+y) \exp(1+y),$$

$$G(1+y)^A \Gamma(u+B, 1+y) \quad \text{and} \quad G(1+y)^A \exp[-(1+y)],$$

converges to zero. Hence

$$\lim_{y \to \infty} F(y, 1) = 0$$

and the proof is completed.

Let us write

$$x\Lambda = \{y \in \mathbf{R}^1 \mid y = \lambda x,\ \lambda \in \Lambda\},$$

$\varphi = 0$ for $\lambda \in (0, \lambda_0)$, $\varphi = \infty$ for $\lambda \in (\lambda_0, \infty)$ and $\varphi = 0$ or ∞ for $\lambda \in (0, \infty)$. The following conditions are required:

CONDITION 1. *There exists a function* $h\colon \Lambda \to]0, \infty[$ *such that for all* $a > 0$ *we have*

$$\text{(i)} \qquad 0 < \int_0^\infty g^2(\lambda)\, h(\lambda)\, \lambda^{p-1} \exp(-a\lambda)\, d\lambda < \infty,$$

$$\text{(ii)} \qquad 0 < \int_0^\infty |g(\lambda)|^i\, h(\lambda)\, \lambda^{s+p-1} \exp(-a\lambda)\, d\lambda < \infty \quad \textit{for } i = 0, 1, 2,$$

where $p \in P$, $s \in S$, $P \subset \mathbf{R}^1$, $S \subset]0, \infty[$, *and* P, S *are some arbitrarily chosen sets.*

CONDITION 2. *There exist functions* $B_i\colon]0, \infty[\to \mathbf{R}^1$ $(i = 0, 1, 2)$ *such that*

$$\text{(i)} \qquad B_2(a) = B_1^2(a)/B_0(a) \quad \textit{for all } a > 0,$$

$$\text{(ii)} \qquad \lim_{a \to \varphi} B_k(a)\, g^k(u/a)\, h(u/a) = A_k(u) \quad \textit{for } k = 0, 1, 2,$$

where A_k $(k = 0, 1, 2)$ *may be an arbitrary function,* $A_0 \not\equiv 0$ *and neither of* A_1 *and* A_2 *is equal to a constant.*

CONDITION 3. *We have*

$$\lim_{a \to \varphi} B_k(a) \int_{a\Lambda} \left| g\left(\frac{u}{a}\right)\right|^k h\left(\frac{u}{a}\right) u^q \exp(-u)\, du = \int_0^\infty A_k(u)\, u^q \exp(-u)\, du < \infty,$$

where $q = p-1$ *for* $k = 2$ *and* $q = p+s-1$ *for* $k = 0, 1, 2$.

Now we write

$$F_1(y, a, p) = \int_{\Lambda a(1+y)} g^2\left(\frac{w}{a(1+y)}\right) h\left(\frac{w}{a(1+y)}\right) w^{s+p-1} \exp(-w)\, dw,$$

$$F_2(y, a, p) = \frac{\left\{\int_{\Lambda a(1+y)} g\left(\frac{w}{a(1+y)}\right) h\left(\frac{w}{a(1+y)}\right) w^{s+p-1} \exp(-w)\, dw\right\}^2}{\int_{\Lambda a(1+y)} h\left(\frac{w}{a(1+y)}\right) w^{s+p-1} \exp(-w)\, dw},$$

and $F(y, a, p) = F_1(y, a, p) - F_2(y, a, p)$.

CONDITION 4. *We have*

$$\lim_{a \to \varphi} \int_0^\infty \frac{y^{s-1} B_2(a)}{(1+y)^{s+p}} F(y, a, p)\, dy = \int_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} \lim_{a \to \varphi} B_2(a) F(y, a, p)\, dy.$$

Let X have the gamma distribution (1). We want to estimate the function $u = g(\cdot)$ of the scale parameter λ under the loss $L(d, g) = g^{-2}(g-d)^2$.

THEOREM. *Suppose that Conditions* 1–4 *hold. If the risk of the estimator* u^* *fulfils the condition*

$$(3)\qquad \sup_{\lambda\in\Lambda} R(u^*, g(\lambda)) \leqslant \sup_{p\in P} \frac{\int\limits_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} \lim\limits_{a\to\varphi} B_2(a) F(y, a, p)\, dy}{\Gamma(s) \int\limits_0^\infty A_2(u) u^{p-1} \exp(-u)\, du},$$

then u^* *is minimax.*

Proof. According to a well-known result on the minimax estimation (see [9], p. 256) it suffices to find a sequence of prior distributions such that Bayes risks of the corresponding Bayes estimators converge to the minimax risk of u^*. Let λ be a random variable with the density

$$\xi(\lambda) = g^2(\lambda) h(\lambda) \lambda^{p-1} \exp(-a\lambda)\Big(\int\limits_\Lambda g^2(\lambda) h(\lambda) \lambda^{p-1} \exp(-a\lambda)\, d\lambda\Big)^{-1},$$

where $\lambda\in\Lambda$. The posterior density of λ is of the form

$$\xi(\lambda|x) = g^2(\lambda) h(\lambda) \lambda^{s+p-1} \exp[-(a+x)\lambda] \times\Big(\int\limits_\Lambda g^2(\lambda) h(\lambda) \lambda^{s+p-1} \exp[-(a+x)\lambda]\, d\lambda\Big)^{-1}.$$

The Bayes estimator for $g(\cdot)$ under the loss mentioned above is given by

$$\tilde{u}(X) = \mathrm{E}_\lambda(g^{-1}|X)/\mathrm{E}_\lambda(g^{-2}|X).$$

Let us calculate the prior risk of the estimator $\tilde{u}$:

$$(4)\qquad R(\xi, \tilde{u}) = \int\limits_\Lambda g^{-2}(\lambda) \int\limits_0^\infty [g(\lambda) - \tilde{u}(x)]^2 f(x, s, \lambda) \xi(\lambda)\, dx\, d\lambda$$

$$= \int\limits_0^\infty [I_0(x) - 2I_{-1}(x)\tilde{u}(x) + I_{-2}(x)\tilde{u}^2(x)]\, dx.$$

Here

$$(5)\qquad I_k(x) = \int\limits_\Lambda g^k(\lambda) f(x, s, \lambda) \xi(\lambda)\, d\lambda.$$

We note that $\tilde{u}(x) = I_{-1}(x)/I_{-2}(x)$. Hence by (4) the risk can be written a

$$(6)\qquad R(\xi, \tilde{u}) = \int\limits_0^\infty [I_0(x) - I_{-1}^2(x)/I_{-2}(x)]\, dx.$$

Applying the formulas (5) and (6) we have

$$(7) \qquad R(\xi, \tilde{u}) = \int_0^\infty x^{s-1} \left\{ \int_\Lambda g^2(\lambda)\, h(\lambda)\, \lambda^{s+p-1} \exp[-(a+x)\lambda]\, d\lambda - \frac{\left(\int_\Lambda g(\lambda)\, h(\lambda)\, \lambda^{s+p-1} \exp[-(a+x)\lambda]\, d\lambda\right)^2}{\int_\Lambda h(\lambda)\, \lambda^{s+p-1} \exp[-(a+x)\lambda]\, d\lambda} \right\} dx \times \left[\Gamma(s) \int_\Lambda g^2(\lambda)\, h(\lambda)\, \lambda^{s-1} \exp(-a\lambda)\, d\lambda\right]^{-1}.$$

Using the substitution $u = a\lambda$ in the integral

$$\int_\Lambda g^2(\lambda)\, h(\lambda)\, \lambda^{s-1} \exp(-a\lambda)\, d\lambda$$

and $y = x/a$ in the other integrals of (7) we get

$$R(\xi, \tilde{u}) = \frac{a^{p+s}}{\Gamma(s) \int_{\Lambda a} g^2(u/a)\, h(u/a)\, u^{p-1} \exp(-u)\, du} \times \int_0^\infty y^{s-1} \left\{ \int_\Lambda g^2(\lambda)\, h(\lambda)\, \lambda^{s+p-1} \exp[-a(1+y)\lambda]\, d\lambda - \frac{\left(\int_\Lambda g(\lambda)\, h(\lambda)\, \lambda^{s+p-1} \exp[-a(1+y)\lambda]\, d\lambda\right)^2}{\int_\Lambda h(\lambda)\, \lambda^{s+p-1} \exp[-a(1+y)\lambda]\, d\lambda} \right\} dy.$$

This equality, after the substitution $w = a(1+y)\lambda$, is equivalent to

$$R(\xi, \tilde{u}) = \int_0^\infty \frac{y^{s-1} B_2(a)}{(1+y)^{s+p}} F(y, a, p)\, dy \times \left[\Gamma(s) \int_{\Lambda a} B_2(a)\, g^2\left(\frac{u}{a}\right) h\left(\frac{u}{a}\right) u^{p-1} \exp(-u)\, du\right]^{-1}.$$

Now let us go to φ with a and calculate

$$\sup_{p \in P} \lim_{a \to \varphi} R(\xi, \tilde{u}).$$

According to the Lehmann Theorem, if the risk of the estimator u^* satisfies the condition

$$\sup_{\lambda \in \Lambda} R(u^*, g(\lambda)) \leqslant \sup_{p \in P} \lim_{a \to \varphi} R(\xi, \tilde{u}),$$

then u^* is minimax. The proof is now completed.

Now we give explicit formulas for the right-hand side of (3) for the class of functions $g(\cdot)$. We present the detailed proof only for Corollary 1 since the remaining proofs are quite similar.

2.1. Rational function. Let

$$g(\lambda) = \sum_{i=n}^{N} b_i \lambda^i \Big(\sum_{i=m}^{M} c_i \lambda^i\Big)^{-1},$$

where b_i and c_i are given constants, $b_n c_m \neq 0$ if $0 < \lambda < \lambda_0$, $b_N c_M \neq 0$ if $\lambda_0 < \lambda < \infty$, and n, m, N, M are integers, $n \leqslant N$, $m \leqslant M$.

COROLLARY 1. *If the risk of the estimator δ satisfies the conditon*

$$\sup_{\lambda \in \Lambda} R\big(g(\lambda), \delta\big) \leqslant \begin{cases} 1 - \dfrac{\Gamma^2(s+N+M-2n)}{\Gamma\big(s+2(N-n)\big)\Gamma\big(s+2(M-n)\big)} & \text{for } \lambda \in (\lambda_0, \infty), \\ 1 - \dfrac{\Gamma^2(s+m-n)}{\Gamma(s)\Gamma\big(s+2(m-n)\big)} & \text{for } \lambda \in (0, \lambda_0), \end{cases}$$

then δ is minimax.

Proof. To prove this corollary it is enough to show that Conditions 1–4 hold. In the case $\Lambda = (\lambda_0, \infty)$ we put

$$h(\lambda) = (c_m \lambda^m + \ldots + c_M \lambda^M)^2,$$

$$B_0(a) = a^{2M}, \quad B_1(a) = a^{N+M}, \quad B_2(a) = a^{2N}.$$

Under the restrictions $p > -2n$ and $s+p > -\min(2m, n+m)$ Conditions 1 and 2 hold immediately and we get

$$A_0(u) = c_M^2 u^{2M}, \quad A_1(u) = b_N c_M u^{N+M}, \quad A_2(u) = b_N^2 u^{2N}.$$

Note that for all a, $0 < a < a_0$, where a_0 is a given constant, we have

$$B_2(a) g^2\left(\frac{u}{a}\right) h\left(\frac{u}{a}\right) = \Big(\sum_{k=n}^{N} b_k u^k a^{N-k}\Big)^2 \leqslant \Big(\sum_{k=n}^{N} |b_k| u^k a_0^{N-k}\Big)^2$$

and $p+2k > 0$ for $k = n, n+1, \ldots, N$. Thus, by the Lebesgue dominated convergence theorem, Condition 3 also holds. Now observe that, after applying the formula

$$(x_1 + \ldots + x_n)^2 \leqslant n(x_1^2 + \ldots + x_n^2),$$

we obtain

$$\begin{aligned} B_2(a) F_1(y, a, p) &= a^{2N} \int_{a\lambda_0(1+y)}^{\infty} \left\{ \sum_{k=n}^{N} b_k \left[\frac{w}{a(1+y)}\right]^k \right\}^2 w^{s+p-1} \exp(-w)\, dw \\ &\leqslant (N-n) \sum_{k=n}^{N} \frac{b_k^2 a^{2(N-k)}}{(1+y)^{2k}} \int_{a\lambda_0(1+y)}^{\infty} w^{s+p+2k-1} \exp(-w)\, dw \\ &\leqslant (N-n) \sum_{k=n}^{N} (1+y)^{-2k} b_k^2 a_0^{2(N-k)} \Gamma(s+p+2k). \end{aligned}$$

Consequently,

$$(8)\qquad \lim_{a\to 0}\int_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} B_2(a) F_1(y, a, p)\,dy$$

$$= \int_0^\infty \frac{y^{s-1}\lim\limits_{a\to 0} B_2(a) F_1(y, a, p)}{(1+y)^{s+p}}\,dy = b_N^2 \Gamma(s)\Gamma(p+2N).$$

On the other hand, from Lemma 2 we get

$$(9)\qquad \lim_{a\to 0}\int_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} B_2(a) F_2(y, a, p)\,dy$$

$$= \int_0^\infty \frac{y^{s-1}\lim\limits_{a\to 0} B_2(a) F_2(y, a, p)}{(1+y)^{s+p}}\,dy = b_N^2 \frac{\Gamma(s)\Gamma(p+2N)\Gamma^2(s+p+N+M)}{\Gamma(s+p+2M)\Gamma(s+p+2N)}.$$

From (8) and (9) we finally infer that Condition 4 holds and

$$\lim_{a\to 0} R(\xi, \tilde{u}) = 1 - \frac{\Gamma^2(s+p+N+M)}{\Gamma(s+p+2M)\Gamma(s+p+2N)}.$$

Hence, using Lemma 1, we obtain

$$\sup_{p>-2n}\left[1-\frac{\Gamma^2(s+p+N+M)}{\Gamma(s+p+2M)\Gamma(s+p+2N)}\right] = 1 - \frac{\Gamma^2(s+N+M-2n)}{\Gamma(s+2(N-n))\Gamma(s+2(M-n))},$$

which completes the proof in the case $\Lambda = (\lambda_0, \infty)$.

Now let $\Lambda = (0, \lambda_0)$ and put

$$h(\lambda) = (c_m\lambda^m + \ldots + c_M\lambda^M)^2,$$

$$B_0(a) = a^{2m}, \qquad B_1(a) = a^{n+m}, \qquad B_2(a) = a^{2n}.$$

The verification of Conditions 1–3 is straightforward. Proceeding as above for all $a > a_0$ we get

$$B_2(a) F_1(y, a, p)$$

$$\leqslant (N-n)\sum_{i=n}^{N} a^{2(n-i)}(1+y)^{-2i}\int_0^{a\lambda_0(1+y)} w^{s+p+2i-1}\exp(-w)\,dw$$

$$\leqslant (N-n)\sum_{i=n}^{N} a^{2(n-i)}(1+y)^{-2i}\Gamma(s+p+2i-1).$$

Thus

$$\lim_{a\to\infty}\int_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} B_2(a) F_1(y, a, p)\,dy = b_n^2\Gamma(s)\Gamma(p+2n).$$

It is easily checked that

$$B_2(a)F_2(y, a, p) = \frac{\left\{\sum_{i=n}^{N}\sum_{k=m}^{M} b_i c_k \frac{\gamma(s+p+i+k, a(1+y))\lambda_0}{[a(1+y)]^{i+k-n-m}}\right\}^2}{(1+y)^{2n}\sum_{i=m}^{M}\sum_{k=m}^{M}\frac{\gamma(s+p+i+k, a(1+y)\lambda_0)}{[a(1+y)]^{i+k-n-m}}}$$

$$\leqslant \frac{(1+y)^{-2n}}{H(a, y)}\left\{\sum_{i=n}^{N}\sum_{k=m}^{M}\frac{|b_i c_k|\,\Gamma(s+p+i+k)}{a_0^{i+k-2m}}\right\}^2$$

$$\leqslant \frac{(1+y)^{-2n}}{\inf\limits_{(a,y)\in A} H(a, y)}\left\{\sum_{i=n}^{N}\sum_{k=m}^{M}\frac{|b_i c_k|\,\Gamma(s+p+i+k)}{a_0^{i+k-n-m}}\right\}^2,$$

where

$$\gamma(x, y) = \int_0^y u^{x-1}\exp(-u)\,du,$$

$$H(y, a) = [a(1+y)]^{2m}\int_0^{a\lambda_0(1+y)} h\left(\frac{w}{a(1+y)}\right)w^{s+p-1}\exp(-w)\,dw$$

and

$$A \doteq \{(a, y)\,|\, a \geqslant a_0,\ y > 0\}.$$

Observe that

$$\inf_{(y,a)\in A} H(a, y) > 0$$

since for every $(a, y)\in A$ the following conditions hold: $H(a, y) > 0$, the function $H(\cdot, \cdot)$ is continuous,

$$\lim_{a\to\infty} H(a, y) = c_M^2\,\Gamma(s+p+2m)$$

and

$$H(a, y) = c_M^2\,\gamma(s+p+2m, a(1+y)\lambda_0) + o[a(1+y)].$$

Then, by the Lebesgue Theorem, we obtain

$$\lim_{a\to\infty}\int_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} B_2(a)F_2(y, a, p)\,dy = b_n^2\frac{\Gamma(s)\,\Gamma(p+2n)\,\Gamma(s+p+n+m)}{\Gamma(s+p+2m)\,\Gamma(s+p+2m)}.$$

Now, an argument similar to the above completes the proof.

2.2. Sum of power functions. Let $g(\lambda) = q_1\lambda^{r_1} + \ldots + q_n\lambda^{r_n}$, where q_i and r_i $(i = 1, \ldots, n)$ are reals, $r_1 < r_2 < \ldots < r_n$, and q_1 and q_2 are not equal to zero.

Corollary 2. *Suppose that the risk of the estimator* δ *satisfies the condition*

$$\sup_{\lambda\in\Lambda} R(g(\lambda), \delta) \leqslant \begin{cases} 1-\dfrac{\Gamma^2(s+r_n-2r_1)}{\Gamma(s-2r_1)\Gamma(s+2r_n-2r_1)} & \text{for } \Lambda=(\lambda_0, \infty), \\ 1-\dfrac{\Gamma^2(s-r_1)}{\Gamma(s)\Gamma(s-2r_1)} & \text{for } \Lambda=(0, \lambda_0). \end{cases}$$

Then δ *is minimax.*

Note that in the proof we take $h(\lambda)=1$, $B_0(a)=1$, $B_1(a)=a^{r_n}$, $B_2(a)=a^{2r_n}$ for $\Lambda=(\lambda_0, \infty)$ and $h(\lambda)=1$, $B_0(a)=1$, $B_1(a)=a^{r_1}$, $B_2(a)=a^{2r_1}$ for $\Lambda=(0, \lambda_0)$. To verify that Condition 4 holds, it is enough to use the simple fact that, for each $x>-1$, $u+r>0$, $u+R>0$, there exists a constant C such that

$$\frac{\Gamma(u+r, a(1+x))\Gamma(u+R, a(1+x))}{\Gamma(u, a(1+x))} \leqslant C$$

(see the Lemma of [6]).

2.3. Analytical functions. Take

$$g(\lambda)=\sum_{i=n}^{\infty} q_i\lambda^i, \qquad \lambda\in(0, \lambda_0),$$

and suppose that, for all $a>a_0$,

(i) $$\sum_{i=n}^{\infty} q_i a^{-s-p-i}\Gamma(s+p+i)<\infty,$$

(ii) $$\sum_{i=n}^{\infty}\sum_{k=n}^{\infty} q_i q_k a^{-p-i-k}\Gamma(s+p+i+k)<\infty.$$

Here a_0 is a given constant and $q_n\neq 0$, $n=0, \pm 1, \pm 2, \ldots$

Corollary 3. *If the risk of the estimator* δ *satisfies the condition*

$$\sup_{\lambda\in\Lambda} R(g(\lambda), \delta)\leqslant 1-\frac{\Gamma^2(s-n)}{\Gamma(s)\Gamma(s-2n)},$$

then δ *is minimax.*

In the proof of Corollary 3 we take $h(\lambda)=1$, $B_0(a)=1$, $B_1(a)=a^n$, and $B_2(a)=a^{2n}$.

2.4. Let $g(\lambda)=1-\exp(b\lambda^c)$, where b and c are arbitrary constants.

Corollary 4. *Suppose that the risk of the estimator* δ *satisfies the condition*

$$\sup_{\lambda\in\Lambda} R(g(\lambda),\delta) \leqslant \begin{cases} 1-\dfrac{\Gamma^2(s+c)}{\Gamma(s)\Gamma(s+2c)} & \textit{for } c>0,\ \Lambda=(0,\lambda_0),\\ 1-\dfrac{\Gamma^2(s-c)}{\Gamma(s)\Gamma(s-2c)} & \textit{for } c<0,\ \Lambda=(\lambda_0,\infty).\end{cases}$$

Then δ is minimax.

In the proof for both cases we use $h(\lambda)=1$ if $b<0$ and $h(\lambda)=\exp(-2b\lambda^c)$ if $b>0$, $B_0(a)=1$, $B_1(a)=a^c$, $B_2(a)=a^{2c}$.

Remark. If the upper bound for minimaxity is equal to zero (see, e.g., Corollary 3 for $n=0$), then we must estimate the function $g(\lambda)-c$, where c is an adequately chosen constant. Thus, if δ is minimax for $g(\lambda)-c$, then $\delta+c$ is minimax for $g(\lambda)$ under the loss $L(d,g)=(g-c)^{-2}(d-g)^2$.

Remark. If there exists at least one estimator the maximal risk of which is equal to the right-hand side of (3), then condition (3) is also necessary for minimaxity.

3. EXAMPLES OF MINIMAX ESTIMATORS

In this section we present some examples of minimax estimators in the gamma and other distributions.

3.1. Minimax estimators in the gamma distribution.

EXAMPLE 1 (minimax estimation of the failure rate in the Erlang distribution). Let X have the Erlang distribution, i.e., the distribution (1), where s is an integer. Then the failure rate (see [16], Section 6.1) is given by the formula

$$g(\lambda)=\frac{t^{s-1}\lambda^s}{(s-1)!}\left(\sum_{k=0}^{s-1}\frac{t^k}{k!}\lambda^k\right)^{-1}.$$

Let us take $\Lambda=(0,\lambda_0)$ and consider the class of estimators of the form

$$\delta(Y)=\frac{t^{s-1}}{(s-1)!}(sn-s-1)\,Y^{-s}+\sum_{i=1}^{m}A_i\,Y^{-s-i},$$

where $Y=\sum_{i=1}^{n}X_i$ and $X_1,\ldots,X_n$ are independent and identically distributed (i.i.d.) random variables (r.v.'s), $n>2$. For such estimators the sufficient condition for minimaxity from Corollary 1 is given by

$$-\frac{\mathrm{E}\delta^2(Y)}{g^2(\lambda)}+2\frac{\mathrm{E}\delta(Y)}{g(\lambda)}-\frac{\Gamma^2(ns-s)}{\Gamma(ns)\Gamma(ns-2s)}\geqslant 0. \tag{10}$$

Denote the left-hand side of (10) by $D(\lambda)$ and observe that

$$D(\lambda) = \lambda^{2ns+1}\left[2A_1 \frac{t^{s-1}}{(s-1)!}\left(\Gamma(ns-s-1) - \frac{\Gamma(ns-2s-1)\,\Gamma(ns-2s+1)}{\Gamma(ns-2s)}\right)+o(1)\right].$$

Thus there exists a constant λ_0 such that $D(\lambda) \geqslant 0$ for all $\lambda \in (0, \lambda_0)$ if and only if the constant A_1 has the same sign as

$$\Gamma(ns-s-1)-\Gamma(ns-2s-1)\,\Gamma(ns-2s+1)/\Gamma(ns-2s).$$

Now we can find the function $\lambda_0(A_1, \ldots, A_n)$ or fixing λ_0 we can obtain the coefficients $A_1, \ldots, A_n$ for which (10) holds, but both methods require numerical calculations of the function $\lambda_0(A_1, \ldots, A_n)$ and the coefficients for different constants s, t and n.

EXAMPLE 2. Let X have the gamma distribution (1). The estimator

$$\left(\frac{u}{s+1} X + bu^2 X^2 + 1\right) a^u$$

is minimax for $g(\lambda) = \lambda a^u/(\lambda - u)$ under the restriction $\lambda > \lambda_0$ and the loss $L(d, g) = g^{-2}(d-g)^2$, where a and u are positive constants, $b < 0$, $\lambda_0 > 0$, b and λ_0 are such that, for all $\lambda > \lambda_0$,

$$-2b\lambda^3 + u\lambda^2\left[b(2s+6) - b^2(s+1)(s+2)(s+3) - \frac{1}{s+1}\right] + 2b(s+2)u\lambda\,[(s+1)(s+3)b-1] - b^2u^3(s+1)(s+2)(s+3) \geqslant 0.$$

Applying the Cardano formulas one can obtain the explicit formula for the function $b(\lambda_0)$.

EXAMPLE 3. Suppose that X has the gamma distribution (1). The estimator

$$(2a)^{-1}[1-\exp(-aX)]$$

is minimax for $(\lambda+a)^{-1}$, $a > 0$, under the loss $L(d, g) = (\lambda+a)^2(d-g)^2$ and the restriction $\lambda > \lambda_0$.

EXAMPLE 4. Assign $g(\lambda) = q_1\lambda^{r_1} + \ldots + q_n\lambda^{r_n}$, where q_i and r_i for $i = 1, \ldots, n$ are reals, $r_1 < r_2 < \ldots < r_n$. First note that the case of $g(\lambda) = \lambda^r$ was considered in [6], so it is omitted. Put $\Lambda = (0, \lambda_0)$ and consider the class of estimators of the form

$$\delta(X) = \sum_{i=1}^{n} q_i A_i X^{-r_i}.$$

Corollary 2 asserts that if the estimator δ satisfies the condition

$$E_\lambda[\delta(X)-\sum_{i=1}^{n} q_i \lambda^{r_i}]^2 \leqslant \left[1-\frac{\Gamma^2(s-r_1)}{\Gamma(s)\Gamma(s-2r_1)}\right](\sum_{i=1}^{n} q_i \lambda^{r_i})^2$$

for all λ $(0<\lambda<\lambda_0)$, then δ is minimax. This condition is equivalent to

$$(11)\quad \sum_{i=1}^{n}\sum_{k=1}^{n} A_i A_k q_i q_k \frac{\Gamma(s-r_i-r_k)}{\Gamma(s)}\lambda^{r_i+r_k}-2(\sum_{i=1}^{n} q_i\lambda^{r_i})\left(\sum_{i=1}^{n} A_i q_i \frac{\Gamma(s-r_i)}{\Gamma(s)}\lambda^{r_i}\right)$$
$$+(\sum_{i=1}^{n} q_i \lambda^{r_i})^2 \frac{\Gamma^2(s-r_1)}{\Gamma(s)\Gamma(s-2r_1)} \leqslant 0$$

for all λ $(0<\lambda<\lambda_0)$. Denote the left-hand side of (11) by $D(\lambda)$ and observe that if we put $A_1=\Gamma(s-r_1)/\Gamma(s-2r_1)$, then

$$\Gamma(s)D(\lambda)\lambda^{-r_1-r_2}=q_1q_2A_2\left[\frac{\Gamma(s-r_1)\Gamma(s-r_1-r_2)}{\Gamma(s-r_2)\Gamma(s-2r_1)}-1\right]+o(1).$$

Hence, for some properly chosen A_i $(i=2,\ldots,n)$ and λ_0, the inequality (11) holds if and only if the constant A_2 has the same sign as

$$\{[\Gamma(s-r_1)\Gamma(s-r_1-r_2)/\Gamma(s-r_2)\Gamma(s-2r_1)]-1\}q_1q_2.$$

As in Example 1, the explicit formula for A_i $(i=2,\ldots,n)$ can be obtained by using numerical calculations.

Now we write

$$S_i=\frac{1}{s+1}\left[1+\left(\frac{3s+5}{(s+2)(s+3)}\right)^{1/2}(-1)^i\right],\quad i=1,2.$$

EXAMPLE 5. Suppose that X is an r.v. with the gamma distribution (1). The estimator $[(s+2)(s+3)]^{-1}q_1X^2+A_2q_2X$ is minimax for $g(\lambda)=q_1\lambda^{-2}+q_2\lambda^{-1}$ under the loss $L(d,g)=g^{-2}(d-g)^2$ and the restriction $0<\lambda<\lambda_0$, where λ_0 is an arbitrary constant and either

$$A_2\in[S_1, S_2]\quad \text{for } q_1q_2>0$$

or

$$A_2\in[0, S_1]\cup[S_2, \infty[\quad \text{for } q_1q_2<0.$$

EXAMPLE 6. The estimator $[(s+2)(s+3)]^{-1}q_1X^2+A_2q_2X$ is minimax for $q_1\lambda^{-2}+q_2\lambda^{-1}$ under the restriction $0<\lambda<\lambda_0$, where λ_0 is a given constant. Here

$$A_2\in[0, S_1]\cup[S_2, \infty[\quad \text{for } q_1q_2>0$$

and

$$A_2\in[S_1, S_2]\quad \text{for } q_1q_2<0,$$

and

$$\lambda_0 = 4A_2 \Big/ \left[(s+3)A_2^2 - 2\frac{s+3}{s+1}A_2 + \frac{1}{s+2}\right] q_1 q_2.$$

The attempt to find an estimator the risk of which fulfils the condition of Corollary 4 was not successful. In the case $\Lambda = (0, \lambda_0)$ we obtained only ε-minimax estimators, e.g., estimators δ^* such that, for each estimator δ,

$$\sup_\lambda R(\lambda, \delta^*) \leqslant \sup_\lambda R(\lambda, \delta) + \varepsilon, \quad \text{where } \varepsilon > 0.$$

One of these ε-minimax estimators for the hazard rate $g(\lambda) = \exp(-\lambda t)$, $0 < \lambda < \lambda_0$ and $t > 0$, in the gamma distribution (1) (where $s = 1$) is equal to

$$\delta^*(X) = \frac{a}{n}\sum_{i=1}^{n} \delta_{X_i}(t) + b,$$

where $\delta_x(t) = 1$ for $x < t$ and is zero otherwise; a and b are suitably chosen constants. The property of ε-minimaxity follows from Corollary 4 and from the fact that

$$\lim_{\lambda \to 0} R(\tilde{\delta}(X), \lambda) = 1 - \frac{\Gamma^2(s+1)}{\Gamma(s)\Gamma(s+2)}.$$

Note that the estimator δ^* appeared in the papers [11] and [14].

3.2. Minimax estimators in some other distributions. In what follows it is shown how the previous results can be applied to estimation in the Pareto, generalized gamma, generalized Laplace and other distributions. Denote by $G(s, \lambda)$ the gamma distribution (1).

3.2.1. *Pareto distribution.* Suppose that $X_1, \ldots, X_n$ are i.i.d. r.v.'s with the density $h(x, a, \lambda) = \lambda a^\lambda x^{-\lambda-1}$, $0 < a < x < \infty$. Note that

$$Y = \sum_{i=1}^{n} \ln(X_i/a_i)$$

has the distribution $G(n, \lambda)$. Hence one can obtain the minimax estimators in the Pareto distribution. For example, the estimator

$$\delta(X_1, \ldots, X_n) = \left[\frac{k}{n+1}\sum_{i=1}^{n} \ln\left(\frac{X_i}{a_i}\right) + bk^2\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{a_i}\right)\right)^2 + 1\right] a^k$$

(see Example 2) is minimax for the moments

$$\mathrm{E}_\lambda X^k = \lambda a^k/(\lambda - k)$$

in the Pareto distribution under the truncation $\lambda > \lambda_0$ (these moments exist if and only if $\lambda > k$) and the loss $L(d, g) = (g-1)^{-1}(d-g)^2$, where λ_0 is the maximal root of the equation

$$-2b\lambda^3+k\lambda^2\left[-b^2(n+1)(n+2)(n+3)+b(2n+6)-\frac{1}{n+1}\right]\cdot$$

$$+2\lambda k^2 b(n+2)[b(n+1)(n+3)-1]-b^2k^3(n+1)(n+2)(n+3)=0.$$

3.2.2. *Generalized Laplace distribution.* Suppose $X_1, \ldots, X_n$ are i.i.d. r.v.'s with the density

$$h(x, k, b)=\frac{k}{2b\Gamma(1/k)}\exp\left(-\frac{|x|^k}{b^k}\right), \quad x\in \boldsymbol{R}^1, \quad k, b>0.$$

It is easily seen that the r.v.

$$Y=\sum_{i=1}^{n}|X_i|^k$$

has the distribution $G(n/k, b^{-k})$. Hence we obtain the minimax estimators in the Laplace ($k=2$) and normal ($k=1$) distributions. For example, in the case of the normal distribution, the estimator

$$\delta_1(Y)=\frac{3}{(n+4)(n+6)}\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]^2+\frac{3}{2}A\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]\mu^2+\mu^4$$

minimax for $E_\sigma X^4=3\sigma^4+3\mu^2\sigma^2+\mu^4$ and the estimator

$$\delta_2(Y)=\frac{12\mu}{(n+4)(n+6)}\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]^2+2\mu^3 A\sum_{i=1}^{n}(X_i-\mu)^2+\mu^5$$

is minimax for $E_\sigma X^5=12\mu\sigma^4+4\mu^2\sigma^2+\mu^5$ of $N(\mu, \sigma)$ under the loss $L(d, g) = g^{-2}(d-g)^2$ and the restriction $0<\sigma_0<\sigma<\infty$, where σ_0 is a given constant (see Example 5). In both cases

$$A\in[A_i, A_2], \quad \text{where } A_i=\frac{2}{n+2}\left[1+(-1)^i\left(\frac{6n+10}{(n+4)(n+6)}\right)^{1/2}\right].$$

Note also that the estimators δ_1 and δ_2 are consistent and asymptotically unbiased.

3.2.3. *Generalized gamma distribution.* Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with the density

$$h(x, p, \alpha, \lambda)=\frac{|\alpha|}{\Gamma(p/\alpha)}\lambda^{p/\alpha}x^{p-1}\exp(-\lambda x^\alpha),$$

where $0<x<\infty$, $p\alpha>0$, $\lambda>0$. Observe that $Y=\sum_{i=1}^{n}X_i^\alpha$ has the distribution $G(np/\alpha, \lambda)$. Hence we get the minimax estimators in the Maxwell, Rayleigh, Weibull and other distributions.

3.2.4. *Particular cases of the beta distribution.* If X has the density $f(x, p) = px^{p-1}$ for $0 < x < 1$ and is zero otherwise, then the r.v. $Y = -\ln X$ has the distribution $G(1, p)$. Thus, applying Example 3 of this section, we infer that the estimator $(1+X)/2$ is minimax for the mean $E_p X = p/(p+1)$ under the loss $L(d, g) = (g-1)^{-2}(d-g)^2$ and the restriction $0 < p_0 < p < \infty$, where p_0 is a given constant.

3.2.5. *Double exponential distribution.* Note that if X has the density

$$h(x, \beta, \lambda) = \lambda\beta e^{\beta x}\exp[-\lambda(\exp(\beta x)-1)], \quad 0 < x < \infty,$$

$\beta, \lambda > 0$ (see [10]), then $Y = -1+\exp(\beta x)$ has the distribution $G(1, \lambda)$.

Proceeding as above, one can obtain minimax estimators in the lognormal, Burr and a few other distributions.

4. ESTIMATION UNDER DIFFERENT LOSS FUNCTIONS

This section contains some sufficient conditions for the loss function $L(d, g) = [W(g)]^{-1}(d-g)^2$ under which every estimator is minimax, i.e., every estimator has the unbounded maximal risk. Suppose first that besides of Conditions 1 (ii), 2, 3 and 4 of the previous section, the following new ones hold:

CONDITION 1 (a). *There exists $p \in P$ such that, for all $a > 0$,*

$$0 < \int_0^\infty W(g(\lambda))h(\lambda)\lambda^{p-1}\exp(-a\lambda)\,d\lambda.$$

CONDITION 5. *There exists $p \in P$ such that*

$$0 < \lim_{a \to \varphi} B_2(a)B_3(a) \int_{A_a} W\left[g\left(\frac{u}{a}\right)\right] h\left(\frac{u}{a}\right) u^{p-1}\exp(-u)\,du$$
$$= \int_0^\infty A_2(u)A_3(u)u^{p-1}\exp(-u)\,du,$$

where

$$A_3(u) = \lim_{a \to \varphi} W\left[g\left(\frac{u}{a}\right)\right] B_3(a)/g^2\left(\frac{u}{a}\right),$$

B_3 may be an arbitrary function.

PROPOSITION. *If $B_3(a)$ converges to infinity as $a \to \varphi$, then every estimator has the unbounded maximal risk.*

Proof. Let us change in the proof of the Theorem of Section 2 the function $h(\cdot)$ by putting

$$h(\lambda) := W[g(\lambda)]\,h(\lambda)g^{-2}(\lambda).$$

An argument similar to that of the Theorem shows that

$$\lim_{a\to\varphi} R(\xi, \tilde{u}) = \frac{\lim\limits_{a\to\varphi} B_3(a) \int\limits_0^\infty \frac{y^{s-1}}{(1+y)^{s+p}} \lim\limits_{a\to\varphi} B_2(a) F(y, a, p)\, dy}{\Gamma(s) \int\limits_0^\infty A_2(u) A_3(u) u^{p-1} \exp(-u)\, du}.$$

Since $\lim\limits_{a\to\varphi} B_3(a) = \infty$, we have

$$\lim_{a\to\varphi} R(\xi, \tilde{u}) = \infty,$$

which, according to the Lehmann Theorem, completes the proof.

We now give a few corollaries which hold for some simple functions $g(\cdot)$ and $W(\cdot)$.

4.1. Let

$$g(\lambda) = \sum_{i=n}^{N} b_i \lambda^i \Big(\sum_{i=m}^{M} c_i \lambda^i\Big)^{-1}.$$

COROLLARY 5. *If $M < N$ for $\Lambda = (\lambda_0, \infty)$ or $m > n$ for $\Lambda = (0, \lambda_0)$, then every estimator is minimax under the quadratic loss $L(d, g) = (d-g)^2$.*

In the proof we put

$$B_3(a) = \begin{cases} a^{2(M-N)} & \text{for } \Lambda = (\lambda_0, \infty), \\ a^{2(m-n)} & \text{for } \Lambda = (0, \lambda_0). \end{cases}$$

4.2. Let $g(\lambda) = \lambda^r$, where r is a real, $r \neq 0$. Suppose that

$$L(d, g) = Q(\lambda)\lambda^h (d-g)^2,$$

where $h \in \boldsymbol{R}^1$ and Q is an arbitrary measurable function such that

$$\inf_{\lambda>0} Q(\lambda) > 0$$

and the limits $\lim\limits_{\lambda\to 0} Q(\lambda)$ and $\lim\limits_{\lambda\to\infty} Q(\lambda)$ exist.

COROLLARY 6. *If $2r+h < 0$ for $\Lambda = (0, \lambda_0)$ and $2r+h > 0$ for $\Lambda = (\lambda_0, \infty)$, then every estimator is minimax.*

In the proof we use $B_3(a) = a^{-2r-h}$ in both cases.

4.3. Suppose that $g(\lambda) = 1 - \exp(b\lambda^c)$, where c is an integer (see Corollary 4) and the loss is given as in the case 4.2.

COROLLARY 7. *If $2c+h < 0$ for $\Lambda = (0, \lambda_0)$ and $c > 0$ or $2c+h > 0$ for $\Lambda = (\lambda_0, \infty)$ and $c < 0$, then every estimator is minimax.*

To prove Corollary 7 it is enough to put $B_3(a) = a^{-2c-b}$ and apply the above Proposition.

References

[1] P. J. Bickel, *Minimax estimation of the mean of a normal distribution when the parameter space is restricted*, Ann. Statist. 9 (1981), pp. 1301–1309.

[2] G. Casella and W. Strawderman, *Estimating a bounded normal mean*, ibidem 9 (1981), pp. 868–876.

[3] Cheng Ping, *Minimax estimates of parameters of distributions belonging to an exponential family*, Chinese Math. 5 (1964), pp. 277–299.

[4] J. K. Ghosh and R. Singh, *Estimation of the reciprocal of the scale parameter of a gamma density*, Ann. Inst. Statist. Math. 22 (1970), pp. 51–55.

[5] I. S. Gradshteyn and J. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York 1965.

[6] M. Kałuszka, *Admissible and minimax estimators of λ^r in the gamma distribution with truncated parameter space*, Metrika 33 (1986), pp. 363–375.

[7] M. W. Katz, *Admissible and minimax estimates of parameters in truncated spaces*, Ann. Math. Statist. 32 (1961), pp. 136–142.

[8] J. Kuks, *Приближенно минимаксная оценка среднего одномерного нормального распределения (An approximately minimax estimator of the mean of a one-dimensional normal distribution)*, Izv. Akad. Nauk ESSR 32 (1983), pp. 268–273.

[9] E. L. Lehmann, *Theory of Point Estimation*, J. Wiley, New York 1983.

[10] P. K. Lloyd and M. Lipov, *Reliability, Management, Methods and Mathematics*, Prentice-Hall, New Jersey 1962.

[11] E. G. Phadia, *Minimax estimation of a cumulative distribution function*, Ann. Statist. 2 (1973), pp. 1149–1157.

[12] D. Ralescu and S. Ralescu, *A class of nonlinear admissible estimators in the one-parameter exponential family*, ibidem 9 (1981), pp. 177–183.

[13] R. Singh, *Admissible estimators of λ^r in gamma distribution with quadratic loss*, Trabajos Estadíst. 23 (1972), pp. 129–134.

[14] S. Trybuła, *Minimax prediction of a sample distribution function*, Zastos. Mat. 16 (1978), pp. 167–174.

[15] N. Yosushi, *An admissible estimation in the one-parameter exponential family with ambiguous information*, Ann. Inst. Statist. Math. 2 (1983), pp. 193–201.

[16] S. Zacks, *The Theory of Statistical Inference*, J. Wiley, New York 1971.

[17] S. Zubrzycki, *Explicit formulas for minimax admissible estimators in some cases of restrictions imposed on the parameter*, Zastos. Mat. 9 (1966), pp. 31–52.

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