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SIMPLE RELATIONS BETWEEN SOME MODELS OF FACTOR ANALYSIS

1. Introduction. According to some opinions (let us quote only Harris [5] out of many), when applying the factor analysis to behavioural research, it is convenient to use a greater number of different models. Thus, for practical applications, a more objective interpretation of results is to be provided, that would be less dependent on the kind of a model applied ([5], p. 370). However, this approach brings about some problems. As one of these problems can be considered the calculation of latent roots and vectors required by individual models. A more general approach to the problem was given in author's papers [2] and [3]. In this paper a less general but more simple relations are derived. Under the assumption of lower bound estimation of the communalities the four-factor analytic models are shown being related to each other only by a change of scales, i.e. by the multiplication by a diagonal transformation matrix.

2. Factor analytic model. The linear model of factor analysis is given by the known equation $z = Af+y$, where z is an observable n-dimensional random vector partitioned into its so-called common part (Af) and unique part or residuum (y). At the same time, f is an m-dimensional ($m \leqslant n$) directly not observable (i.e. latent) factor vector, A is an $(n\times m)$-matrix of factor coefficients (factor loadings). The variables, i.e. the components of z, are conveniently normalized so that $\mathrm{E}(zz') = R$, where R is the correlation matrix. Then it follows from the usual three postulates of independence — of the components of the common part Af and the unique part y, both mutually and between one another — that

$$\mathrm{E}(zz') = AA'+\mathrm{E}(yy'),$$

$$R = AA'+U^2, \tag{1}$$

which is the so-called fundamental factor theorem, where U^2, the so-called matrix of uniqueness, is a diagonal matrix of variances of unique parts y of the vector z. If we adopt the usual representation

$$\operatorname{diag}(AA') = H^2$$

for the so-called matrix of communalities, i.e. of the variances of common parts Af of the vector z, then, evidently

$$H^2 + U^2 = I. \tag{2}$$

The problem of factor analysis is that of finding the matrices A and H^2 (U^2, respectively) such that m would be minimal. During this process $n-m$ is called a *parsimony of factor model*. If H^2 is fixed, A can be found by means of latent roots and vectors of the matrix $R-U^2$, i.e. $A = VC^{1/2}$, where C is a diagonal matrix of m of the greatest latent roots of the matrix $R-U^2$ ordered usually in the descending way, and V is an $(n\times m)$-matrix of column normalized latent vectors associated with them. (This ordering of matrices of the latent roots and vectors will be always kept throughout this paper.)

The model just described with $H^2 < I$ and $m < n$ is called *incomplete* (*reduced*), while with $H^2 = I$ (i.e. $U^2 = 0$) and $m = n$ a so-called *complete* model is involved the parsimony of which is evidently equal to 0. The later model is also known under the term component analysis. Let us mention that in the incomplete model the well-known lower bounds of communalities are most frequently used for estimating them, i.e.

$$H^2 = I - (\operatorname{diag} R^{-1})^{-1}. \tag{3}$$

3. Relation between Rao's and Jöreskog's models. In Rao's model, which leads to a maximum likelihood solution, the unknown factors f are so chosen that their canonical correlation with observable variables z be maximum. Formally, it is the case of a standard canonical analysis of the correlation matrix described, e.g., by Anderson [1]. This analysis, in view of the expression of the submatrix of the composite correlation matrix of the vectors f and y with help of the factor theorem (1), and in view of the three kinds of independence postulated above, leads (see Kaiser and Caffrey [8], equation (7)) to the characteristic equation

$$[(R-U^2) - \nu^2 R]p = 0, \tag{4}$$

where ν^2 are squared canonical correlations. For other purposes the characteristic equation (4) is adapted to

$$[U^{-1}(R-U^2)U^{-1} - \theta^2 I]q = 0, \tag{5}$$

where $\theta^2 = \nu/(1-\nu^2)$ and $q = Up$. The matrix of factor coefficients of Rao's model can then be expressed in the first stage of iteration (see [8], equation (9)),

$$A_r = UQ[\theta^2], \tag{6}$$

where Q and $[\theta^2]$ are matrices of latent vectors and roots belonging to (5).

Nevertheless, the roots of (5) can be expressed by means of

$$[U^{-1}RU^{-1}-\beta^2 I]q = 0, \tag{7}$$

where $\beta^2 = \theta^2+1$.

As it becomes obvious, the matrix (6) of the coefficients of Rao's model can be expressed differently in the first iteration stage

$$A_r = UQ([\beta^2]-I)^{1/2}, \tag{8}$$

where $[\beta^2]$ is a diagonal matrix of the latent roots of the matrix $U^{-1}RU^{-1}$.

Jöreskog (see [6], p. 339 and 345), when deriving his model, introduces the estimate

$$R^* = A^*A^{*\prime}+\Delta^* I, \tag{9}$$

while

$$R^* = D^{1/2}RD^{1/2}. \tag{10}$$

Jöreskog represents there

$$D = \operatorname{diag} R^{-1} \tag{11}$$

and

$$A^* = L^*Z^{*1/2},$$

where L^* and Z^* are latent roots and vectors of the matrix (10), and Δ^* is the positive constant(¹). Then the matrix of factor coefficients of Jöreskog's model is determined (see [6], equation (30)) by

$$A_j = D^{-1/2}A^*. \tag{12}$$

Let us now recall the usual estimation of communalities (3) according to which, provided that (2) holds, it follows from (11) that $D^{1/2} \equiv U^{-1}$. Consequently, Jöreskog's matrix (10) is contained in the characteristic equation (7); hence $L^* \equiv [\beta^2]$ and $Z^* \equiv Q$. Thus Jöreskog's matrix of factor coefficients (12) can be written as

$$A_j = UQ[\beta]. \tag{13}$$

The comparison of (8) and (13) yields then very easily the relation between the matrix of coefficients of Rao's model in the first stage of

(¹) Jöreskog's postulate (9) is led by the earlier efforts of many authors, e.g. Lawley, to derive the factor model from the postulate $R = AA'+\Delta I$, where Δ is a positive constant. This kind of postulate means, however, that the communalities of all the variables are equal, which fails to correspond with the practice. On the other hand, postulate (9) means that the uniqueness of the tests is proportional to their coefficients of multiple alienation. As demonstrated by Jöreskog [7], the constant Δ^* converges to 1 with n increasing.

the iteration cycle and the matrix of coefficients of Jöreskog's model,

$$A_r = A_j T,$$

$$A_j = A_r T^{-1}, \tag{14}$$

where the transformation matrix

$$T = (I - [\beta^{-2}])^{1/2} \tag{15}$$

is diagonal. Thus the matrices of factor coefficients of Rao's (6) and Jöreskog's (12) models are mutually derivable from each other by means of a change of the scales only. At the same time, matrix (15) is a diagonal one of canonical correlations, for $1 - \beta^{-2} = \nu^2$.

4. Relations between Jöreskog's model and Guttman's image theory. Concerning the principle of Guttman's image theory should be stated here only that, according to this theory, the original vector z is partitioned into the so-called image and anti-image part. The particular factor models of each of the two parts of z were given by Guttman in terms of latent roots and vectors of the two succeeding covariance matrices G and Γ,

$$G = R + U^2 R^{-1} U^2 - 2U^2, \tag{16}$$

$$\Gamma = U^2 R^{-1} U^2, \tag{17}$$

where G is the covariance matrix of images and Γ of anti-images of the original variables (R is the original correlation matrix of z).

As it was derived by Harris (see [4], equations (2) and (22)), matrices (16) and (17) can be expressed, under the usual postulate (9), by means of latent roots of matrix (7), which, as we shown above, is identical with Jöreskog's matrix (10). Then

$$G = UQ\left[\frac{(b-1)^2}{b}\right]Q' U, \tag{18}$$

$$\Gamma = UQ\left[\frac{1}{b}\right]Q' U, \tag{19}$$

where $b = \beta^2$ states the relation between Harris' [4] and Kaiser's [8] notation of the latent roots involved. From (18) and (19) the matrix of factor coefficients of the image and anti-image models can be obviously determined fairly easily — so as to satisfy theorem (1) — as

$$A_i = UQ([\beta^2] - I)[\beta^{-1}], \tag{20}$$

$$A_a = UQ[\beta^{-1}]. \tag{21}$$

By the comparison of (20) and (21) with expression (13) for the coefficients of Jöreskog's model, the relations between the matrices of coefficients of the two Guttman's models as well as Jöreskog's model can be easily established,

$$A_a = A_j[\beta^{-2}], \tag{22}$$

$$A_i = A_j(I-[\beta^{-2}]), \tag{23}$$

and so can the interrelation of the former two

$$A_a = A_i([\beta^2]-I)^{-1}. \tag{24}$$

It is evident from expressions (22)-(24) that all three solutions are linked by very simple transfer relations, again by a change of scales only. In (23) the matrix of squared canonical correlations can again be seen, similarly to (15).

5. Possible applications. If any solution of the three factor models — 1. Jöreskog's model, 2. Guttman's image, 3. anti-image model — is known, one can easily obtain the solutions of two remaining models only by multiplication by diagonal matrices according to equations (22), (23) and (24).

If Rao's procedure is programmed, then after its first iteration stage it is possible to obtain solutions of the three above-mentioned models simultaneously (one can use equations (14), (22) and (23)).

References

[1] T. W. Anderson, *An introduction to multivariate statistical analysis*, Wiley, New York 1958.

[2] P. Blahuš, *Comparison and generalization of some linear factor analytic models and their application in the study of motor abilities* (in Czech), Unpublished candidate thesis, Praha 1973.

[3] — *On some similarities of several factor analytic models*, Biometrische Zeitschrift 16 (1974), p. 15-19.

[4] C. W. Harris, *Some Rao-Guttman relationships*, Psychometrika 27 (1962), p. 247-263.

[5] — *On factors and factor scores*, ibidem 32 (1967), p. 363-379.

[6] K. G. Jöreskog, *On the statistical treatment of residuals in factor analysis*, ibidem 27 (1962), p. 335-354.

[7] — *Statistical estimation in factor analysis. A new technique and its foundation*, Almqvist-Wiksell, Stockholm, Uppsala 1963.

[8] H. F. Kaiser and J. Caffrey, *Alpha factor analysis*, Psychometrika 30 (1965), p. 1-14.

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PROSTE RELACJE MIĘDZY NIEKTÓRYMI MODELAMI ANALIZY CZYNNIKOWEJ

STRESZCZENIE

W artykule wykazano, że jeżeli za ocenę wariancji wspólnych czynników wybierze się ich dolne granice, to między macierzą ładunków czynnikowych (tzw. *macierzą nasyceń*) przy czterech modelach analizy czynnikowej istnieją proste relacje transformacyjne. Nawet tylko przez zmianę skali czynników, tj. przez ich pomnożenie przez diagonalną macierz, z każdego modelu można wyprowadzić modele pozostałe. Rozpatrywane są następujące modele: 1. model kanonicznej analizy czynnikowej Rao w jego pierwszym cyklu iteracji, 2. model Jöreskoga z 1962 r., 3. Guttmana *image* i 4. *anti-image model*. Otrzymane relacje transformacyjne można zastosować w praktyce w celu łatwiejszego rozwiązania pozostałych modeli, jeżeli rozwiązanie jednego z nich jest znane. Nie muszą być przy tym ponownie przeprowadzane obliczenia charakterystycznych wartości i wektorów.