ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIX, 2 (1987), pp. 325-333

A. OLEJNICZAK (Wrocław)

CONTINUOUS APPROXIMATION OF TIME-PERIODIC SOLUTIONS OF A LINEAR PARABOLIC EQUATION

0. Introduction. We consider an approximate method of finding T-periodic solutions of the parabolic equation

(1)
$$u_t + Au = f,$$

where A is a linear differential operator of second order in the space variables, with time-dependent coefficients. An approximate method based on the Crank–Nicolson–Galerkin method was discussed in [5]. A similar problem was also studied in [1] by using Fourier expansions. In this paper we start with the semidiscretization method which yields a system of linear ordinary differential equations with T-periodic condition. The solution of this system is approximated by Galerkin's method, and a continuous approximation of the solution of (1) is defined. We prove the existence of the solution and give some estimation of the error.

1. Basic definitions and assumptions. All the considered functions are real-valued. All the derivatives in the sequel are understood to be in the distributional sense. Let $\Omega \subset R^n$ be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ (see [2]) and let T be a fixed positive constant. We assume the operator A is defined in the cylindrical domain $Q = \Omega \times (0, T)$ and let for any $u \in H_1(\Omega)$

$$A(x, t)u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} a_i(x, t)\frac{\partial u}{\partial x_i} + a_0(x, t)u.$$

Let $(x, t) \in Q$. We assume that the coefficients of A are

(i) T-periodic, i.e.

(2)
$$a_{ij}(x, t+T) = a_{ij}(x, t), \qquad a_i(x, t+T) = a_i(x, t), \qquad a_0(x, t+T) = a_0(x, t),$$

(ii) bounded, i.e.

$$|a_{ij}(x,t)| \leqslant \mathcal{N}_0, \quad |a_i(x,t)| \leqslant \mathcal{N}_1, \quad 0 \leqslant m \leqslant a_0(x,t) \leqslant \mathcal{N}_2,$$

(iii) symmetric, i.e. $a_{ij}(x,t) = a_{ji}(x,t)$.

We denote by $(\cdot,\cdot)$ the scalar product on $L^2(\Omega)$. In the sequel we consider the bilinear Dirichlet form of A:

$$a(t; u, v) = \sum_{i,j=1}^{n} \left(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} \left(a_i \frac{\partial u}{\partial x_i}, v\right) + (a_0 u, v). \tag{3}$$

Let $V \subset H_1(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$. We assume that there exists a positive constant δ such that for every $v \in V$ and $t \in (0, T)$ the inequality

$$a(t; v, v) \geqslant \delta \|v\|_1^2 \tag{4}$$

holds. Next, we introduce the following space of T-periodic functions:

$$\tilde{W}(0,T) = \left\{u \in L^2(0,T;V)\colon \frac{\partial u}{\partial t} \in L^2(Q),\ u(\cdot, T) = u(\cdot, 0)\right\} \subset H_1(Q).$$

It is easy to see, by the famous Sobolev lemma, that the set $\tilde{W}(0,T)$ is well defined. Let $f \in L^2(Q)$ be an arbitrary T-periodic function. We shall approximate the solution of the following exact problem:

PROBLEM P. Find a function $u \in \tilde{W}(0,T)$ such that for a.e. $t \in (0,T)$

$$\left(\frac{\partial u}{\partial t}, v\right) + a(t; u, v) = \left(f(\cdot, t), v\right) \quad \text{for every } v \in V. \tag{5}$$

2. Semidiscretization method. Let $V_h \subset V$ be a finite-dimensional linear subspace. We approximate the solution of (5) by requiring that u and v in (5) belong to V_h. In this manner we obtain

PROBLEM P_h. Find a function $u_h \in X = \tilde{W}(0,T) \cap L^2(0,T;V_h)$ such that for $t \in (0,T)$

$$\left(\frac{\partial u_h}{\partial t}, v\right) + a(t; u_h, v) = \left(f(\cdot, t), v\right) \quad \text{for every } v \in V_h. \tag{6}$$

Using the method from [3] it is easy to reduce the estimation of the error to an approximation problem. We have

THEOREM 1. *Let u be a solution of (5). If (2)–(4) are satisfied, then there exists a positive constant $C(u)$ such that*

$$\int_0^T \|u-u_h\|_1^2\,dt \leqslant C(u) \inf_{\tilde{u}\in X} \Big[\int_0^T \|u-\tilde{u}\|_1^2\,dt + \int_0^T \|(u-\tilde{u})_t\|_0^2\,dt\Big].$$

Proof. From (5) and (6) we obtain for any $v \in V_h$ the equality

$$\left(\frac{\partial(u-u_h)}{\partial t}, v\right) + a(t; u-u_h, v) = 0.$$

Let $\tilde{u} \in V_h$ be an arbitrary function. We consider this equality with $v = (u-u_h) - (u-\tilde{u}) \in V_h$. If for any fixed $t \in [0, T]$ we put $e = u-u_h$ and $\eta = u-\tilde{u}$, then we obtain

$$\left(\frac{\partial e}{\partial t}, e\right) + a(t; e, e) = \left(\frac{\partial e}{\partial t}, \eta\right) + a(t; e, \eta).$$

Using the boundedness of the coefficients of A and (4) we get

$$\left(\frac{\partial e}{\partial t}, e\right) + \delta \|e\|_1^2 \leqslant \left(\frac{\partial e}{\partial t}, \eta\right) + C\|e\|_1 \|\eta\|_1.$$

Next, integrating by parts and using the T-periodicity, we obtain

$$\delta \int_0^T \|e\|_1^2 \, dt \leqslant -\int_0^T \left(e, \frac{\partial \eta}{\partial t}\right) dt + C \int_0^T \|e\|_1 \|\eta\|_1 \, dt,$$

which by the Schwarz inequality and the elementary inequality

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \text{for } a, b \in R, \ \varepsilon > 0$$

implies the desired result.

Denoting a basis of V_h by $\{v_i\}_{i=1}^N$ we may write

$$u_h(x, t) = \sum_{i=1}^N \alpha_i(t) v_i(x).$$

It follows from (6) that the coefficients α_i in the above equality are defined as T-periodic solutions of the system of ordinary differential equations

$$\Lambda \dot{\alpha}(t) + \mathscr{A}(t) \alpha(t) = \mathscr{F}(t), \tag{7}$$

where

$$\alpha(t) = \{\alpha_1(t), \alpha_2(t), \ldots, \alpha_N(t)\}^T,$$

$$\Lambda = \{(v_i, v_j)\}_{i,j=1}^N, \qquad \mathscr{A}(t) = \{a(t; v_i, v_j)\}_{i,j=1}^N,$$

$$\mathscr{F}(t) = \{(f(\cdot, t), v_1), (f(\cdot, t), v_2), \ldots, (f(\cdot, t), v_N)\}^T.$$

Some important properties of the coefficients of (7) are collected in the following

Lemma 1. *Let the bilinear form* (3) *satisfy the V-elliptic condition* (4). *If the coefficients of A and the function f satisfy the conditions formulated in Section* 1, *then*

(i) *the matrix* Λ *is symmetric and positive definite,*

(ii) *the matrix* $\mathscr{A}(t)$ *has bounded elements and satisfies the condition*

$$(8) \qquad \sum_{i,j=1}^{N} \mathscr{A}_{ij}(t)\xi_i\xi_j \geqslant d\sum_{i=1}^{N} \xi_i \quad \textit{for all } \xi\in R^N,$$

where $d>0$ *does not depend on* t,

(iii) $(f(\cdot, t), v_i)\in L^2(0, T)$.

Proof. The first condition is obviously valid because Λ is Gramm's matrix. The boundedness of $\mathscr{A}_{ij}(t)$ follows immediately from (2). We prove now (8). For any $\xi\in R^N$ we set

$$z = \sum_{i=1}^{N} \xi_i v_i.$$

For any $t\in(0, T)$ we have

$$\sum_{i,j=1}^{N} \mathscr{A}_{ij}(t)\xi_i\xi_j = a(t; z, z) \geqslant \delta\|z\|_1^2,$$

whence, using the linear independence of $\{v_i\}$, we obtain (8). Finally, for any $t\in(0, T)$ we have

$$\Big|\int_\Omega f(x, t)v_i(x)\,dx\Big| \leqslant \Big\{\int_\Omega f^2(x, t)\,dx\Big\}^{1/2}\Big\{\int_\Omega v_i^2(x)\,dx\Big\}^{1/2};$$

hence

$$\int_0^T \mathscr{F}_i^2(t)\,dt \leqslant \|v_i\|_{L^2(\Omega)}^2\|f\|_{L^2(Q)}^2.$$

In the next section we discuss the existence and some approximation of the solution of (7).

3. System of ordinary differential equations with *T*-periodic condition. Let a natural number N be given. Let $\Lambda\in R^{N\times N}$ be a symmetric and positive definite matrix and let $\mathscr{A}(t)$ be a function matrix of order $N\times N$ with bounded elements satisfying condition (8). For the sake of simplicity we write

$$\|\mathscr{A}\|_\infty = \max_{i,j}\sup_{t\in[0,T]}|\mathscr{A}_{ij}(t)|, \qquad \|\Lambda\|_\infty = \max_{i,j}|\lambda_{ij}|.$$

Next, we introduce the following spaces of vector-valued functions defined on the interval $[0, T]$:

$$H_r^N(0, T) = \{v\colon [0, T]\to R^N\colon v = (v_1, v_2, \ldots, v_N),\ v_i\in H_r(0, T),\ 1\leqslant i\leqslant N\},$$

$$\tilde{H}_r^N(0, T) = \{v\in H_r^N(0, T)\colon v^{(i)}(0) = v^{(i)}(T),\ 0\leqslant i\leqslant r-1\},$$

$$L^{2,N}(0, T) = \{v\colon [0, T]\to R^N\colon v_i\in L^2(0, T)\}.$$

For any vectors $\alpha, \beta \in R^N$ let

$$\langle \alpha, \beta \rangle \stackrel{\mathrm{df}}{=} \sum_{i=1}^{N} \alpha_i \beta_i .$$

The expression

$$(u, v)_{0,N} \stackrel{\mathrm{df}}{=} \int_0^T \langle u(t), v(t) \rangle dt$$

defines an inner-product in $L^{2,N}(0, T)$. Similarly, for $u, v \in H_r^N(0, T)$ the expression

$$(u, v)_{r,N} = \int_0^T \sum_{i=0}^{r} \langle u^{(i)}(t), v^{(i)}(t) \rangle dt$$

defines an inner-product in $H_r^N(0, T)$. We denote by $\|\cdot\|_{r,N}$ the norm induced by $(\cdot,\cdot)_{r,N}$. Moreover, we introduce the seminorm

$$|v|_{r,N} = \left\{ \int_0^T \sum_{i=1}^{N} [v_i^{(r)}]^2 dt \right\}^{1/2} .$$

For any $\alpha \in L^{2,N}(0, T)$ and $\varphi \in \tilde{H}_1^N(0, T)$ we define

$$B(\alpha, \varphi) \stackrel{\mathrm{df}}{=} (\mathscr{A}(\cdot)\alpha, \varphi)_{0,N} - (\Lambda\alpha, \dot{\varphi})_{0,N}, \qquad l_{\mathscr{F}}(\varphi) \stackrel{\mathrm{df}}{=} (\mathscr{F}(\cdot), \varphi)_{0,N},$$

where $\dot{\varphi} = d\varphi/dt$ and $\mathscr{F} \in L^{2,N}(0, T)$. We formulate

PROBLEM Q. Find a function $\alpha \in L^{2,N}(0, T)$ such that

(9) $$B(\alpha, \varphi) = l_{\mathscr{F}}(\varphi) \quad \text{for every } \varphi \in \tilde{H}_1^N(0, T).$$

This problem is a weak formulation of the system of differential equations of the form (7) with T-periodic condition.

The following lemma gives some important properties of the bilinear form $B(\cdot, \cdot)$:

LEMMA 2. *For every $\varphi \in \tilde{H}_1^N(0, T)$ we have*

(i) $B(\varphi, \varphi) \geqslant d\|\varphi\|_{0,N}$,

(ii) $|B(\alpha, \varphi)| \leqslant \sqrt{2}N(\|\Lambda\|_\infty + \|\mathscr{A}\|_\infty)\|\alpha\|_{0,N}\|\varphi\|_{1,N}$ *for every $\alpha \in L^{2,N}(0, T)$.*

Proof. (i) follows from the T-periodicity of φ, inequality (4), and the symmetry of Λ. Indeed, the symmetry of Λ implies the equality

$$\langle \Lambda\varphi, \dot{\varphi} \rangle = \frac{1}{2}\frac{d}{dt} \langle \Lambda\varphi, \varphi \rangle$$

from which, using (8) and the T-periodicity of φ, we obtain

$$B(\varphi, \varphi) = (\mathscr{A}(\cdot)\varphi, \varphi)_{0,N} - (\Lambda\varphi, \dot{\varphi})_{0,N} \geqslant d\|\varphi\|_{0,N}^2 - \frac{1}{2}\langle \Lambda\varphi, \varphi \rangle\big|_0^T = d\|\varphi\|_{0,N}^2 .$$

The second inequality follows from the boundedness of $\mathscr{A}_{ij}(t)$, namely

$$|(\mathscr{A}(\cdot)\alpha, \varphi)_{0,N}| \leqslant \int_0^T \sum_{i,j=1}^N |\alpha_i A_{ij}(t)\varphi_j|\, dt \leqslant \|\mathscr{A}\|_\infty \int_0^T \sum_{i=1}^N |\alpha_i| \sum_{j=1}^N |\varphi_j|\, dt$$
$$\leqslant N\|\mathscr{A}\|_\infty \|\alpha\|_{0,N} \|\varphi\|_{0,N}.$$

Similar arguments lead to

$$|(\Lambda\alpha, \dot{\varphi})_{0,N}| \leqslant N\|\Lambda\|_\infty \|\alpha\|_{0,N} \|\dot{\varphi}\|_{0,N}$$

and (ii) is proved.

Thus, the form $B(\cdot, \cdot)$ is bounded from below and continuous. Using these facts we prove the existence theorem for the solution of (9).

THEOREM 2. *If $\mathscr{F} \in L^{2,N}(0, T)$, then the problem* Q *has a unique solution $\alpha \in \tilde{H}_1(0, T)$. This solution satisfies*

(i) $\Lambda\dot{\alpha} + \mathscr{A}(t)\alpha = \mathscr{F}(t)$ *in the sense* $\mathscr{D}'(0, T)$,

(ii) $\alpha(0) = \alpha(T)$.

Proof. The method is taken from [4]. The uniqueness follows from the inequalities

$$d\|\alpha\|_{0,N}^2 \leqslant B(\alpha, \alpha) = l_{\mathscr{F}}(\alpha) = (\mathscr{F}(\cdot), \varphi)_{0,N} \leqslant \|\mathscr{F}\|_{0,N} \|\alpha\|_{0,N},$$

since for $\mathscr{F} \equiv 0$ we obtain $\alpha \equiv 0$. To prove the existence, we consider a functional $B(\alpha, \varphi)$ with $\varphi \in \tilde{H}_1(0, T)$ arbitrary and fixed. According to Lemma 2, $B(\cdot, \varphi)$ is a continuous functional on $L^{2,N}(0, T)$. By the Riesz–Fréchet theorem we have $B(\alpha, \varphi) = (\alpha, S\varphi)_{0,N}$, where the operator S: $\tilde{H}_1^N(0, T) \to L^{2,N}(0, T)$ is linear. We show that $S(\tilde{H}_1^N(0, T))$ is dense in $L^{2,N}(0, T)$. Suppose that there exists an $\alpha_0 \perp S(\tilde{H}_1^N(0, T))$. Then α_0 is a solution of (9) with vanishing data and, by the uniqueness, we have $\alpha_0 = 0$.

It is sufficient to show that there exists a function $\alpha \in L^{2,N}(0, T)$ for which

$$(10) \qquad (\alpha, S\varphi)_{0,N} = l_{\mathscr{F}}(\varphi) \quad \text{for every } \varphi \in \tilde{H}_1^N(0, T).$$

By Lemma 2 we get

$$d\|\varphi\|_{0,N}^2 \leqslant B(\varphi, \varphi) = (\varphi, S\varphi)_{0,N} \leqslant \|\varphi\|_{0,N} \|S\varphi\|_{0,N},$$

thus

$$\|\varphi\|_{0,N} \leqslant \frac{1}{d}\|S\varphi\|_{0,N}.$$

This means that S is a one-to-one operator and the inverse operator S^{-1} is continuous over the space $\tilde{H}_1^N(0, T)$ equipped with the norm $\|\cdot\|_{0,N}$. We define the functional τ: $S(\tilde{H}_1^N(0, T)) \to R$ by $\tau(S\varphi) = l_{\mathscr{F}}(\varphi)$. Since

$$|l_{\mathscr{F}}(\varphi)| \leqslant \|\mathscr{F}\|_{0,N} \|\varphi\|_{0,N},$$

τ is continuous and τ may be extended by continuity on the whole $L^{2,N}(0, T)$. Let τ denote this extended functional. Using the Riesz–Fréchet theorem once more, we get an $\alpha \in L^{2,N}(0, T)$ satisfying $\tau(v) = (\alpha, v)_{0,N}$ for every $v \in L^{2,N}(0, T)$. Particularly, for $v = S\varphi$ we obtain (10) and α is the desired solution of (9). Setting $\varphi = (0, 0, \ldots, \varphi_s, \ldots, 0)^T$ with $\varphi_s \in C_0^\infty(0, T)$, it is easy to see that α satisfies (i). From the boundedness of the elements of $\mathscr{A}(t)$ we obtain

$$\Lambda\dot{\alpha} = \mathscr{F}(t) - \mathscr{A}(t)\alpha \in L^{2,N}(0, T),$$

thus $\alpha \in H_1^N(0, T)$. Next, for any $\varphi \in \tilde{H}_1^N(0, T)$ we have

$$B(\alpha, \varphi) = (\mathscr{A}(\cdot)\alpha, \varphi)_{0,N} + (\Lambda\alpha, \varphi)_{0,N} - \langle\Lambda\alpha, \alpha\rangle|_0^T = (\mathscr{F}(\cdot), \varphi)_{0,N}$$

and, consequently, $\Lambda(\alpha(0) - \alpha(T)) = 0$. Since the matrix Λ is nonsingular, we obtain (ii).

We are going to approximate the solution of (9). Let W_N be a finite-dimensional subspace of $\tilde{H}_1^N(0, T)$. We formulate the approximate problem as follows:

Problem $Q_{\Delta t}$. Find a function $\beta \in W_N$ such that

$$B(\beta, \varphi) = l_{\mathscr{F}}(\varphi) \quad \text{for every } \varphi \in W_N. \tag{11}$$

The error estimation of this method may be reduced to an approximation problem. We have an analogue of the Cea lemma in elliptic problems [2].

Theorem 3. *There exists a positive constant C such that*

$$\|\alpha - \beta\|_{0,N} \leqslant C \inf_{v \in W_N} \|\alpha - v\|_{1,N}.$$

Proof. Subtracting (11) from (9) with $\varphi \in W_N$ we get $B(\alpha - \beta, \varphi) = 0$. Using Lemma 2 we obtain

$$d\|\alpha - \beta\|_{0,N}^2 \leqslant B(\alpha - \beta, \alpha - \beta) = B(\alpha - \beta, \alpha - v) + B(\alpha - \beta, v - \beta)$$
$$\leqslant C\|\alpha - \beta\|_{0,N}\|\alpha - v\|_{1,N}.$$

In the sequel we restrict our considerations to the case $W_N = W^N$, where $W \subset \tilde{H}_1(0, T)$ is a linear subspace of dimension M. If $\{\varphi_i\}_{i=1}^M$ is a basis of W, then the basis of W_N contains functions ψ_{ij} of the form

$$\psi_{i1}(t) = \begin{bmatrix} \varphi_i(t) \\ 0 \\ \ldots \\ 0 \end{bmatrix}, \quad \psi_{i2}(t) = \begin{bmatrix} 0 \\ \varphi_i(t) \\ \ldots \\ 0 \end{bmatrix}, \quad \ldots, \quad \psi_{iN}(t) = \begin{bmatrix} 0 \\ \ldots \\ 0 \\ \varphi_i(t) \end{bmatrix}$$

for all i $(1 \leqslant i \leqslant M)$. Since

$$\beta = \sum_{i,j} \xi_{ij}\psi_{ij},$$

identity (11) implies the following system of linear equations:

$$(12)\qquad \sum_{i=1}^{M}\sum_{j=1}^{N} \xi_{ij} B(\psi_{ij}, \psi_{kl}) = l_{\mathscr{F}}(\psi_{kl}) \quad (1 \leqslant k \leqslant M,\ 1 \leqslant l \leqslant N).$$

THEOREM 4. *The system of linear equations* (12) *has a unique solution.*

Proof. For any $\eta_{ij} \in R$ we set $z = \sum_{i,j} \eta_{ij}\psi_{ij} \in W_N$. By Lemma 2 we have

$$\sum_{i,j} \eta_{ij} B(\psi_{ij}, \psi_{kl})\eta_{kl} = B(z, z) \geqslant d\|z\|_{0,N}^2 \geqslant d^* \sum_{i,j} \eta_{ij}^2.$$

4. Continuous approximation of the solution of (5). It is evident from Lemma 1 that an approximate solution of (7) may be obtained by Galerkin's method described in the previous section. Let $\beta \in W^N$ denote a solution of (11). A continuous approximation of the solution of the exact problem (P) is defined by the formula

$$(13)\qquad u_h^*(x, t) = \sum_{i=1}^{N} \beta_i(t) v_i(x).$$

We estimate the error of this solution in the case where Ω is a polyhedron in R^n. For this we use spaces of Lagrange-type finite elements [2]. Let $\{\mathscr{T}_h\}$ be a family of regular triangulations of Ω. Let V_h be a finite-element space of Lagrange-type of order $r-1$ corresponding to the triangulation $\mathscr{T}_h$. We have

LEMMA 3. *Let $u \in C^1(\bar{Q})$ be a solution of* (5) *and let $u, u_t \in L^2(0, T; H_r(\Omega))$ for some $r \geqslant n/2$. If Theorem* 1 *holds, then*

$$\int_0^T \|u - u_h\|_1^2 dt \leqslant C(u) h^{2r-2},$$

where h is the diameter of the triangulation $\mathscr{T}_h$.

Proof. Let $\Pi_h\colon V \to V_h$ be the interpolation operator corresponding to the triangulation $\mathscr{T}_h$ (see [2]). This operator satisfies the equality

$$[\Pi_h u(\cdot, t)]_t = \Pi_h u_t(\cdot, t) \quad \text{for every } u \in C^1(\bar{Q}).$$

The approximation theorem [2] implies the inequalities

$$\|u(\cdot, t) - \Pi_h u(\cdot, t)\|_1 \leqslant Ch^{r-1}|u(\cdot, t)|_r$$

and

$$\|u_t(\cdot, t) - \Pi_h u_t(\cdot, t)\|_0 \leqslant Ch^r |u_t(\cdot, t)|_r,$$

valid for all $t \in [0, T]$. Hence the desired estimate is obtained by setting $\tilde{u}(\cdot, t) = \Pi_h u(\cdot, t)$ in Theorem 1.

Similarly, we can estimate the right-hand side of the inequality in Theorem 3. We consider a uniform division of $[0, T]$ with the diameter Δt. Let W be a finite-element space of Lagrange-type of order k which corresponds to this division. We have

LEMMA 4. *If $\alpha \in \tilde{H}_1^N(0, T) \cap H_{k+1}^N(0, T)$ is a solution of* (9) *and $\beta \in W^N$ is an approximate solution of* (11), *then there exists a positive constant C such that*

$$\|\alpha-\beta\|_{0,N} \leqslant C(\Delta t)^k |\alpha|_{k+1,N}.$$

Proof. The result follows directly from the approximation theorem formulated in [2].

Using these facts we may obtain an estimation of the error of the continuous approximation (13).

THEOREM 5. *We assume the above lemmas are valid. The error of the continuous approximation* (13) *satisfies*

$$\int_0^T \|u-u_h^*\|_1^2\, dt \leqslant C(u)\left[h^{2r-2}+(\Delta t)^{2k}\right].$$

Proof. Let $u_h(x, t) = \sum_{i=1}^{N} \alpha_i(t) v_i(x)$ be a solution of (6). We have

$$\int_0^T \|u-u_h^*\|_1^2\, dt \leqslant 2\int_0^T \|u-u_h\|_1^2\, dt + 2\int_0^T \|u_h-u_h^*\|_1^2\, dt.$$

The first term of this inequality is obviously estimated by Lemma 3. For the second one we can use Lemma 4. Namely,

$$\int_0^T \|u_h-u_h^*\|_1^2\, dt = \int_0^T dt \int_\Omega \Big[\sum_{i=1}^{N} \big(\alpha_i(t)-\beta_i(t)\big) v_i(x)\Big]^2 dx \leqslant C_1 \int_0^T \sum_{i=1}^{N} \big(\alpha_i(t)-\beta_i(t)\big)^2 dt$$
$$\leqslant C_1 \|\alpha-\beta\|_{0,N}^2 \leqslant C_1 C(\Delta t)^{2k} |\alpha|_{k+1,N}^2.$$

References

[1] C. Bernardi, *Numerical approximation of periodic linear parabolic problems*, SIAM J. Numer. Anal. 19 (1982), pp. 1196–1207.

[2] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland 1978.

[3] J. Douglas, Jr., and T. Dupont, *Galerkin methods for parabolic equations*, SIAM J. Numer. Anal. 7 (1970), pp. 575–626.

[4] J. L. Lions, *Equations différentielles opérationnelles et problèmes aux limites*, Berlin 1961.

[5] A. Olejniczak, *Crank–Nicolson–Galerkin approximation of the periodic solutions of weakly nonlinear parabolic equations*, Zastos. Mat. 18 (1985), pp. 663–680.

INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
51 151 WROCŁAW

Received on 1984.09.17