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ON A MATHEMATICAL MODEL OF A HEAT EXCHANGE PROCESS IN CONDUCTORS

1. Introduction. In this paper a set of non-linear, first-order, ordinary differential equations describing a heat exchange process in conductors is constructed. It is proved that it has solutions with desired properties (smoothness, asymptotic behaviour, stability). A set of admissible initial conditions is found. Simple estimations of temperatures are obtained. Numerical solutions are presented and comparison with experimental data is performed.

It is well known that if a temperature of a conductor insulation exceeds some critical value, its "life time" decreases rapidly. In practice, this unfavourable fact demands to use more of materials and does not allow to put a curcuit to good account. So, it is an important problem of the electrical curcuit design in the industry to have a good knowledge of the dependence on time of the wire and the insulation temperatures. Up to now, a model based on linear differential equations has been used but it gives a bad approximation of the real process, as it has been shown by comparison with experimental data [3]. We propose here a model based on non-linear differential equations, which describes experimental data more accurately.

2. Qualitative description of the model and derivation of equations. For our model we derive equations from the laws of conservation and transformation of energy. We are interested in that part of energy which is transmitted in the form of heat. Let us consider a system composed of a wire and an insulation, both of unit length. The wire plays the role of the source. It generates the energy Q_s with the velocity equal to

$$\frac{dQ_s}{dt} = I^2 R, \tag{1}$$

where I is the current, and R is the resistance. The part Q_1 of the energy Q_s is used to heat the wire and the remainder energy Q_3 is transferred to the insulation. Then the part Q_2 is used to heat the insulation and the part Q_4 is transferred to the environment. Corresponding velocities are given by the equations

$$\frac{dQ_i}{dt} = S_i C_i \frac{dx_i}{dt}, \quad i = 1, 2, \ldots, \tag{2}$$

$$\frac{dq_i}{dt} = k_i F_i (x_i \delta_{i1} - \varepsilon_{i1} x_2), \tag{3}$$

where

$$\delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \qquad \varepsilon_{ik} = \begin{cases} 1, & i = k, \\ -1, & i \neq k, \end{cases}$$

S_i denotes the cross-section, C_i the specific heat, F_i the area of the sample of the unit length, x_i the relative temperature (the temperature of the environment is assumed to be zero) for the wire if $i = 1$ and for the insulation if $i = 2$, and k_i denotes the coefficient of heat exchange between the wire and its insulation if $i = 1$ and between the insulation and the environment if $i = 2$.

We assume that the dependence of the resistance R, the specific heat C_i and the coefficient k_2 of heat exchange between the insulation and the environment on the temperatures is expressed by the equations

$$R(x_1) = R^0 S_1^{-1}(1 + ax_1), \tag{4}$$

$$C_i(x_i) = C_i^0 (1 + \beta_i x_i), \tag{5}$$

$$k_2(x_2) = k_2^0 (1 + \mu x_2), \tag{6}$$

where R^0, C_i^0 and k_2^0 denote the corresponding quantities at the temperature of the environment, and a, β_i and μ are the thermal coefficients. Using (1)-(6) we obtain the following energy balance equations:

$$I^2 S_1^{-1}(1 + ax_1) = S_1 C_1^0 (1 + \beta_1 x_1) \frac{dx_1}{dt} + k_1 F_1 (x_1 - x_2), \tag{7}$$

$$k_1 F_1 x_1 = S_2 C_2 (1 + \beta_2 x_2) \frac{dx_2}{dt} + k_2^0 F_2 (x_2 + \mu x_2^2). \tag{8}$$

Equations (7) and (8) form the set of the first order ordinary differential equations with the two unknown functions x_1 and x_2 of the time variable t. These equations are non-linear. It is the consequence of the assumption about the temperature dependence of some material constants. All parameters appearing in equations (7) and (8) are positive. Finally, we write them in the normal form with help of the more compact notation

$$\frac{dx_1}{dt} = (a - bx_1 + cx_2)(1 + ex_1)^{-1}, \tag{9}$$

$$\frac{dx_2}{dt} = (b' x_1 - c' x_2 - f x_2^2)(1 + e' x_2)^{-1}, \tag{10}$$

where

(11) $$a = I^2 R^0 S_1^{-2} C_1^0,$$

(12) $$b = (k_1 F_1 - I^2 R^0 a S_1^{-1})(S_1 C_1^0)^{-1},$$

(13) $$c = k_1 F_1 (S_1 C_1^0)^{-1},$$

(14) $$b' = k_1 F_1 (S_2 C_2^0)^{-1},$$

(15) $$c' = (k_1 F_1 + k_2^0 F_2)(S_2 C_2^0)^{-1},$$

(16) $$f = \mu F_2 k_2^0 (S_2 C_2^0)^{-1}.$$

Our model is defined by equations (9) and (10). Hence it is completely described by the set of the coefficients expressed by the material constants in formulas (11)-(16). For the experimental values of the material constants, all coefficients are positive. From now on we assume that the typical values of the material constants are chosen from the set of the experimental values in an arbitrary way and fixed. The dependence of the solutions on the parameters will not be marked explicity in future.

3. Requirements for the solutions. Experimental data allow us to assume that the temperatures of the wire and its insulation are smooth functions of time and the material parameters if the current is constant

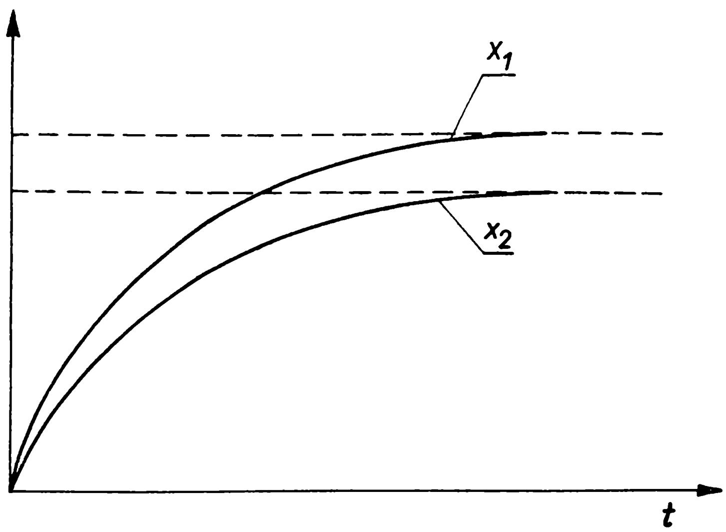

Fig. 1. Typical experimental curves

in time. Typical experimental curves for the direct current are drawn in Fig. 1, the wire temperature x_1 and the insulation temperature x_2 are plotted against time. Under some initial conditions these functions are non-negative, monotonically increasing and tending, as the time t increases to infinity, to some constant value. With the given system (which means that the material constants are fixed) we associate a phase plane defined as a set of points (x_1, x_2), which we call a *state of the system* under consideration. The last property of the experimental curves can be expressed as follows: if the time $t \to \infty$, a state of the system tends to some state with finite, positive coordinates, which we call an *equilibrium state*. Further, it follows from the experiment that an equilibrium state of our system is stable. It means that, after any enough small disturbance of the system being in an equilibrium state, its state returns, as $t \to \infty$, to an equilibrium state value. So, a set of equations which are to be a good model must have, for some set of the coefficients which correspond to the material constants, solutions with the following properties:

1. In the first quadrant of the phase plane there exists an asymptotically stable (in the Lyapunov sense, as $t \to \infty$) equilibrium solution (stationary solution).

2. There exists a set of initial conditions for which the corresponding solutions exist for all $t \geqslant t_0$ and which are
   a. unique,
   b. smooth functions of the parameters and time,
   c. monotonically increasing,
   d. non-negative,
   e. tending to the same stationary solution as $t \to \infty$.

4. Some definitions. The following definitions will be useful:

Definition 1. A solution (x_1, x_2) of equations (9) and (10) whose coordinate functions x_1 and x_2 have properties 2b, c and e for all $t \geqslant t_0$ and all allowed values of the coefficients will be called the *S-solution*.

If, however, property 2c holds only after a sufficiently long lapse of time, (x_1, x_2) will be called *W-solution*.

Definition 2. A closed subset of the phase plane containing an equilibrium point (equilibrium state) of equations (9) and (10) is called the *invariant set* if it has the following property: once any solution falls into this set, it cannot leave this set and it tends to the equilibrium point as $t \to \infty$.

5. Proof of the existence of the solutions with the desired properties.

THEOREM. *For system* (9)-(10) *in the first quadrant of the phase plane, there exists the invariant set W for which the corresponding solutions have property* 2a. *The set W contains a closed subset S for which the corresponding solutions are the S-solutions.*

Solutions corresponding to the set $W - S$ are the W-solutions.

The equilibrium point contained in W is asymptotically stable (*in the Lyapunov sense, as* $t \to \infty$).

Proof. First of all we remark that system (9)-(10) is autonomous. Therefore, our main tool will be the phase plane analysis of the vector field associated with system (9)-(10) and some theorems from the theory of autonomous systems (see [1] and [2]). The proof will be done in several steps.

1° Let us denote the right-hand sides of equations (9) and (10) by $f_1(x; p_1)$ and $f_2(x; p_2)$, respectively, where $x \equiv (x_1, x_2)$ is a point of the phase plane, and $p_1 \equiv (a, b, c)$ and $p_2 \equiv (b', c', f)$ are sets of the coefficients. The functions $f_1(x; p_1)$ and $f_2(x; p_2)$, obviously, belong to C^∞-class in the product of the first quadrant of the coordinate system with the origin placed at the point $(-e^{-1}, -e'^{-1})$ and the sets of the allowed values of the parameters p_1 and p_2, respectively. So, it follows from the well-known theorems that equations (9) and (10) have locally the unique solution satisfying the initial condition $x(t_0) = x_0$ and belonging to the C^∞-class with respect to the time variable and the parameters p_1 and p_2.

2° System (9)-(10) has only two equilibrium points x^e and y^e, i.e. the solutions of the equations

$$f_1(x; p_1) = 0 \quad \text{and} \quad f_2(x; p_2) = 0.$$

Their coordinates are given by the following formulas:

$$x_1^e = ab^{-1} + cb^{-1}x_2^e,$$

$$x_2^e = \{b'c - bc' + [(b'c - bc')^2 + 4abb'f]^{1/2}\}(2bf)^{-1},$$

$$y_1^e = ab^{-1} + cb^{-1}y_2^e,$$

$$y_2^e = \{b'c - bc' - [(b'c - bc')^2 + 4abb'f]^{1/2}\}(2bf)^{-1}.$$

The point x^e belongs to the first quadrant.

3° Let us show that the set

$$W = \{x\colon\ 0 \leqslant x_1 \leqslant x_1^e,\ 0 \leqslant x_2 \leqslant x_2^e\}$$

is invariant.

Indeed, let us draw the curves given by the equations $f_1(x; p_1) = 0$ and $f_2(x; p_2) = 0$.

We consider the subset (see Fig. 2)

$$S = \{x\colon\ x_1 \geqslant 0,\ x_2 \geqslant 0\} \cap \{x\colon\ f_1(x; p_1) \geqslant 0,\ f_2(x; p_2) \geqslant 0\}$$

and draw the vector field $f \equiv (f_1, f_2)$ on the boundary of S. The set S contains the equilibrium point x^e. The arrows of the drawn vector field point into the interior of S. Now, it is clear that, once any solution falls into S, it cannot leave it. In particular, if the initial conditions belong to S, then the whole phase plane trajectory belongs to it. Moreover, if the initial conditions belong to the set $W-S$, then, in view of the sign of the vector-field coordinates and the geometrical relation between S and W (see Fig. 2), after a sufficiently large (but finite) lapse of time, the solutions fall into S.

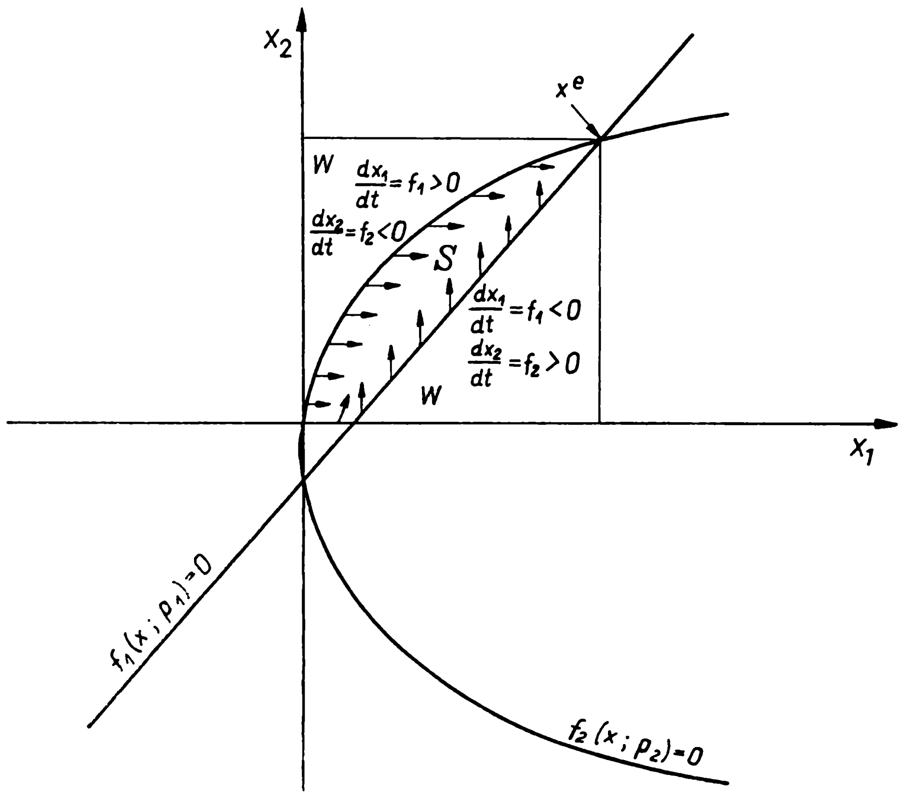

Fig. 2. The phase plane picture of system (9)-(10)

4° In view of 3°, every solution corresponding to W is bounded, so it can be continued onto the whole time interval $[t_0, \infty]$.

Summing up, for every initial condition $x(t_0) = x_0 \in W$, there exists a unique solution defined for all $t \geqslant t_0$.

5° It follows from the uniqueness that the equilibrium point x^e cannot be reached in any finite time.

6° We infer from 3°, 4° and 5° that if the initial condition $x(t_0) = x_0 \in S$, then the corresponding solution is the S-solution, and if $x(t_0) = x_0 \in W \setminus S$, then the corresponding solution is the W-solution.

7° Simple calculations show that all eigenvalues of the Jacobi matrix $f'(x^e)$ have the real parts negative, so the equilibrium solution x^e is asymptotically stable (in the Lyapunov sense, as $t \to \infty$), q.e.d.

For a fixed x_1 the following estimation holds:

$$-ac^{-1}+bc^{-1}x_1 \leqslant x_2 \leqslant -c'(2f)^{-1}+[c'^2+4b'fx_1]^{1/2}(2f)^{-1} \quad \text{if } x \in S.$$

This estimation is useful since x_2 cannot be measured. So it allows one to predict quite well the value of x_2 if x_1 is known from the measurement.

6. Numerical results and a comparison with experimental data. We have used the analog computer Meda which has a unit equal to 10 V. In order to build the program, equations (9) and (10) have been written in the following form:

$$\frac{dx_1}{dt} = A + Bx_1 - Cx_1 \frac{dx_1}{dt} + Dx_2,$$

$$\frac{dx_2}{dt} = A' x_1 - B' x_2 - C' x_2^2 - D' x_2 \frac{dx_2}{dt}.$$

The program which we have used is shown in Fig. 3. However, due to the small unit of our analog computer, the errors were too big and the results we have obtained have not been satisfactory. A computer with the unit equal to 100 V is necessary. The program based on the Procedure Zonnenwald 5 has been built for the digital computer Odra 1304.

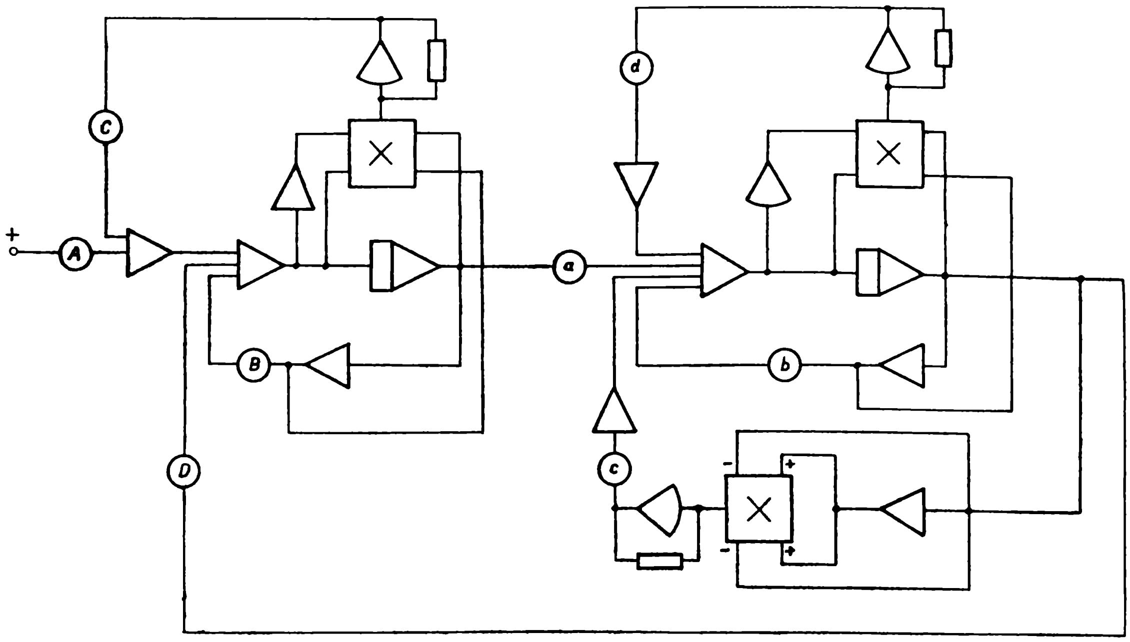

Fig. 3. The analog computer program

Fig. 4 shows that we have obtained excellent agreement with the experimental data.

7. Final remarks. Some explanation connected with Fig. 4 is needed. Not all values of the material constants contained in equations (11)-(16) are known. Some of them are difficult to be measured exactly or even have not been measured, for example the insulation specific heat C_2 and the coefficient β_2 of the temperature dependence on C_2 (see equation (5)). By fitting a model curve to the experimental one it is possible to find these constants.

It is clear that our result can easily be extended to some more complicated vectors fields $f(x)$, for example one can include in (4)-(6) the dependence on quadratic terms x_1^2 or x_2^2.

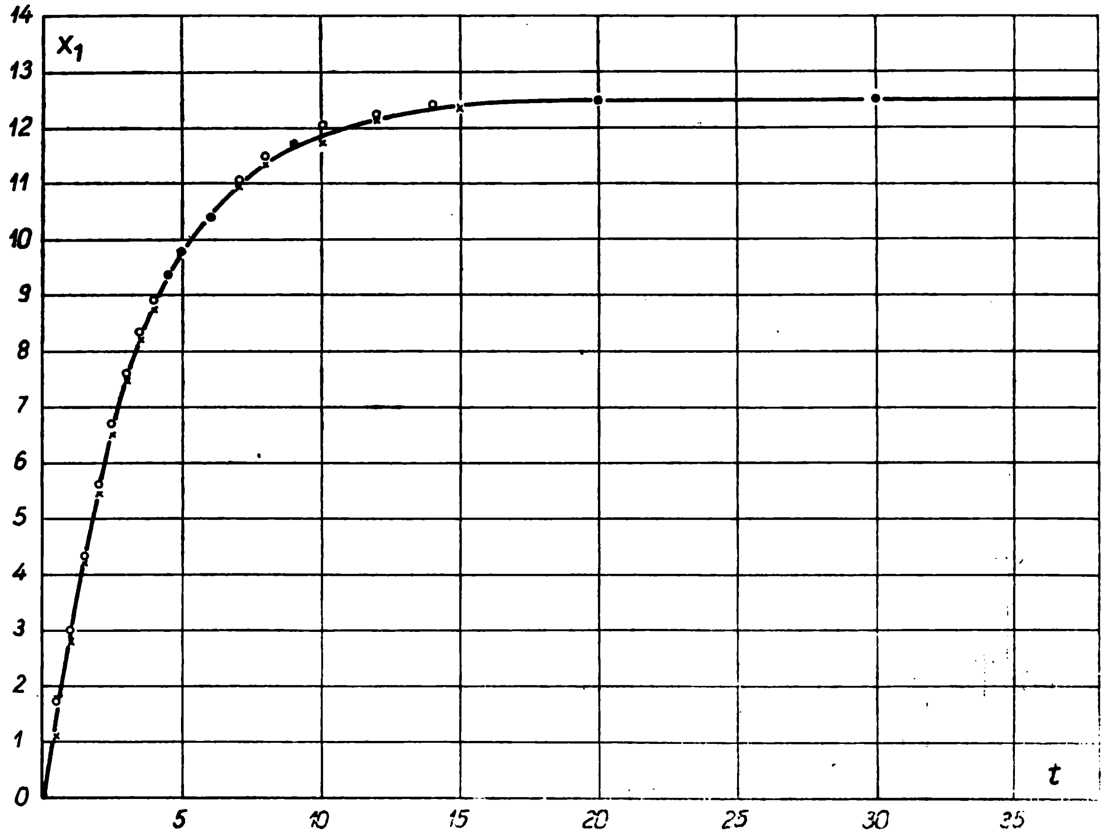

Fig. 4. The dependence of time on the wire temperature under the following conditions: the wire 1 × DY 6 in the air, $I = 32$ A

○ — points of the experimental curve, × — points of the digital computer curve; the temperature unit is 1°C and the time unit is the minute

Further, our interpretation of (9) and (10) as the equations describing the heat exchange process is of no importance. Equally well, they can describe charge exchange processes in electrical curcuits or matter exchange processes in chemical systems or some phenomena in biological systems. In practice, alternating currents are of great importance; these currents, as functions of time, are "step" type functions (Fig. 5). Under some conditions put on the jumps of the function $I(\cdot)$ at the points t_n one can prove, by similar considerations, the existence of solutions which are continuous functions of time and parameters, but which are no longer differentiable functions. The example of such a solution is drawn in Fig. 6.

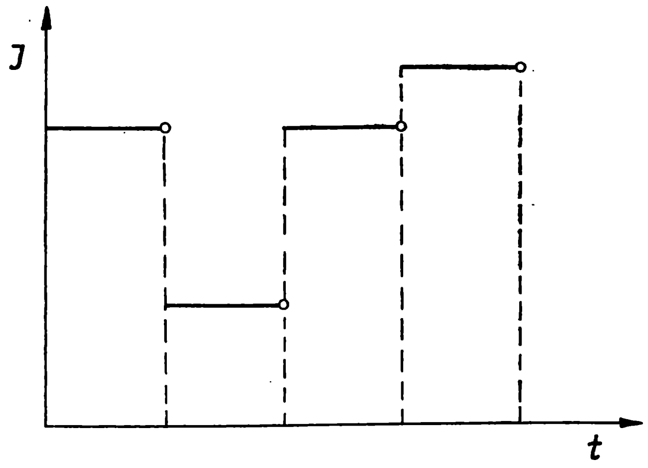

Fig. 5

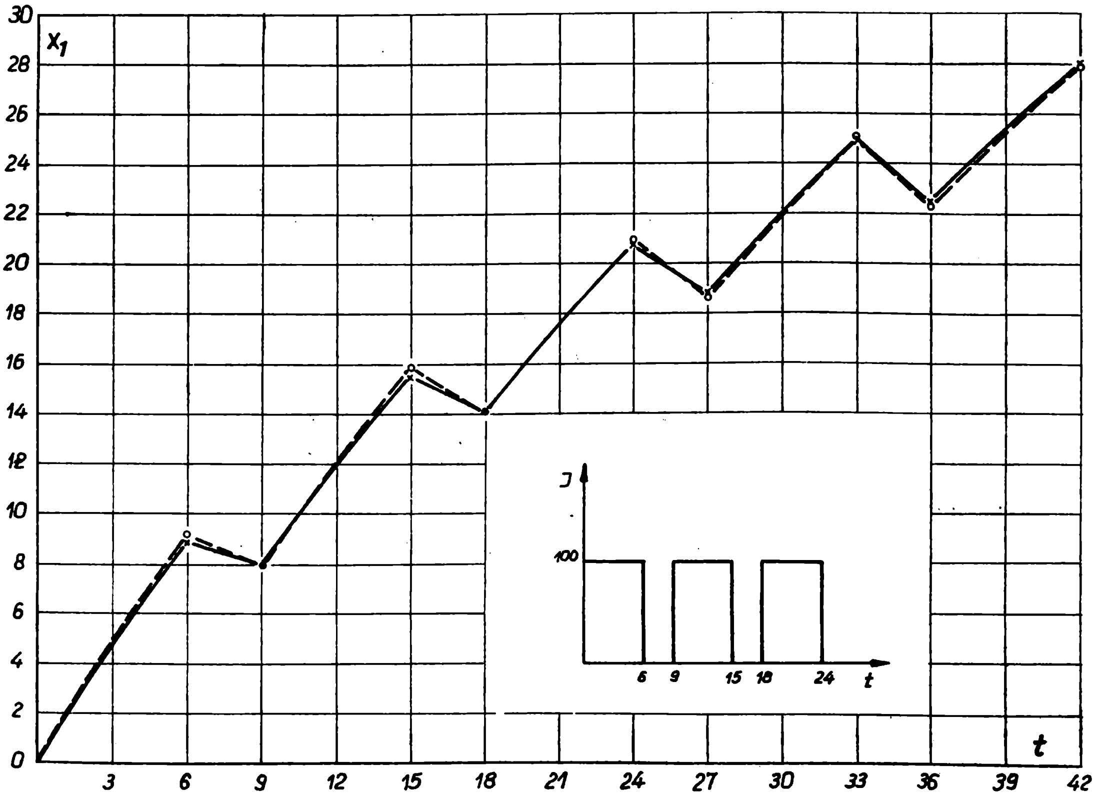

Fig. 6. The dependence of time on the wire temperature under the following conditions: the wire $6 \times$ LY 25, the insulation RL47, the current changes in time

— the experimental curve, - - - the digital computer curve; the current unit is the ampère, and the time unit is the minute

8. Appendix. Here we give the list of the typical values of all parameters which have appeared in the considerations.

The values of the coefficients of equations (9)-(10) under the following conditions: the wire $1 \times DY6$ in the air, $I = 32$ A

a	b	c	e
0.1404	0.1418	0.1423	0.000447
b'	c'	f	e'
0.1492	0.1630	0.0000488	0

The values of the material constants for the wire $1 \times DY6$ in the air

r_1 [cm]	r_2 [cm]	R^0 [Ω cm]	a [°C^{-1}]	C_1^0 [Ws °C^{-1} cm^{-3}]	β_1 [°C^{-1}]
0.138	0.27	0.00000175	0.003863	3.43	0.000447

λ [W °C^{-1}cm^{-1}]	C_2^0 [Ws °C^{-1}cm^{-3}]	β_2 [°C^{-1}]	k_2^0 [W cm^{-2} °C^{-1}]	μ [°C^{-1}]
0.001538	2.3	0	0.00176	0.003538

r_1 and r_2 are the radii of the wire and the insulation, appropriately, and λ is the coefficient of thermal conductivity of the insulation. They have been used in the following formulas to calculate the coefficient k_1:

$$k_1 = \frac{\lambda}{r_1 \ln(r/r_1)},$$

$$r = \exp\left[\frac{r_2^2(2\ln r_2 - 1) - r_1^2(2\ln r_1 - 1)}{2(r_2^2 - r_1^2)}\right].$$

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MATEMATYCZNY MODEL PROCESU WYMIANY CIEPŁA W PRZEWODNIKACH

STRESZCZENIE

Konstruujemy model układu żyła-izolacja, odtwarzający zależność temperatury żyły i izolacji od czasu. Model oparty jest na układzie nieliniowych równań różniczkowych zwyczajnych pierwszego rzędu. Dowodzimy, że układ ten ma rozwiązania o żądanych własnościach (gładkość, zachowanie asymptotyczne, stabilność). Znajdujemy zbiór warunków początkowych, dla których rozwiązania mają te własności. Podajemy proste oszacowanie temperatury izolacji w zależności od temperatury żyły. Porównanie rozwiązań numerycznych z krzywymi doświadczalnymi pokazuje ich bardzo dobrą zgodność.