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ON THE NUMERICAL SOLUTION OF AN ABEL INTEGRAL EQUATION

1. Introduction. Let us consider the Abel integral equation of the form

$$g(t) = 2\int_t^R \frac{sf(s)}{\sqrt{s^2-t^2}}\,ds, \quad t \in [0, R], \tag{1}$$

where f is an unknown function. This equation is of importance in practice, e.g. in the study of the thermodynamic states of axially symmetric radiating plasma columns [3]. It is well known that equation (1) takes the following inversion form:

$$f(s) = -\frac{1}{\pi}\int_s^R \frac{g'(t)}{\sqrt{t^2-s^2}}\,dt, \quad s \in [0, R]. \tag{2}$$

In the sequel we assume that the function g is tabulated at a finite number of points t_i, $0 = t_1 < t_2 < \ldots < t_n = R$. Analogously as in [5], instead of the function f we determine the function f_Δ defined by

$$f_\Delta(s) = -\frac{1}{\pi}\int_s^R \frac{g'_\Delta(t)}{\sqrt{t^2-s^2}}\,dt, \quad s \in [0, R], \tag{3}$$

where g_Δ is a spline function of degree $m = 2l-1$ $(1 < l \leqslant n)$ interpolating the function g on the network Δ determined by t_i. In this paper we present a method for determining approximations f_Δ of f based on (3) and compare our method with the two methods from [2] and [3].

2. Numerical method. It is well known that the function g_Δ can be represented in the form

$$g_\Delta(t) = \sum_{i=0}^{m} a_i t^i + \sum_{j=1}^{n} \beta_j \theta(t, t_j)(t-t_j)^m, \tag{4}$$

where

$$\theta(t, t_j) = \begin{cases} 0 & \text{if } t < t_j, \\ 1 & \text{if } t \geqslant t_j. \end{cases}$$

By substituting (4) into (3) we obtain

$$f_\Delta(s) = \sum_{i=1}^{m} a_i a_i(s) + \sum_{j=1}^{n} \beta_j b_j(s), \tag{5}$$

where

$$a_i(s) = -\frac{i}{\pi}\int_s^R \frac{t^{i-1}}{\sqrt{t^2-s^2}}\,dt, \quad i = 1, 2, \ldots, m,$$

and

$$b_j(s) = -\frac{m}{\pi}\int_s^R \frac{\theta(t, t_j)(t-t_j)^{m-1}}{\sqrt{t^2-s^2}}\,dt, \quad j = 1, 2, \ldots, n.$$

Now, we determine the analytical formulae for the functions $a_i(s)$ and $b_j(s)$. For this purpose we need the integrals

$$\int_u^R \frac{t^{2i-1}}{\sqrt{t^2-s^2}}\,dt = \sum_{v=1}^{i} \frac{\binom{i-1}{v-1}}{2i-2v+1}\, s^{2v-2}\,[(\sqrt{R^2-s^2})^{2i-\ v+1} - (\sqrt{u^2-s^2})^{2i-2v+1}],$$
$$i = 1, 2, \ldots,$$

and

$$\int_u^R \frac{t^{2i}}{\sqrt{t^2-s^2}}\,dt = \sum_{v=1}^{i} \frac{(2i-1)^{[v-1]}}{(2i)^{[v]}}\, s^{2v-2}\,[\sqrt{R^2-s^2}R^{2i-2v+1} - \sqrt{u^2-s^2}u^{2i-2v+1}] +$$
$$+ \frac{(2i-1)^{[i-1]}}{(2i)^{[i]}}\, s^{2i}\ln\frac{R+\sqrt{R^2-s^2}}{u+\sqrt{u^2-s^2}}, \quad i = 0, 1, \ldots,$$

where

$$n^{[i]} = \begin{cases} 1 & \text{if } i \leqslant 0, \\ n(n-2)\ldots(n-2i+2) & \text{otherwise}. \end{cases}$$

Using the integrals given in [4], p. 86-87, we can prove the formulae above by induction.

Now we obtain

$$a_i(s) = -\frac{i}{\pi}\sum_{v=1}^{k} \frac{\binom{k-1}{v-1}}{i-2v+1}\, s^{2v-2}\,(\sqrt{R^2-s^2})^{i-2v+1}, \tag{6}$$
$$i = 2, 4, \ldots, m-1,$$

and

$$(7)\qquad a_i(s) = -\frac{i}{\pi}\sqrt{R^2-s^2}\sum_{v=1}^{k}\frac{(i-2)^{[v-1]}}{(i-1)^{[v]}}s^{2v-2}R^{i-2v}+ + \frac{(i-2)^{[k-1]}}{(i-1)^{[k]}}s^{i-1}\ln\frac{R+\sqrt{R^2-s^2}}{s}, \qquad i = 1, 3, \ldots, m,$$

where $k = \text{entier}(i/2)$.

Since

$$\int_s^R \frac{\theta(t,t_j)(t-t_j)^{m-1}}{\sqrt{t^2-s^2}}\,dt = \begin{cases}\displaystyle\int_s^R \frac{(t-t_j)^{m-1}}{\sqrt{t^2-s^2}}\,dt & \text{if } s \geqslant t_j,\\ \displaystyle\int_{t_j}^R \frac{(t-t_j)^{m-1}}{\sqrt{t^2-s^2}}\,dt & \text{if } s < t_j,\end{cases}$$

we have

$$(8)\qquad b_j(s) = -\frac{m}{\pi}\sum_{i=0}^{m-1}\binom{m-1}{i}(-t_j)^{m-i-1}p_i(s), \qquad j = 1, 2, \ldots, n,$$

where for $i = 0, 2, \ldots, m-1$

$$(9)\qquad p_i(s) = \begin{cases}\displaystyle\sqrt{R^2-s^2}\sum_{v=1}^{k}\frac{(i-1)^{[v-1]}}{i^{[v]}}s^{2v-2} + \\ \displaystyle\quad + \frac{(i-1)^{[k-1]}}{i^{[k]}}s^i\ln\frac{R+\sqrt{R^2-s^2}}{s} & \text{if } s \geqslant t_j,\\ \displaystyle\sum_{v=1}^{k}\frac{(i-1)^{[v-1]}}{i^{[v]}}s^{2v-2}\left(\sqrt{R^2-s^2}R^{i-2v+1} - \sqrt{t_j^2-s^2}\,t_j^{i-2v+1}\right) + \\ \displaystyle\quad + \frac{(i-1)^{[k-1]}}{i^{[k]}}s^i\ln\frac{R+\sqrt{R^2-s^2}}{t_j+\sqrt{t_j^2-s^2}} & \text{if } s < t_j,\end{cases}$$

and for $i = 1, 3, \ldots, m-2$

$$(10)\qquad p_i(s) = \begin{cases}\displaystyle\sum_{v=1}^{k+1}\frac{\binom{k}{v-1}}{i-2v+2}s^{2v-2}\left(\sqrt{R^2-s^2}\right)^{i-2v+2} & \text{if } s \geqslant t_j,\\ \displaystyle\sum_{v=1}^{k+1}\frac{\binom{k}{v-1}}{i-2v+2}s^{2v-2}\left[\left(\sqrt{R^2-s^2}\right)^{i-2v+2} - \left(\sqrt{t_j^2-s^2}\right)^{i-2v+2}\right] \\ & \text{if } s < t_j,\end{cases}$$

and $k = \text{entier}(i/2)$.

We note that

(11) $$\lim_{s\to 0+} a_i(s) = -\frac{iR^{i-1}}{\pi(i-1)}, \quad i = 2, 3, \ldots, m,$$

(12) $$\lim_{s\to 0+} b_1(s) = -\frac{mR^{m-1}}{(m-1)\pi},$$

and

$$\lim_{s\to 0+} a_1(s) = \infty.$$

Therefore, $f_\Delta(s)$ for $s \neq 0$ is determined by equalities (5)-(10) and, additionally, if $a_1 = g'_\Delta(0) = 0$, then it is also defined for $s = 0$ by (5)-(12). In the last case, the value $f_\Delta(0)$ is defined by

$$f_\Delta(0) = \lim_{s\to 0+} f_\Delta(s).$$

Let the number $e(g', \Delta)$ be defined by

$$\sup\{|g'(t) - g'_\Delta(t)|:\ t \in [0, R]\} \leqslant e(g', \Delta).$$

Then from (2) and (3) we obtain the estimation

$$|f(s) - f_\Delta(s)| \leqslant \frac{e(g', \Delta)}{\pi} \ln \frac{R + \sqrt{R^2 - s^2}}{s} \quad \text{for all } s \neq 0.$$

Hence and from the form of $e(g', \Delta)$ (see, e.g., [1]) we infer that for all $s \neq 0$

$$\lim_{|\Delta|\to 0} f_\Delta(s) = f(s), \quad \text{where } |\Delta| = \max\{|t_i - t_{i-1}|:\ i = 2, 3, \ldots, n\}.$$

Moreover, since $e(g', \Delta) = O(|\Delta|^a)$, where $a > 0$, we have

$$\lim_{s,|\Delta|\to 0+} e(g', \Delta)\ln s = 0 \quad \text{and} \quad \lim_{|\Delta|,s\to 0+} f_\Delta(s) = \lim_{s\to 0+} f(s).$$

Therefore, if the solution $f(s)$ is a continuous function for all $s \in [0, R]$, then choosing $a_1 = 0$ we conclude that

$$\lim_{|\Delta|\to 0} f_\Delta(s) = f(s) \quad \text{for all } s \in [0, R].$$

Finally, for all s our method is convergent, and for $s \neq 0$ it has the same order of convergence as the order of convergence of $e(g', \Delta)$ to zero.

3. Numerical results. For calculations of α_i and β_j we use the numerically stable method proposed in [5]. At first, we choose the function g in (1) as in [3], p. 1059. It is given at 31 points $t_{i+1} = i/30$, $i = 0, 1, \ldots, 30$. This function is tabulated in Table 1 from [3] with 3 exact decimal places after the point. In Table 1 we list the error $f(t_i) - f_\Delta(t_i)$ for our method (column I), for the method from [2] (column II) and from [3] (column III). In our calculations we have used $m = 3$ and $g'_\Delta(0) = g'_\Delta(1) = 0$. Note that our method gives the best results. It is remarkable that the solutions $f_\Delta(t_i)$ have the same number of exact places after the point as the given data $g(t_i)$, $i = 1, 2, \ldots, 31$. Therefore, our method is more useful in practice than the others compared here.

TABLE 1

k	I	II	III	k	I	II	III
0	0.0002	−0.0029	0.0018	16	−0.0001	0.0003	0.0000
1	0.0002	−0.0018	0.0003	17	0.0001	−0.0001	0.0000
2	−0.0003	−0.0003	−0.0012	18	0.0000	−0.0000	−0.0003
3	−0.0001	−0.0001	−0.0012	19	−0.0001	0.0000	−0.0004
4	0.0001	0.0002	0.0000	20	−0.0001	0.0000	−0.0003
5	0.0002	0.0007	0.0036	21	0.0001	−0.0004	−0.0002
6	−0.0000	0.0015	−0.0035	22	−0.0000	−0.0002	0.0000
7	0.0001	0.0017	−0.0073	23	0.0000	−0.0005	−0.0001
8	−0.0002	0.0009	−0.0029	24	0.0000	−0.0005	−0.0004
9	0.0001	0.0007	0.0013	25	−0.0000	−0.0005	−0.0004
10	−0.0000	0.0007	0.0029	26	0.0001	−0.0008	−0.0001
11	−0.0001	0.0007	−0.0001	27	−0.0001	−0.0005	0.0003
12	−0.0001	0.0006	−0.0003	28	0.0001	−0.0008	0.0003
13	0.0001	0.0004	−0.0004	29	0.0001	−0.0010	−0.0004
14	0.0000	0.0003	−0.0003	30	0.0000	0.0000	0.0000
15	−0.0001	0.0003	−0.0002				

Secondly, we choose the function g in (1) equal to

$$g(t) = \begin{cases} \dfrac{32}{27}\sqrt{1-t^2}(1-7t^2) + \sqrt{\dfrac{1}{16}-t^2}\left(\dfrac{1}{108}+\dfrac{566}{27}t^2\right) - \\ -24t^4\ln\dfrac{1/4+\sqrt{1/16-t^2}}{t} + \dfrac{32}{9}(t^2+t^4)\ln\dfrac{1+\sqrt{1-t^2}}{1/4+\sqrt{1/16-t^2}} \\ \qquad\qquad \text{if } 0 \leqslant t \leqslant 1/4, \\ \dfrac{32}{27}\left[\sqrt{1-t^2}(1-7t^2) + 3t^2\ln\dfrac{1+\sqrt{1-t^2}}{t}(1+t^2)\right] \\ \qquad\qquad \text{if } 1/4 < t \leqslant 1. \end{cases}$$

For this function g the solution f is given by

$$f(s) = \begin{cases} -32s^3+12s^2+\dfrac{3}{4} & \text{if } 0 \leqslant s \leqslant \dfrac{1}{4}, \\ \dfrac{16}{27}(8s^3-15s^2+6s+1) & \text{if } \dfrac{1}{4} < s \leqslant 1. \end{cases}$$

In Table 2 we list the errors $f(s_i) - f_\Delta(s_i)$ for $s_i = i/10, i = 0, 1, \ldots, 10$. Additionally, we give the time of calculations in seconds. For these calculations we have taken $g'_\Delta(0) = g'_\Delta(1) = 0$, $m = 3$, and t_{i+1} equal to $i/50$ (column I), $i/100$ (column II), $i/200$ (column III) and $i/400$ (column IV) for $i = 0, 1, \ldots$

TABLE 2

s	I	II	III	IV
0.0	$6.7_{10}-5$	$1.0_{10}-5$	$1.5_{10}-6$	$2.2_{10}-7$
0.1	$-3.4_{10}-7$	$-5.2_{10}-8$	$-3.3_{10}-9$	$-4.0_{10}-10$
0.2	$1.9_{10}-6$	$-4.8_{10}-7$	$-3.3_{10}-8$	$-2.3_{10}-9$
0.3	$-3.4_{10}-6$	$-4.3_{10}-8$	$-1.5_{10}-9$	$9.0_{10}-10$
0.4	$-3.0_{10}-8$	$-4.0_{10}-9$	$5.0_{10}-10$	$7.0_{10}-10$
0.5	$4.7_{10}-8$	$3.2_{10}-9$	$1.1_{10}-9$	$1.0_{10}-9$
0.6	$1.2_{10}-7$	$9.7_{10}-9$	$6.0_{10}-10$	$4.0_{10}-10$
0.7	$2.0_{10}-7$	$1.8_{10}-8$	$2.7_{10}-9$	$9.0_{10}-10$
0.8	$3.5_{10}-7$	$3.1_{10}-8$	$3.7_{10}-9$	$-5.0_{10}-10$
0.9	$8.9_{10}-7$	$7.1_{10}-8$	$6.7_{10}-9$	$-3.0_{10}-10$
1.0	0	0	0	0
Time of calculations	34	69	133	268

All the calculations were performed on the ODRA 1204 computer in single precision.

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O NUMERYCZNYM ROZWIĄZYWANIU PEWNEGO CAŁKOWEGO RÓWNANIA ABELA

STRESZCZENIE

W niniejszej pracy przedstawiono numeryczną metodę rozwiązywania równania całkowego Abela (1), dokładniejszą niż metody z prac [2] i [3]. Ponadto udowodniono zbieżność rozwiązania przybliżonego $f_\Delta(s)$ do dokładnego rozwiązania $f(s)$. Zostały także podane dwa przykłady numeryczne.