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QUEUEING SYSTEMS WITH MIXED INPUT STREAM AND FEEDBACK

0. Introduction. Consider the queueing system $(M, GI)/M/\infty$ with feedback between the intensities of the input stream and service, and the number of units being in the system. The input stream in this system is a mixture of two streams: a Poissonian stream with momentary intensity λ_n $(n = 0, 1, \ldots)$ provided n units are in the system, and a GI (Palm) stream with interarrival distribution $H(x)$ and expected value

$$\frac{1}{a} = \int_0^\infty x dH(x) < \infty.$$

Let us assume that the GI-stream is a batched one in which $p_{n,j}$ $(j = 0, 1, \ldots, \sum_{j=0}^{\infty} p_{n,j} = 1,\ n = 0, 1, \ldots)$ denotes the probability of j arrivals in one batch if n units are in the system. Let μ_n $(n = 1, 2, \ldots)$ denote the momentary service intensity in all service channels provided n units are in the system at that moment. Special cases of this system have been considered in Young [10] by the method of some approximations, in Ryba [7] and [8] by the method of extended Markov processes, and in Kuczura [5] and [6] by the method of piecewise Markov processes.

Generally, the mixed input stream is not a GI-stream except when both mixed streams are Poissonian. Thus the process $n(t)$, defined as the number of units in the system at a moment t, is not Markovian with exception of the previously mentioned case. The considered system can, however, be analyzed by the method of imbedded Markov chains. Let $\{S_r\}$ be the sequence of arrivals in the GI-stream. Then the sequence $\{n(S_r - 0)\}$ is an imbedded Markov chain.

Let us notice that the method of imbedded Markov chains gives the characteristics of the process $n(t)$ in selected moments only, thus, the interesting characteristics of the process $n(t)$ in continuous time can be obtained by the additional effort. The stationary probability distributions of the states of the system immediately before arrival moments of the units can be used to find the characteristics of special cases of waiting time. Thus, the investigation of probability distributions of the state of some imbedded Markov chains has some value also in these cases where the probability distribution of the states of the system in continuous time is known.

1. General results. Assume that the process $n(t)$ is stationary and let us write

$$P(n) = \Pr\big(n(t) = n\big), \quad P^-(n) = \Pr\big(n(S_r - 0) = n\big), \quad n = 0, 1, \ldots$$

The following theorem deals with the relations between probabilities $\{P(n)\}$ and $\{P^-(n)\}$ by the method of extended Markov processes (see, for comparison, [3] and [4]):

THEOREM 1. *Consider the system $(M, GI)/M/\infty$ with feedback between the intensities of input stream and service, and the number of units being in the system. If $H(0+) = 0$, then the probabilities $\{P(n)\}$ and $\{P^-(n)\}$, provided they exist, are related by the expressions*

$$\sum_{i=0}^{n-1} \frac{a}{\mu_{i+1}} \varrho_{i+1} \cdots \varrho_{n-1} \sum_{k=0}^{i} r_{k,i-k} P^-(k) = P(n) - \varrho_0 \varrho_1 \cdots \varrho_{n-1} P(0), \quad n = 1, 2, \ldots,$$

(1)

$$P(0) = \frac{1}{\sum_{n=0}^{\infty} \varrho_0 \cdots \varrho_{n-1}} \left[1 - \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \frac{a}{\mu_{i+1}} \varrho_{i+1} \cdots \varrho_{n-1} \sum_{k=0}^{i} r_{k,i-k} P^-(k)\right],$$

where the empty product is 1, $\varrho_n = \lambda_n/\mu_{n+1}$, $n = 0, 1, \ldots$ (if $\mu_{n+1} = \infty$, then $\varrho_n = 0$), and

$$r_{k,m} = \sum_{i=1}^{\infty} p_{k,m+i}, \quad k, m = 0, 1, \ldots$$

Proof. Let $X(t)$ denote the time to the nearest arrival in the GI input stream. The stochastic process $\big(n(t), X(t)\big)$ is Markovian. If it is stationary, then an analysis of the states of the system at the moments $t+h$ ($h > 0$) and t leads us to the following difference equations (see, for comparison, Gnedenko and Kovalenko [1]):

$$\begin{aligned} P(n;\, x) &= \Pr\big(n(t) = n,\, X(t) < x\big) \\ &= (1 - \lambda_n h - \mu_n h)[P(n;\, x+h) - P(n;\, h)] + \\ &+ \sum_{i=0}^{n} P(i;\, h) p_{i,n-i} H(x + \theta h) + \lambda_{n-1} h [P(n-1;\, x+h) - P(n-1;\, h)] + \\ &+ \mu_{n+1} h [P(n+1;\, x+h) - P(n+1;\, h)] + o(h), \end{aligned}$$

$$0 < \theta < 1, \; n = 0, 1, \ldots \; (P(-1;\, x) \equiv 0, \mu_0 = 0, \lambda_{-1} = 0).$$

An appropriate transformation and taking limits with $h \to 0$ lead us to

(2) $$P'(n;\ x) - P'(n;\ 0) - (\lambda_n + \mu_n)P(n;\ x) + \sum_{i=0}^{n} P'(i;\ 0) p_{i,n-i} H(x) + \\ + \lambda_{n-1} P(n-1;\ x) + \mu_{n+1} P(n+1;\ x) = 0, \qquad n = 0, 1, \ldots$$

If $x \to \infty$ holds, then $P(n;\ x) \to P(n)$ and $P'(n;\ x) \to 0$. Taking the limit as $x \to \infty$, we obtain from (2)

(3) $$-(\lambda_n + \mu_n)P(n) + \mu_{n+1}P(n+1) + \lambda_{n-1}P(n-1) - P'(n;\ 0) + \\ + \sum_{i=0}^{n} p_{i,n-i} P'(i;\ 0) = 0, \qquad n = 0, 1, \ldots$$

For the stationary GI (Palm) stream, we have

$$P(x) = \Pr\{X(t) < x\} = \sum_{n=0}^{\infty} P(n;\ x) = a \int_0^x \big(1 - H(u)\big)\,du,$$

whence

$$P'(n;\ 0) = \lim_{h \to 0} \frac{1}{h} P(n;\ h)$$

$$= \lim_{h \to 0} \frac{1}{h} \Pr\{X(t) < h\} \Pr\{n(t) = n \mid X(t) < h\} = aP^-(n), \qquad n = 0, 1, \ldots$$

From this and from (3), equality (1) follows. This completes the proof of Theorem 1.

COROLLARY 1. *If in the $(M, GI)/M/\infty$ system defined in Theorem 1 the GI input stream is simple, i.e. if*

$$p_{n,i} = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

then

$$r_{k,i-k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Hence

$$\sum_{i=0}^{n-1} \frac{a}{\mu_{i+1}} \varrho_{i+1} \cdots \varrho_{n-1} P^-(i) = P(n) - \varrho_0 \cdots \varrho_{n-1} P(0), \qquad n = 1, 2, \ldots,$$

$$P(0) = \frac{1}{\sum_{n=0}^{\infty} \varrho_0 \cdots \varrho_{n-1}} \left[1 - \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \frac{a}{\mu_{i+1}} \varrho_{i+1} \cdots \varrho_{n-1} P^-(i)\right].$$

COROLLARY 2. *If in the $(M, GI)/M/N$ system with losses the GI input stream is simple and*

$$\mu_k = \begin{cases} k\mu, & k = 1, 2, \ldots, N, \\ \infty, & k = N+1, N+2, \ldots, \end{cases}$$

$$\lambda_k = \lambda, \quad k = 0, 1, \ldots,$$

then

$$\frac{a}{\mu}\sum_{i=0}^{n-1}\frac{i!}{n!}\varrho^{n-i-1}P^{-}(i) = P(n) - \frac{\varrho^n}{n!}P(0), \quad n = 1, 2, \ldots, N,$$

$$P(0) = \frac{1}{\sum\limits_{n=0}^{N}(\varrho^n/n!)}\left[1 - \frac{a}{\mu}\sum_{n=0}^{N-1}\sum_{i=0}^{N-n-1}\frac{n!}{(n+i+1)!}\varrho^i P^{-}(n)\right],$$

where $\varrho = \lambda/\mu$ (see [8]).

COROLLARY 3. *If in the $(M, GI)/M/1$ system (without feedback) the GI arrival stream is batched, i.e. if $\lambda_k = \lambda$, $\mu_{k+1} = \mu$, $p_{k,j} = p_j$ with $k, j = 0, 1, \ldots$, then*

$$\frac{a}{\mu}\sum_{i=0}^{n-1}\varrho^{n-i-1}\sum_{k=0}^{i} r_{i-k}P^{-}(k) = P(n) - \varrho^n P(0), \quad n = 1, 2, \ldots,$$

$$P(0) = (1-\varrho)\left[1 - \frac{a}{\mu}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\varrho^{n-i-1}\sum_{k=0}^{i} r_{i-k}P^{-}(k)\right],$$

where

$$r_m = \sum_{i=1}^{\infty} p_{m+i}, \quad m = 0, 1, \ldots$$

2. Repairman problem with priorities. In Takács' machine repair system (see Takács [9]), the idle time periods of the repairman can be used for service of additional units. Assume that in the considered system the machines have preemptive priority over the additional units; the unit whose service has been interrupted by a machine failure rejoins the queue. In such a system the additional units do not influence the machine repair process and service of additional units can be done in the time intervals in which all machines are working.

Let in Takács' machine repair system there are $N+1$ machines and let λ denote the failure intensity. Let ν_n be the momentary input intensity of additional units provided n units are in the system at that moment. Consider the state of the process $n(t)$, defined as the number of additional units in the system. It is convenient to investigate this process in the modified time t', i.e. during the time of all machines being working. The time intervals in which the repairman is busy with servicing machines are contracted to points; this transformation gives also the batch arrival of additional units which, really, have arrived during the repairman's busy period. In the modified time t', the input stream of additional units is a batched Poisson stream with feedback between the arrival intensity and the number of units being in the system.

Let $p_{n,j}$ $(n, j = 0, 1, \ldots)$ denote the probability of j arrivals in one batch, e.g. in the repairman busy period with servicing machines under the condition that n units are in the system on the beginning of that period. The intensity of non-empty batches in the input stream is equal to $\nu_n + (1 - p_{n,0})(N+1)\lambda$ and the distribution of the number units in one batch is given by the formula

$$p_{n,i}^{+} = \begin{cases} \dfrac{\nu_n + p_{n,1}(N+1)\lambda}{\nu_n + (1-p_{n,0})(N+1)\lambda}, & i = 1, \\[2ex] \dfrac{p_{n,i}(N+1)\lambda}{\nu_n + (1-p_{n,0})(N+1)\lambda}, & i = 2, 3, \ldots \end{cases}$$

Let μ_n be the momentary intensity of service of additional units provided n units are in the system. In our assumptions the system of additional units in the modified time t' is an $M/M/1$ system with batched input stream and feedback of the intensities of input stream and service, and the state of the system. This system can be analyzed by Markov processes in the usual way. In the case of system without feedback, e.g. $\nu_n = \nu$ and $\mu_{n+1} = \mu_1$, $n = 0, 1, \ldots$, the problem has been considered by Jankiewicz [2].

3. Cyclic system of additional units. Assume that we have M additional units which work, break down and are repaired in Takács' machine repair system. Let the working time of those units be exponentially distributed with parameter ν and let the service time be exponentially distributed with parameter μ_1. By these assumptions the input stream of additional units is Poissonian with momentary intensity $\nu_n = (M-n)\nu$, $n = 0, 1, \ldots, M$, provided n additional units are in the system at that moment. To give a complete description of the system we find the probability distributions $\{p_{n,j}\}$, $n = 0, 1, \ldots$ Of course, the probability of j arrivals in one batch (during the repairman's busy period with servicing machines), under the condition that n units are in the system, is given by the formula

$$p_{n,j} = \int_0^\infty \binom{M-n}{j} (1-e^{-\nu x})^j e^{-\nu x(M-n-j)} dG_N(x),$$

$$j = 0, 1, \ldots, M-n, \quad n = 0, 1, \ldots, M,$$

with

$$\sum_{j=0}^{M-n} p_{n,j} = 1, \quad n = 0, 1, \ldots, M,$$

where $G_N(x)$ denotes the distribution function of the busy period in Takács' machine repair system (see [9]).

On the basis of the above-given observations we can state the following simple result:

THEOREM 2. *Consider the repair system with cyclic additional units. The stationary probabilities $\{P^-(n)\}$ of the number of additional units immediately before the beginnings of the repairman's busy periods satisfy the system of equations*

$$\frac{(N+1)\lambda}{\mu_1} \sum_{i=0}^{n-1} \frac{(M-i-1)!}{(M-n)!} \left(\frac{\nu}{\mu_1}\right)^{n-i-1} \sum_{k=0}^{i} r_{k,i-k} P^-(k)$$
$$= P(n) - \frac{M!}{(M-n)!} \left(\frac{\nu}{\mu_1}\right)^n P(0), \quad n = 1, 2, \ldots, M,$$

$$P(0) = \frac{1}{\sum_{n=0}^{M} [M!/(M-n)!](\nu/\mu_1)^n} \left[1 - \frac{(N+1)\lambda}{\mu_1} \sum_{n=1}^{M} \sum_{i=0}^{n-1} \frac{(M-i-1)!}{(M-n)!} \times \right.$$
$$\left. \times \left(\frac{\nu}{\mu_1}\right)^{n-i-1} \sum_{k=0}^{i} r_{k,i-k} P^-(k)\right],$$

where the probabilities $\{P(n)\}$ satisfy the system of equations

$$-(M-n)\nu P(n) + \mu_1 P(n+1) - (N+1)\lambda \sum_{k=0}^{n} r_{k,n-k} P(k) = 0,$$
$$n = 0, 1, \ldots, M-1,$$

with

$$\sum_{n=0}^{M} P(n) = 1,$$

where

$$r_{k,m} = \sum_{j=1}^{M-k} p_{k,m+j}, \quad m = 0, 1, \ldots, M-k, \ k = 0, 1, \ldots, M.$$

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SYSTEMY OBSŁUGI MASOWEJ Z MIESZANKĄ STRUMIENI ZGŁOSZEŃ I SPRZĘŻENIEM ZWROTNYM

STRESZCZENIE

W pracy rozważa się systemy obsługi masowej, w których strumień zgłoszeń jest mieszanką dwóch strumieni: poissonowskiego i grupowego strumienia Palma. Zakłada się, że intensywności strumienia zgłoszeń i obsługi zależą od stanu systemu. Główny rezultat polega na znalezieniu relacji między rozkładami prawdopodobieństwa stanu systemu w ciągłym czasie i włożonego łańcucha Markowa, zdefiniowanego w chwilach sygnałów strumienia Palma. Rezultaty zastosowano do analizy problemu konserwatora z priorytetem bezwzględnym.