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TIME-OPTIMAL CONTROL OF A CERTAIN SECOND-ORDER NONOSCILLATORY SYSTEM

In a time-optimal control problem we are given a differential equation $\ddot{x} = f(x, \dot{x}, u, v)$, where u and v are control parameters taking their values from the sets $\langle \alpha, \beta \rangle$ and $\langle -1, 1 \rangle$, respectively. We shall construct a synthesis, being optimal in the class of measurable functions, for a certain case of this system. To determine the optimal synthesis we have to express the optimal control by the function $(u(x), v(x))$ depending on the state x of the system and defined on any region G, $0 \in G \subset R^2$, such that for each $x_0 \in G$ the corresponding response $x(t)$ of the system moves from the initial state x_0 to the origin O in a minimal interval of time.

The problem of construction of an optimal synthesis was solved in [1], [4] and [11] for the case of linear differential equations and in [3], [4], [8] and [9] for certain nonlinear equations. In [2] Boltjanskiĭ defined a regular synthesis and proved its optimality for the class of piecewise continuous controls. The author's result is an extension of a result of Boltjanskiĭ in [4], where the optimal synthesis for $\alpha = \beta$ was given.

1. Introduction. Consider the differential system

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = f(x^1, x^2, u, v), \tag{1.1}$$

where $f(x^1, x^2, u, v)$ together with $\partial f/\partial x^1$, $\partial f/\partial x^2$, $\partial f/\partial u$ and $\partial f/\partial v$ are continuous functions in $R^2 \times W$, and

$$W = \{(u, v) : u \in \langle \alpha, \beta \rangle, v \in \langle -1, 1 \rangle\} \tag{1.2}$$

is a compact subset of R^2. For any two measurable functions $(u(t), v(t))$, $t \in R$, taking their values from W (called an *admissible control*) the differential system (1.1) has a unique absolutely continuous solution $x(t)$ defined on $\langle t_0, t_1 \rangle$ which satisfies (1.1) almost everywhere in $\langle t_0, t_1 \rangle$. This is a conclusion of Caratheodory's existence theorem for differential equation systems [6].

Moreover, the function f satisfies the following assumptions:

Z1. There exist $(\bar{u}, \bar{v}) \in W$, $-1 < \bar{v} < 1$, such that $f(0, 0, \bar{u}, \bar{v}) = 0$.

Z2(a) $x^2 \partial f(x^1, x^2, u, v)/\partial u < 0$, $x^1 \in R$, $x^2 \in R \setminus \{0\}$, $(u, v) \in W$.

Z2(b) $\partial f(x^1, x^2, u, v)/\partial v > 0$, $(x^1, x^2) \in R^2$, $(u, v) \in W$.

Z3. Each solution of the system (1.1) corresponding to any admissible control can be extended over R.

The condition Z3 is satisfied if, for example, f has bounded partial derivatives $\partial f/\partial x^1$ and $\partial f/\partial x^2$ on $R^2 \times W$. It follows from the condition Z1 that the origin O is the stability point of the system (1.1).

Let $x(t)$ be a solution of (1.1) corresponding to an optimal control $(u(t), v(t))$ passing from any initial state $x_0 \in R^2$ to the final state O on $\langle t_0, t_1 \rangle$. The Pontryagin maximum principle [5] implies for $x(t)$ and $(u(t), v(t))$ that there exists a nontrivial solution $\psi(t) = (\psi_1(t), \psi_2(t))$ of the system

$$\dot{\psi}_1 = -\frac{\partial f}{\partial x^1}\psi_2, \qquad \dot{\psi}_2 = -\psi_1 - \frac{\partial f}{\partial x^2}\psi_2 \tag{1.3}$$

defined on $\langle t_0, t_1 \rangle$ such that on $\langle t_0, t_1 \rangle$

$$\max_{(u,v)\in W} \left[\psi_1(t)x^2(t) + \psi_2(t) f\big(x^1(t), x^2(t), u, v\big)\right] = \psi_1(t)x^2(t) + \psi_2(t) f\big(x^1(t), x^2(t), u(t), v(t)\big). \tag{1.4}$$

A function $x(t)$ such that $\dot{x}(t) = 0$ almost everywhere on the interval $\langle t_1, t_2 \rangle$, $t_1 < t_2$, is called *singular*. We can exclude from our considerations every singular solution of (1.1) to be nonoptimal.

The control $(u(t), v(t))$ is called *extremal* on $\langle t_0, t_1 \rangle$ if the response $x(t)$ of (1.1) corresponding to it is not singular and there exists a nontrivial solution of (1.3) such that (1.4) holds for $t \in \langle t_0, t_1 \rangle$.

Then the search for any optimal control can be restricted to the examination of the set of extremal controls. From (1.4) and Z2 we obtain the necessary condition for the extremality of control.

If $\psi_2 \neq 0$, then

$$v = \operatorname{sgn} \psi_2. \tag{1.5}$$

Moreover, if $x^2 \neq 0$, then

$$u = \begin{cases} \alpha & \text{if } v \operatorname{sgn} x^2 > 0, \\ \beta & \text{if } v \operatorname{sgn} x^2 < 0. \end{cases} \tag{1.6}$$

Now we prove a few auxiliary lemmas.

LEMMA 1.1. *If $\psi(t) = (\psi_1(t), \psi_2(t))$ is a nontrivial solution of (1.3), then zero points of the function $\psi_2(t)$ are isolated.*

Proof. Notice at first that $\psi_2(t) \equiv 0$ fails on an interval $\langle t_1, t_2\rangle$, $t_1 < t_2$, because otherwise $\psi_1(t) \equiv 0$ on this interval; this contradicts the nontriviality of $\psi(t)$. Also the set of zeros of $\psi_2(t)$ cannot be dense on any interval $\langle t_1, t_2\rangle$, $t_1 < t_2$.

Now we show that between two zeros of $\psi_2(t)$ a zero of $\psi_1(t)$ lies. Let t_1, t_2 $(t_1 < t_2)$ be two zeros of ψ_2. Then there exists an interval $\langle t_1', t_2'\rangle \subset \langle t_1, t_2\rangle$ such that $\psi_2(t_1') = \psi_2(t_2') = 0$ and, for instance, $\psi_2(t) > 0$ for $t \in (t_1', t_2')$. By the nontriviality of $\psi(t)$ we have $\psi_1(t_1') \neq 0$. Notice that $\psi_1(t_1') < 0$. Indeed, if $\psi_1(t_1') > 0$, then, by (1.3), $\dot{\psi}_2(t) < 0$ almost everywhere in the neighbourhood of the point t_1', and since $\psi_2(t_1') = 0$, we obtain $\psi_2(t) < 0$ on the right-hand neighbourhood of t_1'. This contradicts the assumption $\psi_2(t) > 0$ for $t \in (t_1', t_2')$. Similarly we obtain $\psi_1(t_2') > 0$, and thus the function $\psi_1(t)$ takes the value 0 on the interval (t_1', t_2').

Let us assume now that the set of zeros of $\psi_2(t)$ has a condensation point p and let $\{t_n\}$ be a sequence of zeros of $\psi_2(t)$ which converges to p. Without loss of generality we may assume that $\{t_n\}$ is a monotonic sequence. Then there exists a sequence $\{t_n'\}$ tending to p such that $\psi_1(t_n') = 0$. Therefore $\psi_1(p) = \psi_2(p) = 0$, which contradicts the nontriviality of the function $\psi(t)$.

LEMMA 1.2. *The coordinate $x^2(t)$ of the solution $x(t) = \big(x^1(t), x^2(t)\big)$ of the system* (1.1) *corresponding to any extremal control takes the value* 0 *at isolated points.*

Proof. Notice at first that the function $x^2(t)$ cannot be equal to 0 on an interval $\langle t_1, t_2\rangle$ of positive length because otherwise $\big(\dot{x}^1(t), \dot{x}^2(t)\big) = 0$ on this interval, which contradicts the nonsingularity of $x(t)$. Therefore, the set of zeros of $x^2(t)$ is not dense in any interval $\langle t_1, t_2\rangle$, $t_1 < t_2$.

Let $\langle s_1, s_2\rangle$ be an interval on which the function $\psi_2(t)$ does not change its sign. Then, because of (1.5), $v(t) = \text{const} = v_0$ for $t \in \langle s_1, s_2\rangle$. We prove that the set of zeros of $x^2(t)$ in the interval $\langle s_1, s_2\rangle$ is finite.

We show at first that if $\langle t_1, t_2\rangle \subset \langle s_1, s_2\rangle$ and $x^2(t_1) = x^2(t_2) = 0$, and, for instance, $x^2(t) > 0$ for $t \in (t_1, t_2)$, then there exists a point $\tau \in (t_1, t_2)$ such that $\dot{x}^2(\tau) = 0$. Indeed, by (1.6) we have $u(t) = \text{const} = u_0$ in the interval $\langle t_1, t_2\rangle$, and so $x(t)$ is a function of class C^1 on this interval. By the assumption about the function $x^2(t)$ there exists $\tau \in (t_1, t_2)$ such that $\dot{x}^2(\tau) = 0$.

Let $P = \{t \in \langle s_1, s_2\rangle : x^2(t) = 0\}$. The equation $\dot{x}^1 = x^2$ implies $\dot{x}^1(t) = 0$ for $t \in P$. Assume now that P is an infinite set and let τ be a condensation point of P. It is clear that $\tau \in P$ because P is a closed set. Futhermore, P is a boundary set, thus there exists a sequence of intervals $I_n = (t_1^n, t_2^n) \subset \langle s_1, s_2\rangle$, separate in pairs, such that $x^2(t_1^n) = x^2(t_2^n) = 0$, $x^2(t) \neq 0$ for $t \in I_n$ and $t_1^n \to \tau$, $t_2^n \to \tau$. As was shown before, in each interval I_n there exists a point τ_n such that $\dot{x}^2(\tau_n) = 0$.

Since on the intervals I_n we have $u(t) = \text{const} = \alpha$ or β, it is possible to select a subsequence (for simplicity still denoted by $\{I_n\}$) such that either $u(t) = \alpha$ on $\bigcup I_n$ or $u(t) = \beta$ on $\bigcup I_n$. Let, for example, $u(t) = \alpha$ for $t \in \bigcup I_n$. Then, by $f(x^1(\tau_n), x^2(\tau_n), \alpha, v_0) = \dot{x}^2(\tau_n) = 0$, by the continuity of the functions $x^1(t)$, $x^2(t)$ and by the condition $\tau_n \to \tau$, we obtain $f(x^1(\tau), 0, \alpha, v_0) = 0$. Since, by Z2(a), $f(x^1(\tau), 0, u, v_0) = 0$ for each $u \in \langle \alpha, \beta \rangle$, we infer that $(x^1(\tau), 0)$ is a stationary point and for every control strategy $u(t)$ we have $x^1(t) = \text{const}$ and $x^2(t) \equiv 0$ on the interval $\langle s_1, s_2 \rangle$, which contradicts the nonsingularity of $x(t)$.

Denote by E the set of vertices of the rectangle W. The following conclusion follows from (1.5), (1.6) and from Lemmas 1.1 and 1.2.

LEMMA 1.3. *Each extremal control is a piecewise constant function. It has a finite number of switches on every interval of finite length and takes its values from E.*

Let $x(t, x_0, u_0, v_0) = x(t)$ and $\psi(t, \psi^0, u_0, v_0) = \psi(t)$ be solutions of the systems (1.1) and (1.3), respectively, with a constant control $(u(t), v(t)) = (u_0, v_0) \in E$ and with the initial conditions $x(0, x_0, u_0, v_0) = x_0$ and $\psi(0, \psi^0, u_0, v_0) = \psi^0$, respectively. Moreover, let $\Gamma_{u_0,v_0}(t_0, p_0)$ be a trajectory defined by

$$\Gamma_{u_0,v_0}(t_0, p_0) = \{x \in R : x = x(t-t_0, p_0, u_0, v_0), t < t_0\}.$$

Notice that if $\dot{x}(t_1, p_0, u_0, v_0) \neq 0$ for any $t_1 \in R$, then $\dot{x}(t, p_0, u_0, v_0) \neq 0$ for each $t \in R$ and $x(t)$ is a nonsingular function. A trajectory corresponding to an extremal control is said to be *extremal*.

LEMMA 1.4. *Any segment of the trajectory $\Gamma_{\beta,1}(0, 0)$ and any segment of the trajectory $\Gamma_{\beta,-1}(0, 0)$ are the only extremal trajectories which reach O at the moment $t = 0$ without variation of the control strategy. These segments are included in $\{x\colon x^1 > 0, x^2 < 0\}$ and $\{x\colon x^1 < 0, x^2 > 0\}$, respectively.*

Proof. Lemma 1.3 and (1.6) imply that the only segments of the trajectory corresponding to the constant controls $(\alpha, -1)$ or $(\beta, 1)$ on the half-space $\{x\colon x^2 < 0\}$ and $(\alpha, 1)$ or $(\beta, -1)$, respectively, on the half-space $\{x\colon x^2 > 0\}$ are extremal. From Z1 and Z2 we obtain

$$f(0, 0, \beta, 1) > 0, \quad f(0, 0, \alpha, -1) < 0$$

and

$$f(0, 0, \beta, -1) < 0, \quad f(0, 0, \alpha, 1) > 0,$$

thus an extremal trajectory lying in the half-space $\{x\colon x^2 < 0\}$ and reaching O at $t = 0$ must take the shape of $\Gamma_{\beta,1}(0, 0)$; similarly, an extremal trajectory reaching O from the half-space $\{x\colon x^2 > 0\}$ must take the shape of $\Gamma_{\beta,-1}(0, 0)$.

The segments $\Gamma_{\beta,1}(0, 0) \cap \{x\colon x^2 < 0\}$ and $\Gamma_{\beta,-1}(0, 0) \cap \{x\colon x^2 > 0\}$ of the trajectories $\Gamma_{\beta,1}(0, 0)$ and $\Gamma_{\beta,-1}(0, 0)$, respectively, are nonsingular. Thus, if $x^2(t, 0, \beta, 1) < 0$ for $t \in (t_1, 0)$ and $x^2(t, 0, \beta, -1) > 0$

for $t \in (t_2, 0)$, then $\dot{x}^1(t, 0, \beta, 1) < 0$ for $t \in (t_1, 0)$ and $\dot{x}^1(t, 0, \beta, -1) > 0$ for $t \in (t_2, 0)$, and so $x^1(t, 0, \beta, 1) > 0$ on $\langle t_1, 0)$ and $x^1(t, 0, \beta, -1) < 0$ on $\langle t_2, 0)$. Moreover, the trajectories $\Gamma_{\beta,1}(0, 0)$ and $\Gamma_{\beta,-1}(0, 0)$ are extremal on any left-hand neighbourhood at $t = 0$ with respect to the functions $\psi(t, (1, 0), \beta, 1)$ and $\psi(t, (-1, 0), \beta, -1)$, respectively, because by (1.3) we have

$$\dot{\psi}_2(0, (1, 0), \beta, 1) = -1 < 0 \quad \text{and} \quad \dot{\psi}_2(0, (-1, 0), \beta, -1) = 1 > 0.$$

They are extremal on the intervals

$$(t_1', 0) = \{t \in (t_1, 0) : \psi_2(t, (1, 0), \beta, 1) > 0\}$$

and

$$(t_2', 0) = \{t \in (t_2, 0) : \psi_2(t, (-1, 0), \beta, -1) < 0\},$$

respectively.

Consider the time-control system

$$\dot{x} = f(x, u), \tag{1.7}$$

where $x \in R^n$, $u \in U$, $f(x, u)$ and the derivatives $\partial f/\partial x$ and $\partial f/\partial u$ are continuous functions on $R^n \times U$, and U is a compact convex set.

THEOREM 1.1. *If there exists an optimal synthesis $v(x)$ of the system (1.7) on a region G in the class of piecewise continuous controls and the optimal time $T(x)$ is a continuous function on G, then this synthesis is also optimal in the class of measurable controls.*

The proof of this theorem will be preceded by a lemma.

Suppose that equation (1.7) has a solution $z(t, u)$ which can be extended on the interval $\langle t_0, t_1\rangle$ for each measurable control strategy taking its values from U.

LEMMA 1.5. *For each number $\varepsilon > 0$ there exists such a number $\delta > 0$ that the inequality*

$$\int_{t_0}^{t_1} |u_1(\tau) - u_2(\tau)|\, d\tau < \delta$$

implies

$$|z(t, u_1) - z(t, u_2)| < \varepsilon \quad \textit{for } t \in \langle t_0, t_1\rangle.$$

The proof of this lemma is based on the idea of the proof of the theorem on the continuous dependence of solutions of differential equations on the parameter variable [11].

Proof of Theorem 1.1. Let $\omega(x) = -T(x)$, $x \in G$. Of course, the function $\omega(x)$ is continuous on G and, for each piecewise continuous control $u(t)$ moving the response $x(t)$ of (1.7) from the state $x \in G$ to the origin O on the time interval $\langle t_0, t_1\rangle$, the inequality

$$\omega(0) - \omega(x) \leqslant t_1 - t_0 \tag{1.8}$$

holds.

Now let $u_1(t)$ be a measurable control taking its values from U, moving the point x to O on the time interval $\langle t_0, t_1\rangle$ and let $x_1(t)$ be a solution of (1.7) corresponding to this control. For each $\eta > 0$ Luzin's theorem implies the existence of a piecewise continuous control $u_2(t)$ with values from U such that

$$\int_{t_0}^{t_1} |u_1(\tau)-u_2(\tau)|\,d\tau < \eta. \tag{1.9}$$

By Lemma 1.5 for each $\delta > 0$ there exists $\eta > 0$ such that the response $x_2(t)$ of (1.7) with the initial condition $x_2(t_1) = 0$ corresponding to the control $u_2(t)$ satisfies the inequality $|x_1(t)-x_2(t)| < \delta$ for $t \in \langle t_0, t_1\rangle$. Then also

$$|x_2(t_0)-x_0| < \delta. \tag{1.10}$$

By the continuity of $\omega(x)$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|\omega\big(x_2(t_0)\big)-\omega(x_0)\big| < \varepsilon,$$

since, by (1.8)-(1.10), for each $\varepsilon > 0$ we obtain the inequality

$$\omega(0)-\omega(x) \leqslant t_1-t_0+\varepsilon,$$

which implies $T(x) \leqslant t_1-t_0$. Since the class of measurable functions contains the class of piecewise continuous functions, $T(x)$ is the optimal time also in the class of measurable controls. This completes the proof of Theorem 1.1.

2. The optimal synthesis for a certain nonoscillatory system.

2.1. In this section we consider the differential system (1.1) with the restriction (1.2) for values of admissible controls and with the function $f(x^1, x^2, u, v)$ satisfying the assumptions Z1-Z3 and the following additional condition:

Z4. There exists a function $\varphi(x^1, x^2, u, v) \in C^1(R^2\times W)$ such that (a) for each $x^1 \in R$ and $(u, v) \in W$

$$\frac{\partial\varphi}{\partial u}(x^1, 0, u, v) \geqslant 0,$$

(b) for each constant control $\big(u(t), v(t)\big) = (u_0, v_0)$, $(u_0, v_0) \in E$, and for each $(x^1, x^2) \in R^2$ the inequality

$$-\frac{\partial f}{\partial x^1}-\frac{\partial f}{\partial x^2}\varphi+\frac{\partial\varphi}{\partial x^1}x^2+\frac{\partial\varphi}{\partial x^2}f+\varphi^2 \leqslant 0$$

holds.

The class of systems satisfying the conditions Z1-Z4 is nonempty because it contains the linear systems

$$\dot{x}^1 = x^2, \qquad \dot{x}^2 = -x^1 - ux^2 + v, \tag{2.1}$$

where in the definition (1.2) of W we take $\alpha \geqslant 2$. Then the condition Z4 is satisfied for the function $\varphi \equiv -1$.

The forthcoming three lemmas describe properties of solutions of the systems (1.1) and (1.3).

Let $x(t) = \big(x^1(t), x^2(t)\big)$ be a nonsingular solution of (1.1) with a constant control $\big(u(t), v(t)\big) = (u_0, v_0)$, $(u_0, v_0) \in E$, and let $\psi(t) = \big(\psi_1(t), \psi_2(t)\big)$ be a nontrivial solution of the system (1.3) corresponding to this control.

LEMMA 2.1. *The function $\psi_2(t)$ takes the value zero not more than once.*

LEMMA 2.2. *The coordinate $x^2(t)$ takes the value zero not more than once.*

LEMMA 2.3. *Let $\psi(t)$ be a nontrivial solution of* (1.3) *corresponding to the control* (*which is discontinuous at time τ*)

$$\big(u(t), v(t)\big) = \begin{cases} (\beta, v_0) & \text{for } t < \tau, \\ (\alpha, v_0) & \text{for } t \geqslant \tau, \end{cases}$$

where $v_0 \in \{-1, 1\}$, and corresponding to a solution of (1.1), *suitable for this control, with the condition $x^2(\tau) = 0$. Then the function $\psi_2(t)$ takes the value zero not more than once.*

The proofs of two first lemmas are almost identical with the proofs of similar lemmas in [4]. The proof of the last lemma is similar to the proofs of Lemmas 2.1 and 2.2. From Z4(a) it is easy to see that the auxiliary function

$$\eta(t) = \psi_1(t) + \varphi\big(x^1(t), x^2(t), u(t), v(t)\big)\cdot\psi_2(t) \qquad \text{for } \psi_2(\tau) < 0$$

and the function

$$\eta(t) = -\psi_1(t) - \varphi\big(x^1(t), x^2(t), u(t), v(t)\big)\cdot\psi_2(t) \qquad \text{for } \psi_2(\tau) > 0,$$

which are discontinuous at τ, are nonincreasing at τ.

2.2. Consider now the trajectories $\Gamma_{\beta,1}(0, 0)$ and $\Gamma_{\beta,-1}(0, 0)$.

LEMMA 2.4. *The curves $\Gamma_{\beta,1}(0, 0)$ and $\Gamma_{\beta,-1}(0, 0)$ are extremal, their projection on the axis Ox^1 is a one-to-one mapping and*

$$\Gamma_{\beta,1}(0, 0) \subset \{x\colon x^1 > 0, x^2 < 0\}, \; \Gamma_{\beta,-1}(0, 0) \subset \{x\colon x^1 < 0, x^2 > 0\}. \tag{2.2}$$

Proof. By the conditions Z1 and Z2 we have $f(0, 0, \beta, 1) > 0$ and $f(0, 0, \beta, -1) < 0$, thus (2.2) and the existence of a projection of these trajectories on the axis Ox^1 follow from Lemma 2.2 and from the equation $\dot{x}^1 = x^2$.

The curves $\Gamma_{\beta,1}(0,0)$ and $\Gamma_{\beta,-1}(0,0)$ are extremal according to $\psi\big(t,(1,0),\beta,1\big)$ and $\psi\big(t,(-1,0),\beta,-1\big)$, respectively $(t<0)$, because by Lemma 2.1 the functions $\psi_2\big(t,(1,0),\beta,1\big)$ and $\psi_2\big(t,(-1,0),\beta,-1\big)$

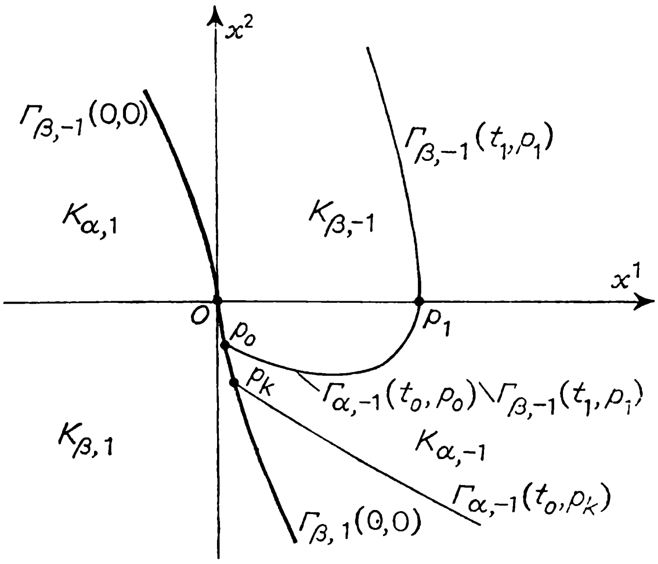

Fig. 1

take the value zero not more than once. Thus from the shape of the system (1.3) and from the initial conditions we obtain

$$\psi_2\big(t,(1,0),\beta,1\big)>0 \quad\text{and}\quad \psi_2\big(t,(-1,0),\beta,-1\big)<0 \quad\text{for } t<0.$$

Then the control $\big(u(t),v(t)\big)=(\beta,1)$, the functions $\psi\big(t,(1,0),\beta,1\big)$ and $x(t,0,\beta,1)$ satisfy condition (1.4) of the maximum principle for $t<0$. Condition (1.4) is also satisfied for $t<0$ if

$$\big(u(t),v(t)\big)=(\beta,-1),$$
$$\psi(t)=\psi\big(t,(-1,0),\beta,-1\big) \quad\text{and}\quad x(t,0,\beta,-1).$$

Write now

$$\tilde{\Gamma}_{u_0,v_0}(t_0,p_0)=\{(t,x)\colon x=x(t-t_0,p_0,u_0,v_0),\, t<t_0\}.$$

Let $(t_0,p_0)\in\tilde{\Gamma}_{\beta,1}(0,0)$. Then $p_0^2<0$ and the trajectory $\Gamma_{\alpha,-1}(t_0,p_0)$ is nonsingular for $t<t_0$. Since, by Z2,

$$f(p,\beta,1)>f(p,\beta,-1)>f(p,\alpha,-1) \quad\text{for } p^2<0,$$

every trajectory $\Gamma_{\alpha,-1}(t,p)$ for $(t,p)\in\tilde{\Gamma}_{\beta,1}(0,0)$ reaches the curve $\Gamma_{\beta,1}(0,0)$ on its right side at a nonzero angle. Since, by Lemma 2.2, $\Gamma_{\alpha,-1}(t_0,p_0)$ intersects the axis Ox^1 not more than once, p_0 is the unique common point of the trajectory $\Gamma_{\beta,1}(0,0)$ and the arc $\Gamma_{\alpha,-1}(t_0,p_0)$.

According to Lemma 2.2 two cases can occur:

(i) $\Gamma_{a,-1}(t_0, p_0) \subset \{x\colon x^2 < 0\}$,

(ii) there exists a unique point $q \in \{x\colon x^2 = 0\}$ such that

$$\Gamma_{a,-1}(t_0, p_0) \cap \{x\colon x^2 = 0\} = \{q\}.$$

The inequality $\dot{x}^1 = x^2$ implies $q^1 > 0$.

In case (i) the trajectory

$$(2.3) \qquad \big(\Gamma_{\beta,1}(0, 0) \setminus \Gamma_{\beta,1}(t_0, p_0)\big) \cup \Gamma_{a,-1}(t_0, p_0)$$

is described by the function

$$(2.4) \qquad x(t) = \begin{cases} x(t-t_0, p_0, \beta, 1) & \text{for } t \in \langle t_0, 0), \\ x(t-t_0, p_0, a, -1) & \text{for } t \in (-\infty, t_0), \end{cases}$$

which is a solution of the system (1.1) with the control

$$\big(u(t), v(t)\big) = \begin{cases} (\beta, 1) & \text{for } t \in \langle t_0, 0), \\ (a, -1) & \text{for } t \in (-\infty, t_0). \end{cases}$$

The trajectory (2.3) lies completely in the fourth quadrant of the phase plane.

We show that this trajectory is an extremal one in the interval $(-\infty, 0)$ according to the nontrivial solution of (1.3):

$$\psi(t) = \begin{cases} \psi\big(t-t_0, (-1, 0), \beta, 1\big) & \text{for } t \in \langle t_0, 0), \\ \psi\big(t-t_0, (-1, 0), a, -1\big) & \text{for } t \in (-\infty, t_0). \end{cases}$$

This follows from the fact that $\dot{\psi}_2(t_0) = -\psi_1(t_0) = 1$, $\psi_2(t_0) = 0$, and thus, by Lemma 2.1,

$$\psi_2\big(t, (-1, 0), \beta, 1\big) > 0 \quad \text{for } t \in \langle t_0, 0)$$

and

$$\psi_2\big(t, (-1, 0), a, -1\big) < 0 \quad \text{for } t < t_0.$$

In case (ii) there exists a unique moment $t_1 < t_0$ at which the response (2.4) intersects the axis Ox^1. Let

$$q = x(t_1-t_0, p_0, a, -1) \quad \text{and} \quad s = \psi\big(t_1-t_0, (-1, 0), a, -1\big).$$

Consider the trajectory

$$\big(\Gamma_{\beta,1}(0, 0) \setminus \Gamma_{\beta,1}(t_0, p_0)\big) \cup \big(\Gamma_{a,-1}(t_0, p_0) \setminus \Gamma_{a,-1}(t_1, q)\big) \cup \Gamma_{\beta,-1}(t_1, q)$$

described by the equation

$$x(t) = \begin{cases} x(t, 0, \beta, 1) & \text{for } t \in \langle t_0, 0), \\ x(t-t_0, p_0, a, -1) & \text{for } t \in \langle t_1, t_0), \\ x(t-t_1, q, \beta, -1) & \text{for } t \in (-\infty, t_1). \end{cases}$$

The function $x(t)$ satisfies the differential system (1.1) with the control

$$\big(u(t), v(t)\big) = \begin{cases} (\beta, 1) & \text{for } t \in \langle t_0, 0), \\ (\alpha, -1) & \text{for } t \in \langle t_1, t_0), \\ (\beta, -1) & \text{for } t \in (-\infty, t_1) \end{cases}$$

and it is an extremal one according to the nontrivial solution of (1.3):

$$\psi(t) = \begin{cases} \psi\big(t-t_0, (-1, 0), \beta, 1\big) & \text{for } t \in \langle t_0, 0), \\ \psi\big(t-t_0, (-1, 0), \alpha, -1\big) & \text{for } t \in \langle t_1, t_0), \\ \psi(t-t_1, s, \beta, -1) & \text{for } t \in (-\infty, t_1). \end{cases}$$

This follows from the considerations for case (i) and from Lemma 2.3.

Moreover, the trajectory $\Gamma_{\alpha,-1}(t_0, p_0)$ reaches the axis Ox^1 with a nonzero velocity and $f(q^1, 0, \beta, -1) < 0$. Thus, by condition Z2(a), we have $f(q^1, 0, \alpha, -1) < 0$. Consequently, it follows from Lemma 2.2 that the trajectory $\Gamma_{\beta,-1}(t_1, q)$ lies in the upper half-space.

Notice that the trajectory $\Gamma_{\alpha,-1}(0, 0)$ reaches the axis Ox^1 from the upper half-space with a nonzero velocity at a right angle, because Z1 and Z3(b) imply $f(0, 0, \alpha, -1) < 0$. By the theorem on the continuous dependence of solutions of differential equations on the initial conditions, the trajectory $\Gamma_{\alpha,-1}(t_0, p_0)$ intersects the axis Ox^1 when $p_0 \in \Gamma_{\beta,1}(0, 0)$ is sufficiently near to O. Thus case (ii) holds for any part of the curve $\Gamma_{\beta,1}(0, 0)$.

In case (ii) one only needs to consider the linear case (2.1). Then the rays

$$l_1 = \left\{(x^1, x^2):\ x^1 > 1, x^2 = \frac{-\beta-\sqrt{\beta^2-4}}{2}\,(x^1-1)\right\}$$

and

$$l_2 = \left\{(x^1, x^2):\ x^1 > -1, x^2 = \frac{-\alpha-\sqrt{\alpha^2-4}}{2}\,(x^1+1)\right\}$$

intersect each other and are trajectories of the system (2.1) with the constant control corresponding to $(\beta, 1)$ or $(\alpha, -1)$. The points of $\Gamma_{\beta,1}(0, 0)$ sufficiently near to O lie between the rays l_1 and l_2, thus according to the theorem on the existence of unique solutions of differential equations the curve $\Gamma_{\beta,1}(0, 0)$ is on the left side of the ray l_1. Then the ray l_2 crosses this curve at some point p_0. By the theorem on the existence of a unique trajectory, $\Gamma_{\alpha,-1}(t_0, p_0)$ lies on the ray l_2 and it does not intersect the axis Ox^1.

If case (i) holds, then there exists a point $p_k \in \Gamma_{\beta,1}(0, 0)$ such that for each point $p_0 \in \Gamma_{\beta,1}(0, 0)$ lying above p_k the trajectory $\Gamma_{\alpha,-1}(t_0, p_0)$,

$(t_0, p_0) \in \tilde{\Gamma}_{\beta,1}(0, 0)$, crosses the axis Ox^1. If the point p_0 equals p_k or lies below p_k, the trajectory $\Gamma_{\alpha,-1}(t_0, p_0)$ is completely included in the fourth quadrant of the phase plane.

The extremal trajectories for the points $(t_0, p_0) \in \tilde{\Gamma}_{\beta,-1}(0, 0)$ are constructed similarly.

Notice that the above-described trajectories have not any other common point with the exception of the possible common segment with the curve $\Gamma_{\beta,1}(0, 0)$ or $\Gamma_{\beta,-1}(0, 0)$, because their extensions for $p_0' \neq p_0''$ are solutions of the same differential autonomous equation with different initial conditions.

Notice also that every extremal trajectory reaching O must be one of the above trajectories. Indeed, by Lemma 1.4, an extremal trajectory reaching O must finish in a segment of the trajectory $\Gamma_{\beta,1}(0, 0)$ or $\Gamma_{\beta,-1}(0, 0)$ and, by (1.6), an extremal control in the lower half-space can be only of shape $(\beta, 1)$ or $(\alpha, -1)$ and in the upper half-space must take the form $(\beta, -1)$ or $(\alpha, 1)$. Moreover, according to Lemmas 2.1, 2.2 and 2.3, the extremal control has not more than two switches. Hence the above-described curves are the unique extremal trajectories which reach O.

2.3. Denote by G the set which consists of 0 and of all points of trajectories described in 2.2.

If

$$\lim_{t\to-\infty} x^1(t, 0, \beta, 1) = \infty, \tag{2.5}$$

then, by the theorem on the continuous dependence of solutions of differential equations on the initial conditions and by $\dot{x}^1 = x^2$, the whole positive semi-axis x^1 is included in G. Notice that if there exists a point $(t_0, p_k) \in \tilde{\Gamma}_{\beta,1}(0, 0)$ such that $\Gamma_{\alpha,-1}(t_0, p_k)$ does not cross the axis Ox^1, then (2.5) holds. If, however,

$$\lim_{t\to-\infty} x^1(t, 0, \beta, 1) = a < \infty,$$

then by $\dot{x}^1 = x^2$ we have

$$\lim_{t\to-\infty} x^2(t, 0, \beta, 1) = 0.$$

Thus, by Z2 each trajectory $\Gamma_{\alpha,-1}(t_0, p_0)$, $(t_0, p_0) \in \tilde{\Gamma}_{\beta,1}(0, 0)$, intersects the axis Ox^1 and the interval $\langle 0, a)$ of the positive semi-axis Ox^1 is contained in G. Since by Z2 we have also $f(0, 0, \beta, -1) < f(a, 0, \beta, 1) = 0$, the trajectory $\Gamma_{\beta,-1}\big(t_0, (a, 0)\big)$, $t_0 \in R$, is nonsingular and it bounds the set G on the right side, where $G \cap \big(\{(a, 0)\} \cup \Gamma_{\beta,-1}\big(t_0, (a, 0)\big)\big) = \emptyset$. The curve $\Gamma_{\beta,1}(0, 0) \cup \{0\} \cup \Gamma_{\beta,-1}(0, 0)$ divides G into two separate parts.

By the theorem on the continuous dependence of solutions of differential equations on the initial conditions one can easily prove, similarly as in [4], that G is an open set.

Write the following notation:

$$K_{\alpha,-1} = \bigcup_{(t,p)\in\tilde{\Gamma}_{\beta,1}(0,0)} \Gamma_{\alpha,-1}(t,p)\cap\{x:\ x^2<0\},$$

$$K_{\alpha,1} = \bigcup_{(t,p)\in\tilde{\Gamma}_{\beta,-1}(0,0)} \Gamma_{\alpha,1}(t,p)\cap\{x:\ x^2>0\},$$

$$K_{\beta,-1} = \bigcup_{(t,p)\in\tilde{\Gamma}_{\beta,1}(0,0)} \Gamma_{\beta,-1}\big(s(t,p),\ q(t,p)\big),$$

$$K_{\beta,1} = \bigcup_{(t,p)\in\tilde{\Gamma}_{\beta,-1}(0,0)} \Gamma_{\beta,1}\big(s(t,p),\ q(t,p)\big),$$

where $s(t,p)$ is the moment at which the trajectory $\Gamma_{\alpha,-1}(t,p)$ or $\Gamma_{\alpha,1}(t,p)$, respectively, comes out of the point $q(t,p)$ on the axis Ox^1. The sets $K_{\alpha,-1}$, $K_{\alpha,1}$, $K_{\beta,-1}$ and $K_{\beta,1}$ are open.

We can now define in the region G the control as a function depending upon the state x of the control system in the following way:

(2.6)

$$\big(u(x),v(x)\big) = \begin{cases} (\alpha,1) & \text{for } x\in K_{\alpha,1}\cup(G\cap\{x:\ x^1<0,\ x^2=0\}),\\ (\beta,1) & \text{for } x\in K_{\beta,1}\cup\Gamma_{\beta,1}(0,0),\\ (\alpha,-1) & \text{foı } x\in K_{\alpha,-1}\cup(G\cap\{x:\ x^1>0,\ x^2=0\}),\\ (\beta,-1) & \text{for } x\in K_{\beta,-1}\cup\Gamma_{\beta,-1}(0,0),\\ (\bar{u},\bar{v}) & \text{for } x=0. \end{cases}$$

Then each solution of the system (1.1) with the initial condition $x_0\in G\setminus\{0\}$ reaches O and is equal to the respective part of the trajectory described in 2.2.

The set $\Gamma_{\beta,1}(0,0)\cup\Gamma_{\beta,-1}(0,0)\cup(G\cap\{x:\ x^2=0\})$ consisting of 0 and of the line of switches of the control $\big(u(x),v(x)\big)$ is a piecewise smooth manifold of dimension less than 2.

We consider the origin O as a 0-dimensional cage (a curvilinear quadrangle) of the second type. A 1-dimensional cage of the first type consists of the curves $\Gamma_{\beta,1}(0,0)$ and $\Gamma_{\beta,-1}(0,0)$. The axis $\{x:\ x^2=0,\ x^1\neq 0\}\cap G$ is a 1-dimensional cage of the second type. The sets $K_{\alpha,1}$, $K_{\alpha,-1}$, $K_{\beta,1}$, $K_{\beta,-1}$ are 2-dimensional cages of the second type. Let

$$P^0=\{0\},\quad P^1=\Gamma_{\beta,1}(0,0)\cup\Gamma_{\beta,-1}(0,0)\cup(\{x:\ x^2=0\}\cap G),\quad P^2=G.$$

For each point $x_0\in G\setminus\{0\}$ a univocal function $\omega(x)$ is defined so that the solution of (1.1) with the condition $x\big(\omega(x_0)\big)=x_0$ reaches O exactly at $t=0$ and the function $T(x_0)=-\omega(x_0)$ is the time of control. Assuming $\omega(0)=0$, we obtain a function defined and continuous on the whole set G. The continuity of $\omega(x)$ follows from the theorem on the continuous dependence of solutions of differential equations on the initial conditions and from the construction of trajectories in 2.2. Moreover, $\omega(x)$ is continuously differentiable on the set $K_{\alpha,1}\cup K_{\alpha,-1}\cup K_{\beta,1}\cup K_{\beta,-1}$.

Similarly as in [4], one can show that G is the set of all points from which the response $x(t)$ of (1.1) corresponding to an admissible control moves to O.

Thus all conditions of regularity of the synthesis are satisfied. From the Boltjanskiĭ theorem [4] we get

CONCLUSION 2.1. *The synthesis* (2.6) *of control is optimal in* R^2 *in the class of piecewise continuous functions.*

Theorem 1.1 implies

CONCLUSION 2.2. *The synthesis* (2.6) *of control is optimal in* R^2 *in the class of measurable functions.*

Remark. In the case of the linear equation $\ddot{x}+u\dot{x}+x = v$, considered in [7], the parameter v has the character of an outer force while u is taken to mean the coefficient of environmental resistance. The same physical interpretation holds for the nonlinear case considered in this paper.

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CZASOWO-OPTYMALNE STEROWANIE PEWNYM NIEOSCYLACYJNYM UKŁADEM DYNAMICZNYM DRUGIEGO RZĘDU

STRESZCZENIE

W pracy rozważa się zagadnienie syntezy sterowań optymalnych dla nieliniowego układu drugiego rzędu $\ddot{x} = f(x, \dot{x}, u, v)$ z dwoma parametrami sterującymi u i v. Część pierwsza zawiera sformułowanie zagadnienia oraz własności sterowań i trajektorii ekstremalnych, niezbędne dla dalszych rozważań. W części drugiej znajduje się konstrukcja syntezy optymalnej dla obiektu spełniającego pewne warunki wykluczające oscylacje. W przypadku takiego układu, sterowania optymalne $u(t)$ i $v(t)$ mają odpowiednio co najwyżej dwa lub jedno przełączenie.