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# LIMITING PROPERTIES OF THE $k$-TH RECORD VALUES

**1. Definitions.** Let $\{X_m\}$, $m = 1, 2, \ldots$, be a sequence of independent random variables with common cumulative distribution function $F(x)$, continuous with density $f(x)$. Denote by

$$Z_1^{(m)} \leqslant Z_2^{(m)} \leqslant \ldots \leqslant Z_m^{(m)} \tag{1}$$

the order statistics in the considered sequence. First, we consider the sequence $\{Z_{m-k+1}^{(m)}\}$, $m = k, k+1, \ldots$, of the $k$-th order statistics, where $k$ is a constant integer. Note that the sequence $\{Z_{m-k+1}^{(m)}\}$ is non-decreasing. By elimination of repetitions in it, we get a strictly increasing subsequence which is called a *sequence of $k$-th record values*.

Strictly speaking, we define the indices $\{L_k(n)\}$, $n = 1, 2, \ldots$, of record values in the sequence $\{Z_{m-k+1}^{(m)}\}$ by

$$L_k(1) = 1, \quad L_k(n+1) = \min\{j\colon Z_{L_k(n)}^{(L_k(n)+k-1)} < Z_j^{(j+k-1)}\}, \; n = 1, 2, \ldots,$$

and hence the sequence of the $k$-th record values by

$$Y_n^{(k)} = Z_{L_k(n)}^{(L_k(n)+k-1)}, \quad n = 1, 2, \ldots \tag{2}$$

**2. Distribution function of the $k$-th record value.** Consider the sequence of random vectors

$$(Z_{L_k(n)}^{(L_k(n)+k-1)}, Z_{L_k(n)+1}^{(L_k(n)+k-1)}, \ldots, Z_{L_k(n)+k-1}^{(L_k(n)+k-1)}), \quad n = 1, 2, \ldots, \tag{3}$$

formed by the $k$ last terms in sequence (1) with $m = L_k(n)+k-1$. The first component of these vectors is the $k$-th record value.

LEMMA 1. *Sequence* (3) *is a Markov chain. The probability density $f_n(z_1, \ldots, z_k)$ of vector* (3) *satisfies the recurrent system of equations*

$$f_1(z_1, \ldots, z_k) = \begin{cases} k!f(z_1)f(z_2)\ldots f(z_k), & z_1 < z_2 < \ldots < z_k, \\ 0 & \textit{otherwise}, \end{cases}$$

$$f_{n+1}(z_1, \ldots, z_k) = \begin{cases} \displaystyle\sum_{i=1}^{k} \int_{-\infty}^{z_1} f_n(y, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k) \frac{f(z_i)}{1-F(y)}\, dy, & \\ & z_1 < z_2 < \ldots < z_k, \\ 0 & \textit{otherwise}, \end{cases}$$

*where* $n = 1, 2, \ldots$

THEOREM 1. *The probability density* $f_n^{(k)}(y)$ *of the k-th record value is of the form*

$$f_n^{(k)}(y) = \frac{k}{(n-1)!}\left[-k\log\big(1-F(y)\big)\right]^{n-1}\big(1-F(y)\big)^{k-1}f(y), \qquad n = 1, 2, \ldots$$

Proof. By induction, the densities $f_n(z_1, \ldots, z_k)$, $n = 1, 2, \ldots$, satisfy the equations

$$f_n(z_1, z_2, \ldots, z_k) = \begin{cases} k!\, g_n(z_1)f(z_1)f(z_2)\ldots f(z_k), & z_1 < z_2 < \ldots < z_k, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where

$$g_1(z) = 1,$$

$$g_{n+1}(z) = k\int_{-\infty}^{z} g_n(y)\frac{f(y)}{1-F(y)}\,dy, \qquad n = 1, 2, \ldots$$

It is easy to verify that

$$g_n(z) = \frac{1}{(n-1)!}\left[-k\log\big(1-F(z)\big)\right]^{n-1}, \qquad n = 1, 2, \ldots$$

Then, by (4), (2) and (3), we have

$$f_n^{(k)}(z) = \int_z^{\infty}\int_{z_2}^{\infty}\ldots\int_{z_{k-1}}^{\infty}\frac{k!}{(n-1)!}\left[-k\log\big(1-F(z)\big)\right]^{n-1}f(z)f(z_2)\ldots f(z_k)\,dz_k\ldots dz_2$$
$$= \frac{k}{(n-1)!}\left[-k\log\big(1-F(z)\big)\right]^{n-1}\big((1-F(z)\big)^{k-1}f(z),$$

whence the proof of the theorem is complete.

COROLLARY 1. *The probability density of the record value is of the form*

$$f_n^{(1)}(y) = \frac{1}{(n-1)!}\left[-\log\big(1-F(y)\big)\right]^{n-1}f(y)$$

(see Karlin [2]).

COROLLARY 2. *The cumulative distribution function of the k-th record value is of the form*

$$F_n^{(k)}(y) = \int_0^{-k\log(1-F(y))}\frac{u^{n-1}}{(n-1)!}e^{-u}\,du, \qquad n = 1, 2, \ldots$$

**3. Limiting distributions.** While considering the limiting properties of the $k$-th record values we use the ideas from the papers by Tata [4] and Resnick [3], where the case $k = 1$ is considered. Here the key role is played by the function

$$R^{(k)}(y) = -k\log\big(1-F(y)\big).$$

THEOREM 2. *In order that there exist normalizing constants $a_n > 0$, $b_n$ and a non-degenerate distribution function $\Phi(x)$ such that*

$$\lim_{n\to\infty} \Pr\{Y_n^{(k)} < a_n y + b_n\} = \Phi(y) \quad \textit{in continuity points of } \Phi(y)$$

*it is necessary and sufficient that there exists a non-decreasing function $g(y)$ (possibly infinite valued) with more than one point of increase such that*

$$\lim_{n\to\infty} \big(R^{(k)}(a_n y + b_n) - n\big)/\sqrt{n} = g(y) \quad \textit{in continuity points of } g(y).$$

*In addition, $\Phi(y) = N\big(g(y)\big)$, where $N(x)$ is the standard normal distribution.*

Proof. Since

$$\Pr\{R^{(k)}(Y_n^{(k)}) < y\} = \int_0^y \frac{u^{n-1}}{(n-1)!} e^{-u} du,$$

we get

$$\lim_{n\to\infty} \Pr\big\{R^{(k)}(Y_n^{(k)}) < \sqrt{n}y + n\big\} = N(y).$$

From this statement and from the equation

$$\Pr\{Y_n^{(k)} < a_n y + b_n\} = \Pr\big\{\big(R^{(k)}(Y_n^{(k)}) - n\big)/\sqrt{n} < \big(R^{(k)}(a_n y + b_n) - n\big)/\sqrt{n}\big\}$$

we obtain theorem 2.

Resnick [3] stated the form of the function $g(y)$ in the case $k = 1$. Analogously, for the $k$-th record values, it can be proved that the function $g(y)$ is of the form

$$g(y) = -\log\big(-\log G(y)\big),$$

where $G(y)$ is the cumulative distribution function belonging to one of the three types of limit distributions of extremal order statistics (see Gnedenko [1]).

THEOREM 3. *The limit distribution of the $k$-th record values belongs to one of the three types of distributions*

$$\Phi_1(y) = N(y), \qquad -\infty < y < \infty, \tag{5}$$

$$\Phi_2(y) = \begin{cases} 0, & y < 0, \\ N(\log y^a) & y \geq 0,\ a > 0, \end{cases} \tag{6}$$

$$\Phi_3(y) = \begin{cases} N\big(\log(-y)^{-a}\big), & y < 0,\ a > 0, \\ 1, & y \geq 0. \end{cases} \tag{7}$$

Finally, we determine the domains of attraction of types (5)-(7). From the dual theorem (see [3], theorem 4.1) we obtain immediately the following

THEOREM 4. *The distribution function* $F(y)$ *belongs to the domain of attraction of the distribution* $N\big(-\log(-\log G(y))\big)$ *if and only if* $1-\exp\big(-\sqrt{R^{(k)}(y)}\big)$ *belongs to the domain of attraction of the limit distribution function of extremal order statistics* $G(y)$.

### References


[1] B. W. Gnedenko, *Sur la distribution limite du terme maximum d'une série aléatoire*, Ann. Math. 44 (1943), p. 423-453.

[2] S. Karlin, *A first course in stochastic processes*, Academic Press, New York 1966.

[3] S. I. Resnick, *Limit laws for record values*, J. Stoch. Proc. Their Appl. 1 (1973), p. 67-82.

[4] M. N. Tata, *On outstanding values in a sequence of random variables*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 12 (1969), p. 9-20.



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## WŁASNOŚCI GRANICZNE STATYSTYK $k$-REKORDOWYCH

### STRESZCZENIE

W pracy podaje się definicję, rozkład prawdopodobieństwa oraz własności graniczne rozkładu prawdopodobieństwa $k$-tych statystyk rekordowych w ciągach niezależnych zmiennych losowych o tym samym rozkładzie.