ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVII, 4 (1983)

LIDIA INGLOT (Wrocław)

POINT PROCESSES OF MINIMAL ORDER STATISTICS

1. Introduction. Let

$$\mathscr{X} = \{X_n : n \in \boldsymbol{Z}\} \tag{1}$$

be a stationary sequence of random variables with distributions $F(x) = P(X_n < x)$, where $\boldsymbol{Z}$ is the set of integer numbers. We denote by $\mathscr{F}^k_{-\infty}(\mathscr{X})$ the σ-field generated by the sequence $\{X_n : n \leqslant k\}$, where $k \in \boldsymbol{Z}$.

Suppose that $\{u_n = u_n(\tau)\}$ is a sequence of real numbers satisfying the following condition:

$$F(u_n) = \tau/n + o(1/n) \quad \text{for every } \tau > 0. \tag{2}$$

The main assumption in this paper is that the sequence (1) satisfies Berman's condition of order 1.

Definition 1. The stationary sequence $\mathscr{X} = \{X_n : n \in \boldsymbol{Z}\}$ satisfies *Berman's condition of order* 1 if for

$$\varphi(x) = \mathrm{E}\left|P\left(X_1 < x \mid \mathscr{F}^0_{-\infty}(\mathscr{X})\right) - F(x)\right| \tag{3}$$

we have

$$\lim_{n\to\infty} n\varphi(u_n) = 0, \tag{4}$$

where u_n is defined to satisfy (2).

Definition 1 is a special case of the definition of [1], p. 503 (see also [2], p. 62).

Remark that every independent sequence $\{X_n : n \in \boldsymbol{Z}\}$ of random variables with identical distributions trivially satisfies Berman's condition.

The next useful tool in our consideration is the following

LEMMA 1. *Suppose that $\mathscr{X} = \{X_n : n \in \boldsymbol{Z}\}$ is a stationary sequence of random variables with distributions $F(x)$. Then*

$$\left|P\Big(\bigcap_{j=1}^{r} \{X_{r_j} \geqslant x\}\Big) - \big(1 - F(x)\big)^n\right| \leqslant n\varphi(x),$$

where φ is a function defined to satisfy (3) *and* $r_j < r_{j+1}$.

Proof. Let $\{\hat{X}_n: n \in \boldsymbol{Z}\}$ be a stationary sequence associated with $\{X_n: n \in \boldsymbol{Z}\}$, e.g., a sequence of independent random variables with identical distributions $F(x) = P(X_n < x) = P(\hat{X}_n < x)$. For fixed n and x, define the events A_{r_j} and B_{r_j}, where $j \in \boldsymbol{Z}$, as follows:

$$A_{r_j} = \{X_{r_j} \geqslant x\} \quad \text{and} \quad B_{r_j} = \{\hat{X}_{r_j} \geqslant x\}.$$

The difference $P\left(\bigcap_{j=1}^{n} A_{r_j}\right) - P\left(\bigcap_{j=1}^{n} B_{r_j}\right)$ can be expressed as the sum of differences of probabilities:

$$\begin{aligned}
P\Big(\bigcap_{j=1}^{n} A_{r_j}\Big) - P\Big(\bigcap_{j=1}^{n} B_{r_j}\Big) &= \Big[P\Big(\bigcap_{j=1}^{n} A_{r_j}\Big) - P\Big(\bigcap_{j=1}^{n-1} A_{r_j} \cap B_{r_n}\Big)\Big] + \\
&+ \Big[P\Big(\bigcap_{j=1}^{n-1} A_{r_j} \cap B_{r_n}\Big) - P\Big(\bigcap_{j=1}^{n-2} A_{r_j} \cap \bigcap_{j=n-1}^{n} B_{r_j}\Big)\Big] + \ldots + \\
&+ \Big[P\Big(\bigcap_{j=1}^{k} A_{r_j} \cap \bigcap_{j=k+1}^{n} B_{r_j}\Big) - P\Big(\bigcap_{j=1}^{k-1} A_{r_j} \cap \bigcap_{j=k}^{n} B_{r_j}\Big)\Big] + \ldots + \\
&+ \Big[P\Big(A_{r_1} \cap \bigcap_{j=2}^{n} B_{r_j}\Big) - P\Big(\bigcap_{j=1}^{n} B_{r_j}\Big)\Big].
\end{aligned}$$

Remark that $\bigcap_{j=1}^{k-1} A_{r_j}$ is in $\mathscr{F}_{-\infty}^{r_{k-1}}(\mathscr{X}) \subset \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})$. Hence

$$\begin{aligned}
&\Big|P\Big(\bigcap_{j=1}^{k} A_{r_j} \cap \bigcap_{j=k+1}^{n} B_{r_j}\Big) - P\Big(\bigcap_{j=1}^{k-1} A_{r_j} \cap \bigcap_{j=k}^{n} B_{r_j}\Big)\Big| \\
&= \Big| \int\limits_{\bigcap_{j=1}^{k-1} A_{r_j}} \Big[P\Big(A_{r_k} \cap \bigcap_{j=k+1}^{n} B_{r_j} \,\big|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\Big) - P\Big(\bigcap_{j=k}^{n} B_{r_j} \,\big|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\Big)\Big] dP \Big| \\
&= \Big| \int\limits_{\bigcap_{j=1}^{k-1} A_{r_j}} \big(P(B_{r_1})\big)^{n-k} \Big[P\big(A_{r_k} \,|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\big) - P(B_{r_k})\Big] dP \Big| \\
&\leqslant \mathrm{E}\big|P\big(A_{r_k} \,|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\big) - P(B_{r_k})\big|.
\end{aligned}$$

Since $\{X_n: n \in \boldsymbol{Z}\}$ is stationary, so is the sequence of random variables $\left\{\big|P\big(A_{r_k} \,|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\big) - P(B_{r_k})\big|,\ k \in \boldsymbol{Z}\right\}$; hence

$$\mathrm{E}\big|P\big(A_{r_k} \,|\, \mathscr{F}_{-\infty}^{r_k-1}(\mathscr{X})\big) - P(B_{r_k})\big| = \mathrm{E}\big|P\big(A_1 \,|\, \mathscr{F}_{-\infty}^{0}(\mathscr{X})\big) - P(B_1)\big|.$$

Thus

$$\Big|P\Big(\bigcap_{j=1}^{n} A_{r_j}\Big) - P\Big(\bigcap_{j=1}^{n} B_{r_j}\Big)\Big| \leqslant n\varphi(x).$$

The proof of Lemma 1 is an obvious modification of the proof of Theorem 2.1 from [1].

2. The main theorem. Let $\boldsymbol{N}$ be the set of natural numbers. For each $n \in \boldsymbol{N}$ we define the discrete parameter process $\eta_n(t)$ for $t = j/n$ and $j \in \boldsymbol{Z}$ as $\eta_n(j/n) = X_j$. Thus for a given n the process η_n is obtained from $\{X_n\colon n \in \boldsymbol{Z}\}$ by time scale changes.

Definition 2. The point process N_n, $n \in \boldsymbol{N}$, defined as

$$N_n(B) = \sum_{j/n \in B} \mathbf{1}_{\{\eta_n(j/n) < u_n\}} \tag{5}$$

for every Borel subset B of the real line, where $\mathbf{1}_A(\cdot)$ is the indicator function of the set A, will be called a *point process of minimal order statistics*.

The point processes M_n may be properly regarded as random elements either in the space of integer-valued increasing step functions on the real line or in the space $\mathcal{N}$ of integer-valued Borel measures on the real line. In either case the space is metric under the "vague topology" (e.g., generated in $\mathcal{N}$ by the functions $\mu \to \int f d\mu$ for continuous f with bounded support, cf. [3]) and we may consider convergence in distribution of such random elements ($M_n \xrightarrow{d} M$ will be used to indicate this convergence).

The following result is a special case of a theorem of Kallenberg [3].

THEOREM 1. *Let M_n, $n \in \boldsymbol{N}$, be point processes on the real line and let M be a point process without multiple events and such that $M(\{a\}) = 0$ (a. s.) for every fixed real a. Assume that*

(i) $P\big(M_n(B) = 0\big) \to P\big(M(B) = 0\big)$ *for all sets B of the form* $\bigcup_{i=1}^{r} (a_i, b_i]$, $a_1 < b_1 < \ldots < a_r < b_r$;

(ii) $\limsup_{n\to\infty} \mathrm{E} M_n(a, b] \leqslant \mathrm{E} M(a, b]$ *for all finite* $a < b$.

Then $M_n \xrightarrow{d} M$.

Now we use this theorem to obtain the main result of this paper. The idea of proof of the following theorem is taken from [4].

THEOREM 2. *Let $\{X_n\colon n \in \boldsymbol{Z}\}$ be a stationary sequence of random variables which satisfies Berman's conditions* (3) *and* (4). *Suppose that $\{u_n\colon n \in \boldsymbol{N}\}$ satisfies condition* (2). *If N_n is the point process defined in* (5), *then*

$$N_n \xrightarrow{d} N,$$

where N is a Poisson process with parameter τ.

LEMMA 2. *Let $\{X_n\colon n \in \boldsymbol{Z}\}$ satisfy the assumption of Theorem 2 and let $\{u_n\colon n \in \boldsymbol{N}\}$ satisfy condition* (2). *Then for every $a > 0$ we have*

$$\lim_{n\to\infty} P\big(\min_{1 \leqslant j \leqslant [an]} X_i < u_n\big) = 1 - e^{-a\tau}.$$

Proof. Let

$$Z_n = \min_{1 \leqslant i \leqslant n} X_i \quad \text{and} \quad A_{in} = \{X_i \geqslant u_n\}.$$

Remark that if $a = 1$, then

$$\lim_{n\to\infty} P(Z_n < u_n) = 1 - e^{-\tau}, \tag{6}$$

which follows easily from the equalities

$$|P(Z_n < u_n) - (1-e^{-\tau})| \leqslant \Big|\prod_{i=1}^{n} P(A_{in}) - e^{-\tau}\Big| + \Big|P\big(\bigcap_{i=1}^{n} A_{in}\big) - \prod_{i=1}^{n} P(A_{in})\Big|$$
$$= \big|\big(1-F(u_n)\big)^n - e^{-\tau}\big| + \big|P\big(\bigcap_{i=1}^{n} A_{in}\big) - \big(1-F(u_n)\big)^n\big|$$
$$= \big|\big(1-\tau/n + o(1/n)\big)^n - e^{-\tau}\big| + n\varphi(u_n).$$

Now assume that $a > 0$. From (6) we obtain

$$\lim_{n\to\infty} P\big(Z_{[an]} < u_{[an]}(a\tau)\big) = 1 - e^{-a\tau}.$$

Hence it is necessary only to show that

$$\lim_{n\to\infty} \big[P\big(Z_{[an]} < u_{[an]}(a\tau)\big) - P\big(Z_{[an]} < u_n(\tau)\big)\big] = 0. \tag{7}$$

But the proof of (7) is an obvious modification of the proof of Theorem 2.1 from [4]. Clearly, (7) proves Lemma 2.

Proof of Theorem 2. To prove the theorem it is enough to show that the sequence $\{N_n\colon n \in \mathbf{N}\}$ satisfies the assumptions of Theorem 1. The assumption (ii) of this theorem holds trivially:

$$\mathrm{E}N_n(a, b] = ([bn]-[an])F(u_n) \leqslant n(b-a)\big(\tau/n + o(1/n)\big) = \mathrm{E}N(a, b].$$

To check (i) suppose that $B = (a, b]$ for some $a < b$. Then

$$P\big(N_n(a, b] = 0\big) = P(Z_{[bn]-[an]} \geqslant u_n).$$

Let $h > 0$ and let n be sufficiently large. Then

$$[(b-a)n] \leqslant [bn]-[an] \leqslant [(b-a)n]+1 \leqslant [(b-a+h)n]$$

and, consequently,

$$P(Z_{[(b-a+h)n]} \geqslant u_n) \leqslant P(Z_{[bn]-[an]} \geqslant u_n) \leqslant P(Z_{[(b-a)n]} \geqslant u_n).$$

By Lemma 2, the outside terms have the limits $e^{-\tau(b-a+h)}$ and $e^{-\tau(b-a)}$, respectively, and since h is arbitrary, we have

$$\lim_{n\to\infty} P\big(N_n(a, b] = 0\big) = e^{-\tau(b-a)}.$$

Now let $B = \bigcup_{i=1}^{r} (a_i, b_i]$, $a_1 < b_1 < \ldots < a_r < b_r$, and put

$$E_j = ([a_j n]+1, [a_j n]+2, \ldots, [b_j n]),$$

$$Z(E_j) = \min\{X_i\colon i \in E_j\},$$

$$C_{jn} = \{Z(E_j) \geqslant u_n\}, \qquad A_{in} = \{X_i \geqslant u_n\}.$$

Then

$$P\big(N_n(B) = 0\big) = P\Big(\bigcap_{j=1}^{r} C_{jn}\Big)$$

$$= \prod_{j=1}^{r} P\big(N_n(a_j, b_j] = 0\big) + \Big[P\Big(\bigcap_{j=1}^{r} C_{jn}\Big) - \prod_{j=1}^{r} P(C_{jn})\Big].$$

Now it is enough to show that

$$\lim_{n\to\infty}\Big[P\Big(\bigcap_{j=1}^{r} C_{jn}\Big) - \prod_{j=1}^{r} P(C_{jn})\Big] = 0.$$

Using Lemma 1 we obtain

$$\Big|P\Big(\bigcap_{j=1}^{r} C_{jn}\Big) - \prod_{j=1}^{r} P(C_{jn})\Big| = \Big|P\Big(\bigcap_{j=1}^{r} \bigcap_{i\in E_j} A_{in}\Big) - \prod_{j=1}^{r} P\Big(\bigcap_{i\in E_j} A_{in}\Big)\Big|$$

$$\leqslant \Big|P\Big(\bigcap_{j=1}^{r} \bigcap_{i\in E_j} A_{in}\Big) - \prod_{j=1}^{r} \prod_{i\in E_j} P(A_{in})\Big| +$$

$$+ \Big|\prod_{j=1}^{r} \prod_{i\in E_j} P(A_{in}) - \prod_{j=1}^{r} P\Big(\bigcap_{i\in E_j} A_{in}\Big)\Big|$$

$$\leqslant \varphi(u_n)\Big(\sum_{j=1}^{r}([b_j n] - [a_j n])\Big) + \prod_{j=1}^{r}\big(\varphi(u_n)([b_j n] - [a_j n])\big)$$

$$\leqslant \big(\varphi(u_n)n\big)\big(r \max_{1\leqslant j\leqslant r}(b_j - a_j) + r/n\big) + \big(\varphi(u_n)n\big)^r \big(\max_{1\leqslant j\leqslant r}(b_j - a_j) + 1/n\big)^r.$$

Thus, if B is a finite sum on the segment $(a_i, b_i]$, then

$$\lim_{n\to\infty} P\big(N_n(B) = 0\big) = \prod_{j=1}^{r} \exp\{-\tau(b_j - a_j)\} = \exp\{-\tau m(B)\},$$

where m is the Lebesgue measure. Therefore, using Theorem 1 we obtain Theorem 2.

3. Application. Suppose that the sequence $\{X_n\colon n \in \boldsymbol{Z}\}$ satisfies the assumption of Theorem 2 and that $\{N_n\colon n \in \boldsymbol{N}\}$ is defined as above. From Theorem 2 we obtain the following fact:

For every Borel set B whose boundary has Lebesgue measure zero $(m(\partial B) = 0)$ we have

(8) $$\lim_{n\to\infty} P\big(N_n(B) = r\big) = e^{-\tau m(B)} \frac{\big(\tau m(B)\big)^r}{r!}.$$

Moreover, the joint distribution of any finite number of $N_n(B_1), \ldots, N_n(B_k)$ corresponding to disjoint B_i (with $m(\partial B_i) = 0$ for each i) converges to the product of the corresponding Poisson probabilities.

Let Z_n^k be the $(n-k+1)$-st order statistic for $\{X_n: n \in \boldsymbol{Z}\}$. Remark that $\{N_n(0,1] < k\} = \{Z_n^k \geqslant u_n\}$. Hence and from (8) we have

$$\lim_{n\to\infty} P(Z_n^k < u_n) = 1 - \sum_{i=0}^{k-1} e^{-\tau} \frac{\tau^i}{i!}.$$

References

[1] S. M. Berman, *Limit theorems for the maximum term in stationary sequence*, Ann. Math. Statist. 35 (1964), p. 502-516.

[2] W. Dziubdziela, *Rozkłady graniczne ekstremalnych statystyk pozycyjnych*, Matematyka Stosowana 9 (1977), p. 45-71.

[3] O. Kallenberg, *Characterization and convergence of random measures and point processes*, Z. Wahrscheinlichkeitstheorie und Verwandte Gebiete 27 (1973), p. 9-21.

[4] M. R. Leadbetter, *Weak convergence of high level exceedances by a stationary sequence*, ibidem 34 (1976), p. 11-15.

Received on 4. 6. 1979

LIDIA INGLOT (Wrocław)

PROCESY PUNKTOWE MINIMALNYCH STATYSTYK POZYCYJNYCH

STRESZCZENIE

Praca poświęcona jest teorii rozkładów granicznych ekstremalnych statystyk pozycyjnych w stacjonarnym (w węższym sensie) ciągu zmiennych losowych spełniających warunek Bermana. Zdefiniowany został proces minimalnych statystyk pozycyjnych i pokazano jego zbieżność do procesu Poissona. W dowodzie twierdzenia i lematu wykorzystano metody podane przez Leadbettera w [4].
