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ON THE CONVERGENCE OF BHATTACHARYYA BOUNDS IN THE MULTIPARAMETER CASE

1. Introduction. Blight and Rao [2] have considered the Bhattacharyya bounds for the unbiased estimation of a parametric function $\tau(\theta)$ when the sampling distribution is a member of an exponential family with density $f(t;\theta)$, with respect to a σ-finite measure ν, which has the property

$$\frac{\partial \log\{f(t;\theta)\}}{\partial\theta} = V^{-1}(\theta)(t-\theta),$$

where $V(\theta) = C_0+C_1\theta+C_2\theta^2$ for some constants C_0, C_1 and C_2.

Seth [5] has proved that the Bhattacharyya matrix for this family is diagonal. Shanbhag [6] has proved this family to be equivalent within a linear transform to the family composed of the normal, gamma, Poisson, binomial and negative binomial distributions. He has also shown that the distribution assumptions are necessary as well as sufficient for the diagonality of the Bhattacharyya matrix. Using these results Blight and Rao [2] have shown that the Bhattacharyya bounds converge to the variance of the minimum variance unbiased estimator of the function $\tau(\theta)$. This paper deals with the multiparameter case.

Assume that a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ has a joint probability density function

$$f(\boldsymbol{x};\boldsymbol{\theta}) = f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_r)$$

with respect to a σ-finite measure μ, and let $K(\boldsymbol{X})$ be an unbiased estimator of a real function $\tau(\boldsymbol{\theta}) = \tau(\theta_1, \theta_2, \ldots, \theta_r)$ with finite variance. Let $k>1$ be an integer. Put

$$\iota = (i_1, i_2, \ldots, i_r), \quad 0 \leqslant i_j,\ 0 < i_1+i_2+\ldots+i_r \leqslant k,$$

and $\iota' = (i'_1, i'_2, \ldots, i'_r)$ with similar properties. Bhattacharyya [1] has proved the following:

Assume that

(1) the functions $f(\boldsymbol{x};\boldsymbol{\theta})$ and $\tau(\boldsymbol{\theta})$ have all partial derivatives with respect to $\theta_1, \theta_2, \ldots, \theta_r$ of order up to k,

$$f^{(\iota)} = \frac{\partial^{i_1+i_2+\ldots+i_r} f(\boldsymbol{x};\boldsymbol{\theta})}{\partial\theta_1^{i_1}\partial\theta_2^{i_2}\ldots\partial\theta_r^{i_r}}, \qquad \tau^{(\iota)} = \frac{\partial^{i_1+i_2+\ldots+i_r}\tau(\boldsymbol{\theta})}{\partial\theta_1^{i_1}\partial\theta_2^{i_2}\ldots\partial\theta_r^{i_r}},$$

and they can be calculated under the integral with respect to $\boldsymbol{x}$;

(2) the expectations

$$J(\iota, \iota';\boldsymbol{\theta}) = \mathrm{E}_{\boldsymbol{\theta}}\left\{\frac{f^{(\iota)}}{f(\boldsymbol{X};\boldsymbol{\theta})}\frac{f^{(\iota')}}{f(\boldsymbol{X};\boldsymbol{\theta})}\right\}$$

exist and are finite; they form the matrix $\|J(\iota, \iota';\boldsymbol{\theta})\|$ called the *generalized Bhattacharyya matrix*;

(3) there exists an inverse matrix

$$\|J[\iota, \iota';\boldsymbol{\theta}]\| = \|J(\iota, \iota';\boldsymbol{\theta})\|^{-1};$$

(4) the function $K(\boldsymbol{x})f(\boldsymbol{x};\boldsymbol{\theta})$ is differentiable with respect to $\theta_1, \theta_2, \ldots, \theta_r$ under the integral with respect to $\boldsymbol{x}$ at least k times.

Then

$$\operatorname{var}\{K(\boldsymbol{X})\} \geqslant \sum \tau^{(\iota)}\tau^{(\iota')}J[\iota, \iota';\boldsymbol{\theta}],$$

where summation is running over all ι, ι'.

The right-hand side of this inequality is called the *k-th generalized Bhattacharyya bound.*

To make calculations simpler, in the sequel we consider the case $r = 2$. All results can easily be extended for more than two dimensions. Using the result of Bhattacharyya [1] and assuming that the complete sufficient vector statistic has independent components whose distributions satisfy the regularity conditions of Blight and Rao [2] we prove that the sequence of the generalized Bhattacharyya bounds converges to the variance of the best unbiased estimator of $\tau(\boldsymbol{\theta})$ when $k \to \infty$. As an application of our result we give, in Section 3, the Bhattacharyya bounds for the variance of the minimum variance unbiased estimator of $\Pr(Y < X)$ when independent samples are taken from one-parameter exponential distributions.

2. The main result. Let $\boldsymbol{T} = (T_1, T_2)$ be a random vector, where T_i are independent random variables with probability density functions $f_i = f_i(t_i;\theta_i)$ with respect to a σ-finite measure μ_i $(i = 1, 2)$. Thus $\boldsymbol{T}$ has the probability density function

$$f = f(\boldsymbol{t};\boldsymbol{\theta}) = f_1(t_1;\theta_1)f_2(t_2;\theta_2) \tag{1}$$

with respect to measure $\mu = \mu_1 \times \mu_2$. We assume the following regularity conditions:

I. $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Omega = \Omega_1 \times \Omega_2$, where Ω_i $(i = 1, 2)$ are open intervals on the real line.

II. The distributions of the random variables T_i belong to an exponential family with the property

$$\frac{\partial \log f_i}{\partial \theta_i} = V_i^{-1}(\theta_i)(t_i - \theta_i) \qquad (\theta_i \in \Omega_i),$$

where T_i is a complete sufficient statistic for θ_i and $V_i(\theta_i) = C_0^{(i)} + C_1^{(i)}\theta_i + C_2^{(i)}\theta_i^2$ for some constants $C_0^{(i)}$, $C_1^{(i)}$, $C_2^{(i)}$ $(i = 1, 2)$. Hence $\boldsymbol{T} = (T_1, T_2)$ is the complete sufficient statistic for $\boldsymbol{\theta} = (\theta_1, \theta_2)$.

III. The density $f(\boldsymbol{t}; \boldsymbol{\theta})$ can be differentiated with respect to θ_1 and θ_2 under the integral with respect to $\boldsymbol{t}$ any number of times.

IV. The density $f(\boldsymbol{t}; \boldsymbol{\theta})$ and the parametric function $\tau(\boldsymbol{\theta})$ admit a convergent Taylor series expansion of two variables θ_1, θ_2 at each $\boldsymbol{\theta} \in \Omega$ for almost all $\boldsymbol{t}$.

Let $\iota = (i_1, i_2)$, $0 \leqslant i_j$ $(j = 1, 2)$, $0 < i_1 + i_2 \leqslant k$ $(k = 1, 2, \ldots)$, and let $\iota' = (i_1', i_2')$ with similar properties. Let

$$u(\iota) = u_{(i_1, i_2)}(\boldsymbol{t}; \boldsymbol{\theta}) = \frac{1}{f(\boldsymbol{t}; \boldsymbol{\theta})} \frac{\partial^{i_1 + i_2} f(\boldsymbol{t}; \boldsymbol{\theta})}{\partial \theta_1^{i_1} \partial \theta_2^{i_2}}$$

and $J(\iota, \iota') = J(\iota, \iota'; \boldsymbol{\theta}) = \mathrm{E}_{\boldsymbol{\theta}}\{u(\iota) u(\iota')\}$. From the results of Seth [5] and Blight and Rao [2] two lemmas follow.

LEMMA 1. *If $f(\boldsymbol{t}; \boldsymbol{\theta})$ is of form* (1) *and conditions* I, II *and* III *are satisfied, then*

$$u(\iota) = u_{i_1}^{(1)} u_{i_2}^{(2)}, \quad \textit{where } u_{i_j}^{(j)} = \frac{1}{f_j} \frac{\partial^{i_j} f_j}{\partial \theta_j^{i_j}} \qquad (j = 1, 2),$$

and

$$J(\iota, \iota') = \begin{cases} \{J_{i_1}^{(1)}\}^2 \{J_{i_2}^{(2)}\}^2 & \textit{if } \iota = \iota', \textit{ i.e.}, i_1 = i_1', \ i_2 = i_2', \\ 0 & \textit{otherwise}, \end{cases} \tag{2}$$

where $\{J_{i_j}^{(j)}\}^2 = \{J_{i_j}^{(j)}(\theta_j)\}^2 = \mathrm{E}_{\theta_j}\{u_{i_j}^{(j)}\}^2$ $(j = 1, 2)$ *are Bhattacharyya functions* (see [2]) *and* $\{J_0^{(j)}\}^2 = 1$ $(j = 1, 2)$.

It follows from formula (2) that the generalized Bhattacharyya matrix $\|J(\iota, \iota')\|$ is diagonal.

LEMMA 2. *Let* $\Phi(0) = 1$ *and*

$$\Phi(\iota) = \Phi(\iota; \boldsymbol{t}, \boldsymbol{\theta}) = u(\iota) \{J(\iota, \iota')\}^{-1/2},$$

$$\{\iota = (i_1, i_2),\ 0 \leqslant i_j,\ 0 < i_1 + i_2 \leqslant k,\ k = 1, 2, \ldots\}.$$

If conditions I, II *and* III *are satisfied, then the set* $\{\Phi(0), \Phi(\iota)\}$ *is orthonormal in* $\mathscr{L}_2(\boldsymbol{t}, \boldsymbol{\theta})$, *the space of all functions of* $\boldsymbol{t} = (t_1, t_2)$ *having finite second moments with respect to the density* $f(\boldsymbol{t}; \boldsymbol{\theta})$.

Notice that $\Phi(\iota) = \Phi_{i_1}^{(1)}\Phi_{i_2}^{(2)}$, where $\Phi_{i_j}^{(j)} = u_{i_j}^{(j)}\{J_{i_j}^{(j)}\}^{-1}$ $(j = 1, 2)$ are defined in [2] for the one-parameter case.

Similarly as in [2] we prove now the main theorem.

THEOREM 1. *Let $K = K(T)$ be the minimum variance unbiased estimator of the function $\tau(\theta)$ with finite variance. If conditions* I-IV *are satisfied and $K(t)f(t;\theta)$ is differentiable with respect to θ_1 and θ_2 under the integral with respect to t any number of times, then for $\theta \in \Omega$*

$$\operatorname{var}(K) = \sum_{k=1}^{\infty}\sum_{j=0}^{k}\left\{\frac{\partial^k \tau(\theta)}{\partial\theta_1^j\partial\theta_2^{k-j}}\frac{1}{J_j^{(1)}(\theta_1)J_{k-j}^{(2)}(\theta_2)}\right\}^2.$$

Proof. Let $C_0(\theta) = \tau(\theta)$ and

$$C_\iota(\theta) = \frac{\partial^{i_1+i_2}\tau(\theta)}{\partial\theta_1^{i_1}\partial\theta_2^{i_2}}\frac{1}{J_{i_1}^{(1)}J_{i_2}^{(2)}}.$$

Differentiating both sides of the equality

$$\int K(t)f_1(t_1;\theta_1)f_2(t_2;\theta_2)d\mu(t) = \tau(\theta)$$

i_1 times with respect to θ_1 and i_2 times with respect to θ_2 and using the definition of $\Phi(\iota)$ we get

$$\int K(t)\Phi(\iota)f(t;\theta)d\mu(t) = C_\iota(\theta).$$

Since $K \in \mathscr{L}_2(t,\theta)$ and $\{1, \Phi(\iota)\}$ is orthonormal, by the Bessel inequality (see [4], p. 151) we have

$$C_0^2(\theta) + \sum_\iota C_\iota^2(\theta) < \infty \quad (\theta \in \Omega),$$

where the summation is running over all possible $\iota = (i_1, i_2)$ $(0 \leqslant i_j,$ $0 < i_1+i_2 \leqslant k,\ k = 1, 2, \ldots)$. It therefore follows from the Riesz-Fischer theorem (see [4], p. 153) that for each $\theta \in \Omega$ there exists a function $K_\theta(t) \in \mathscr{L}_2(t,\theta)$ such that

$$\mathrm{E}_\theta\{K_\theta^2(T)\} = C_0^2(\theta) + \sum_\iota C_\iota^2(\theta), \tag{3}$$

where

$$C_\iota(\theta) = \int K_\theta(t)\Phi(\iota)f(t;\theta)d\mu(t).$$

Hence we have

$$\frac{\partial^{i_1+i_2}\tau(\theta)}{\partial\theta_1^{i_1}\partial\theta_2^{i_2}} = \int K_\theta(t)\frac{\partial^{i_1+i_2}f(t;\theta)}{\partial\theta_1^{i_1}\partial\theta_2^{i_2}}d\mu(t).$$

Let $\boldsymbol{\theta} = \boldsymbol{\theta}_0 = (\theta_{10}, \theta_{20})$ be fixed and evaluate the expectation $\mathrm{E}_{\boldsymbol{\theta}}\{K_0(\boldsymbol{T})\}$ of $K_0(\boldsymbol{T}) = K_{\boldsymbol{\theta}_0}(\boldsymbol{T})$, i.e.,

$$\mathrm{E}_{\boldsymbol{\theta}}\{K_0(\boldsymbol{T})\} = \int K_0(\boldsymbol{t}) f(\boldsymbol{t}; \boldsymbol{\theta})\, d\mu(\boldsymbol{t}) = \int K_0(\boldsymbol{t}) \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial f}{\partial \theta_1} g + \frac{\partial f}{\partial \theta_2} h\right)^{(k)} d\mu(\boldsymbol{t}),$$

where $g = \theta_1 - \theta_{10}$, $h = \theta_2 - \theta_{20}$ and

$$\left(\frac{\partial f}{\partial \theta_1} g + \frac{\partial f}{\partial \theta_2} h\right)^{(k)} = \frac{\partial^k f}{\partial \theta_1^k} g^k + \binom{k}{1} \frac{\partial^k f}{\partial \theta_1^{k-1} \partial \theta_2} g^{k-1} h + \ldots + \binom{k}{k} \frac{\partial^k f}{\partial \theta_2^k} h^k .$$

Hence, putting

$$u^0_{(i_1, i_2)} = u^0_{(i_1, i_2)}(\boldsymbol{t}, \boldsymbol{\theta}_0) = \frac{1}{f} \left.\frac{\partial^{i_1 + i_2} f}{\partial \theta_1^{i_1} \partial \theta_2^{i_2}}\right|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0},$$

we obtain

$$\begin{aligned} (4) \qquad \mathrm{E}_{\boldsymbol{\theta}}\{K_0(\boldsymbol{T})\} &= \int K_0(\boldsymbol{t}) \sum_{k=0}^{\infty} \frac{1}{k!} \Big\{ u^0_{(k,0)} g^k + \binom{k}{1} u^0_{(k-1,1)} g^{k-1} h + \ldots + \\ &\quad + \binom{k}{k} u^0_{(0,k)} h^k \Big\} f(\boldsymbol{t}; \boldsymbol{\theta}_0)\, d\mu(\boldsymbol{t}) \\ &= \int K_0(\boldsymbol{t}) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} u^0_{(j,k-j)} g^j h^{k-j} f(\boldsymbol{t}; \boldsymbol{\theta}_0)\, d\mu(\boldsymbol{t}). \end{aligned}$$

Let us consider

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \left\{ \int \left| K_0(\boldsymbol{t}) \frac{1}{k!} \binom{k}{j} u^0_{(j,k-j)} g^j h^{k-j} f(\boldsymbol{t}; \boldsymbol{\theta}_0) \right| d\mu(\boldsymbol{t}) \right\}.$$

By the Schwarz inequality this is less than or equal to

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{j=0}^{k} \left\{ \int K_0^2(\boldsymbol{t}) f(\boldsymbol{t}; \boldsymbol{\theta}_0)\, d\mu(\boldsymbol{t}) \right\}^{1/2} \left[\int \{u^0_{(j,k-j)}\}^2 f(\boldsymbol{t}; \boldsymbol{\theta}_0)\, d\mu(\boldsymbol{t}) \right]^{1/2} \frac{1}{k!} \binom{k}{j} |g|^j |h|^{k-j} \\ &= \left\{ C_0^2(\boldsymbol{\theta}_0) + \sum_{\iota} C_{\iota}^2(\boldsymbol{\theta}_0) \right\}^{1/2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} |g|^j |h|^{k-j} \times \\ &\quad \times \left[\int \{u^{(1)}_{i_1}(\theta_{10})\}^2 \{u^{(2)}_{i_2}(\theta_{20})\}^2 f_1(t_1; \theta_{10}) f_2(t_2; \theta_{20})\, d\mu(\boldsymbol{t}) \right]^{1/2} \\ &= \left\{ C_0^2(\boldsymbol{\theta}_0) + \sum_{\iota} C_{\iota}^2(\boldsymbol{\theta}_0) \right\}^{1/2} \left\{ \sum_{j=0}^{\infty} \frac{|\theta_1 - \theta_{10}|^j}{j!} J_j^{(1)}(\theta_{10}) \right\} \left\{ \sum_{r=0}^{\infty} \frac{|\theta_2 - \theta_{20}|^r}{r!} J_r^{(2)}(\theta_{20}) \right\}. \end{aligned}$$

The last expression is finite in an open rectangle containing θ_0. This follows from (3) and from the convergence of these series for the one-parameter case (see [2]). The integration and summation operators in (4) may be interchanged. Thus, for θ belonging to an open rectangle containing θ_0, we have

$$\begin{aligned}\mathrm{E}_\theta\{K_0(T)\} &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}g^j h^{k-j}\int K_0(t)\,u^0_{(j,k-j)}f(t;\,\theta_0)\,d\mu(t)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}g^j h^{k-j}\frac{\partial^k\tau(\theta)}{\partial\theta_1^j\,\partial\theta_2^{k-j}} = \tau(\theta).\end{aligned}$$

Hence $\mathrm{E}_\theta\{K_0(T)-K(T)\} = 0$. Therefore, from the completeness of T it follows that $K_0 = K$ almost everywhere and

$$\operatorname{var}(K\,|\,\theta_0) = \operatorname{var}(K_0\,|\,\theta_0) = \sum_\iota C_\iota^2(\theta_0).$$

Since θ_0 is arbitrary, this completes the proof of the theorem.

3. Application. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent samples. Assume that X_i $(i = 1, 2, \ldots, n)$ have the one-parameter exponential distribution with density

$$f(x;\,\theta_1) = \begin{cases}\theta_1^{-1}\exp(-x/\theta_1) & \text{if } x>0,\\ 0 & \text{otherwise,}\end{cases}$$

and that Y_j $(j = 1, 2, \ldots, m)$ have the distribution with density $f(y;\,\theta_2)$. The statistic

$$T = (T_1, T_2) = \Big(\sum_i X_i, \sum_j Y_j\Big)$$

is sufficient and complete for $\theta = (\theta_1, \theta_2)$. Tong [7], [8] and Johnson [3] have derived the minimum variance unbiased estimation of the function

$$\tau(\theta) = \theta_1(\theta_1+\theta_2)^{-1} = \Pr(Y<X)$$

which takes the form

$$\hat{P} = \begin{cases}\displaystyle\sum_{i=0}^{n-1}(-1)^i\frac{(n-1)!\,(m-1)!}{(n-1-i)!\,(m-1-i)!}\left(\frac{T_2}{T_1}\right)^i & \text{if } T_2\leqslant T_1,\\[2ex] \displaystyle 1-\sum_{i=0}^{m-1}(-1)^i\frac{(m-1)!\,(n-1)!}{(m-1-i)!\,(n-1-i)!}\left(\frac{T_1}{T_2}\right)^i & \text{if } T_2>T_1.\end{cases}$$

Using our result we can obtain the variance of $\hat{P}$. It is easy to verify that

$$\frac{\partial^{i_1+i_2}\tau(\theta)}{\partial\theta_1^{i_1}\,\partial\theta_2^{i_2}} = \frac{(i_1+i_2-1)!\,(-1)^{i_1+i_2}\,(i_2\theta_1-i_1\theta_2)}{(\theta_1+\theta_2)^{i_1+i_2-1}}. \tag{5}$$

Blight and Rao [2] have derived the Bhattacharyya functions for the exponential distribution

$$\{J_i^{(1)}(\theta_1)\}^2 = \frac{(n+i-1)!\,i!}{(n-1)!\,\theta_2^{2i}}$$

and similarly for

$$\{J_i^{(2)}(\theta_2)\}^2 = \frac{(m+i-1)!\,i!}{(m-1)!\,\theta_2^{2i}}.$$

Hence

$$\operatorname{var}(\hat{P}) = \sum_{k=1}^{\infty}\sum_{j=0}^{k}\left\{\frac{\partial^k\tau(\theta)}{\partial\theta_1^j\,\partial\theta_2^{k-j}}\right\}^2 \frac{(n-1)!\,(m-1)!\,(\theta_1\theta_2)^{2k}}{(n+j-1)!\,(m+k-j-1)!\,j!\,(k-j)!}.$$

Using (5) and putting $\varrho = \theta_1/\theta_2$ we get

$$\operatorname{var}(\hat{P}) = \sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{\binom{k}{j}^2}{\binom{n+j-1}{j}\binom{m+k-j-1}{k-j}}\left\{\frac{j(1+\varrho)-k}{k}\right\}^2\frac{\varrho^{2(k-j)}}{(1+\varrho)^{2k+1}}.$$

Table 1 contains the values of the first four generalized Bhattacharyya bounds B_k for $\varrho = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and $n = m = 5$; $n = m = 10$; $n = 5, m = 10$; $n = 20, m = 10$. It is seen that the convergence is fairly fast in all cases.

TABLE 1. Generalized Bhattacharyya bounds for the best unbiased estimator of $\Pr(Y < X)$ in the exponential case multiplied by 10^3

ϱ	$n = m = 5$				$n = m = 10$			
	B_1	B_2	B_3	B_4	B_1	B_2	B_3	B_4
1/4	10.24	11.76	12.16	12.26	0.51	1.01	1.06	1.07
1/2	19.74	19.94	20.29	20.36	9.87	10.42	10.47	10.48
3/4	23.99	26.08	26.41	26.45	11.99	12.56	12.62	12.62
1	25.00	27.08	27.40	27.43	12.50	13.07	13.11	13.11
	$n = 5,\ m = 10$				$n = 20,\ m = 10$			
	B_1	B_2	B_3	B_4	B_1	B_2	B_3	B_4
1/4	0.77	1.80	2.12	2.22	3.84	3.98	3.99	3.99
1/2	14.81	16.52	19.47	20.13	7.40	7.63	7.65	7.65
3/4	17.99	19.53	19.74	19.78	8.99	9.30	9.31	9.31
1	18.75	21.30	21.46	21.50	9.38	9.73	9.76	9.76

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O ZBIEŻNOŚCI OGRANICZEŃ BHATTACHARYYI W PRZYPADKU WIELOPARAMETROWYM

STRESZCZENIE

W pracy rozpatrzono uogólnione ograniczenia Bhattacharyyi dla nieobciążonej estymacji funkcji parametrycznej wielu zmiennych, gdy wektor statystyk dostatecznych ma niezależne składowe o rozkładach z jednoparametrowej rodziny wykładniczej. Udowodniono, że ciąg uogólnionych ograniczeń Bhattacharyyi jest zbieżny do wariancji najlepszego nieobciążonego estymatora funkcji parametrycznej. Wynik ten zastosowano do obliczenia wariancji nieobciążonego estymatora z jednostajnie minimalną wariancją prawdopodobieństwa $\Pr(Y < X)$, gdzie X i Y są niezależnymi zmiennymi losowymi o rozkładach wykładniczych.
