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ON SYSTEM RELIABILITY UNDER RANDOM LOAD OF ELEMENTS

0. In this note the reliability of elements with random failure rate function and the systems consisting of such elements are considered. We suppose that it forms a probabilistic description of elements and systems having some repertoire of tasks or working in a random environment when the tasks or environmental conditions are randomly varying, causing an unequal load of elements. Assuming that the failure rate function of an element is a semi-Markov process we find the reliability function of an element and its limiting properties. Considering systems of elements with randomly varying failure rate function of elements we prove that if the failure rate function of elements depends upon the random environment, then the working times of elements of the system are random variables by mixture positively dependent and if this failure rate function of elements depends upon the number of working elements in the system, then the working times of elements are associated random variables. These properties can be used to estimate the system reliability by the marginal reliabilities of elements (see [10] and [3]).

1. Reliability of an element. Let us consider an element of a system and assume that its load varies, for example, depending on the task performed by the system, on the state of other elements of the system or on the random environment. Strictly speaking, we assume that the failure rate function of an element is a random process; the aim of our considerations is the reliability function of an element and its limiting properties. It is easy to give examples of such systems (see [9]), e.g. the motor of a fishing ship used for the motion of the ship during the trip, for the manoeuvres in the port and for the driving of the board processing plants, and also the combine harvester used for harvest and threshing, stationary threshing, etc.

Let $\Lambda(t)$, $t \geqslant 0$, denote the semi-Markov process described on a finite state set $\Lambda = \{\lambda_j\colon j \in J\}$, where $J = \{1, 2, \ldots, m\}$ is defined in Pyke's sense [8] by the regeneration moments t_n, $n \in N$, where $N = \{0, 1, \ldots\}$, $t_0 = 0$, and the homogeneous Markov chain $\Lambda_n = \Lambda(t_n)$, $n \in N$. This process is characterized by the pair $(F_i, (p_{ij}))$, where F_i denotes the probability distribution function of the random variable $t_n - t_{n-1}$ under the condition $\Lambda_{n-1} = \lambda_i$ and (p_{ij}) denotes the transition probability matrix of the Markov chain Λ_n, $n \in N$. We usually assume that there exist an expected value μ_i of the probability distribution function F_i and a limiting probability distribution σ_j, $j \in J$, of the chain Λ_n, $n \in N$. Under this assumption the limiting probabilities of the process $\Lambda(t)$, $t \geqslant 0$, are of the form

$$q_i = \mu_i \sigma_i \Big/ \sum \mu_j \sigma_j, \qquad i \in J.$$

Here and in the sequel, having a summation over the set J, we omit the summation bounds.

Let $\Lambda(t)$, $t \geqslant 0$, be the failure rate process of an element and let Z denote its working time. The reliability function of an element under the initial condition $\Lambda_0 = \lambda_i$ is now of the form

$$(1) \quad P_i(x) = \Pr(Z > x \mid \Lambda_0 = \lambda_i) = \mathrm{E}\Big(\exp\Big[-\int_0^x \Lambda(u)\,du\Big] \Bigm| \Lambda_0 = \lambda_i\Big), \qquad i \in J.$$

THEOREM 1. *If the failure rate function of an element is a semi-Markov process $\Lambda(t)$, $t \geqslant 0$, then the reliability functions P_i, $i \in J$, satisfy the system of equations*

$$(2) \quad P_i(x) = \exp[-\lambda_i x]\big(1 - F_i(x)\big) + \int_0^x \sum \exp[-\lambda_i u] p_{ij} P_j(x-u)\, dF_i(u), \qquad i \in J.$$

Passing in (2) to the Laplace transform

$$P_i^*(s) = \int_0^\infty \exp[-sx] P_i(x)\,dx, \qquad f_i^*(s) = \int_0^\infty \exp[-sx]\,dF_i(x), \qquad i \in J,$$

we obtain

COROLLARY 1. *The functions $P_i^*(s)$, $i \in J$, satisfy the system of linear equations*

$$\sum \big(\delta_{ij} - p_{ij} f_i^*(s+\lambda_i)\big) P_j^*(s) = \frac{1 - f_i^*(s+\lambda_i)}{s+\lambda_i}, \qquad i \in J,$$

where δ_{ij} is Kronecker's delta.

Example 1. Consider an m-element system with independent working times of the elements and common exponential probability distribution function with parameter λ. Let $m(t)$, $t \geqslant 0$, denote the number of working elements at the moment t and let $\Lambda(t) = \lambda_{m(t)}$, given the sequence $\lambda_0, \lambda_1, \ldots, \lambda_m$. Considering $\Lambda(t)$, $t \geqslant 0$, as the failure rate process of an element we have

$$F_i(x) = 1-\exp[-i\lambda x], \quad p_{ij} = \delta_{ij+1}, \quad i, j \in J.$$

Hence, using Theorem 1 for the reliability functions $P_i(x)$, we get the following system of equations:

$$P_i(x) = \exp[-(\lambda_i+i\lambda)x]+\int_0^x \exp[-(\lambda_i+i\lambda)u]i\lambda P_{i-1}(x-u)\,du, \quad i \in J,$$
$$P_0(x) = \exp[-\lambda_0 x].$$

The function $P_m(x)$ can be found by the m-fold integration of exponential functions. It is easy to see that if $\lambda_i = \lambda$, $i = 0, 1, \ldots, m$, then $P_m(x) = \exp[-\lambda x]$.

COROLLARY 2. *The expected values* $v_i = \mathrm{E}(Z \mid \Lambda_0 = \lambda_i)$, $i \in J$, *satisfy the system of equations*

$$\sum \left(\delta_{ij}-p_{ij}f_i^*(\lambda_i)\right)v_j = \frac{1-f_i^*(\lambda_i)}{\lambda_i}, \quad i \in J.$$

THEOREM 2. *If the failure rate function of an element is the semi-Markov process* $\Lambda(t)$, $t \geqslant 0$, *then the limiting distribution* $\{\sigma_j\}$ *of the chain* $\{\Lambda_n\}$ *with* $n \to \infty$ *exists and does not depend on the initial state* $\Lambda_0 = \lambda_i$, *and if* $\lambda_j = a_j\lambda$, $j \in J$, $\lambda \to 0$, *then*

$$\lim_{\lambda\to 0} P_i(x/\lambda) = \exp[-ax], \quad i \in J,$$

where

$$a = \sum a_j\mu_j\sigma_j \Big/ \sum \mu_j\sigma_j.$$

Proof of Theorem 1. Let Y denote the random variable exponentially distributed with parameter λ_i which does not depend upon t_1. To find the reliability function $\Pr(Z > x \mid \Lambda_0 = \lambda_i)$ we consider two events: the first of them is $t_1 > x$, $Y > x$, and the second one is $0 < t_1 \leqslant x$, $Y > t_1$; we assume also that the element does not fail in the interval $(t_1, x]$. Hence

$$\Pr(Z > x \mid \Lambda_0 = \lambda_i) = \Pr(t_1 > x,\ Y > x \mid \Lambda_0 = \lambda_i)+$$
$$+\int_0^x \sum \Pr(Y > u)\Pr(\Lambda_1 = \lambda_j \mid \Lambda_0 = \lambda_i)\Pr(Z > x-u \mid \Lambda_0 = \lambda_j)\,dF_i(u).$$

Substituting the notation we have Theorem 1.

Proof of Theorem 2. Let $A(t) = \Lambda(t)/\lambda$. Independently of the initial condition $A(0)$ we have (see [1])

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T A(u)\,du = \sum a_j q_j = a$$

with probability 1. Hence, using the continuity of the exponential function, we get

$$\lim_{\lambda\to 0} P_i\left(\frac{t}{\lambda}\right) = \lim_{\lambda\to 0} \mathrm{E}\left(\exp\left[-\int_0^{t/\lambda} \Lambda(u)\,du\right] \mid \Lambda_0 = \lambda_i\right)$$

$$= \lim_{\lambda\to 0} \mathrm{E}\left(\exp\left[-t\frac{\lambda}{t}\int_0^{t/\lambda} A(u)\,du\right] \mid A(0) = a_i\right)$$

$$= \exp\left[-t\lim_{T\to\infty}\frac{1}{T}\int_0^T A(u)\,du \mid \Lambda(0) = \lambda_i\right] = \exp[-at].$$

2. A particular case. Assume that $\Lambda(t)$, $t \geqslant 0$, is a Markov process with failure rate matrix (Θ_{ij}), where

$$\Theta_{ii} = \Theta_i = \sum_{i\neq j}\Theta_{ij}, \quad i \in J.$$

Considering the Markov process as a semi-Markov process, we get

$$F_i(x) = 1-\exp[-\Theta_i x], \quad p_{ij} = \Theta_{ij}/\Theta_i \text{ for } i\neq j, \quad p_{ii} = 0, \quad i,j \in J.$$

It follows from Theorem 1 that

$$P_i(x) = \exp[-(\lambda_i+\Theta_i)x] + \int_0^x \sum_{j\neq i}\Theta_i \exp[-(\lambda_i+\Theta_i)u]p_{ij}P_j(x-u)\,du, \quad i\in J. \tag{3}$$

Hence

$$P_i(x) = \exp[-\Theta_i^* x] + \int_0^x \sum_{j\neq i}\exp[-\Theta_i^* u]\Theta_{ij}^* P_j(x-u)\,du,$$

where

$$\Theta_{ij}^* = \Theta_{ij} \text{ for } i\neq j, \quad \Theta_i^* = \lambda_i+\Theta_i, \quad i,j\in J. \tag{4}$$

Consider the Markov process $\Lambda^*(t)$, $t \geqslant 0$, described on $m+1$ states $0,1,\ldots,m$ with the absorbing state 0. For the transition rate matrix (4), where in addition

$$\Theta_{i0}^* = \lambda_i, \quad \Theta_0^* = 0, \quad \Theta_{0i}^* = 0, \quad i\in J,$$

and for the transition probabilities $P_{ij}^*(t)$ for the process $\Lambda^*(t)$, $t \geqslant 0$ we get from (3) (see [2]) the following

COROLLARY 3. *If T is the absorbing time of the process $\Lambda(t)$, $t \geqslant 0$, then*

$$P_i(x) = \Pr(Z > x \mid \Lambda_0 = \lambda_i) = \Pr\left(T > x \mid \Lambda^*(0) = i\right)$$

$$= 1-P_{i0}^*(x) = \sum_{j=1}^m P_{ij}^*(x).$$

3. Generalization. Assume that $\Lambda(t)$, $t \geqslant 0$, is a piecewise Markov process defined on the set Λ, characterized by the triple $\big(F_k, (p_{ij}), \Theta_{ij}^{(k)}\big)$, where (see [5] and [6]) by F_k we denote the probability distribution function of length of the Markov segment under the initial condition $\Lambda(0) = \lambda_k$, by (p_{ij}) — the transition probability matrix at the moment of regeneration of the process, and by $\Theta_{ij}^{(k)}$ — the transition rate matrix of the process in the Markov segment under the initial condition of this segment. Let μ_k denote the expected value of the probability distribution F_k, and $\big(P_{ij}^{(k)}(t)\big)$ — the transition probability matrix of the Markov process with the transition rate matrix $(\Theta_{ij}^{(k)})$. Then $\Lambda_n = \Lambda(t_n)$, $n \in N$, is an embedded Markov chain with the transition probability matrix (P_{ij}), where

$$P_{ij} = \int_0^\infty \sum_k P_{ik}^{(i)}(u) p_{kj} dF_i(u), \qquad i, j \in J.$$

Denote by $\{\sigma_j\}$ the limiting probabilities of the embedded Markov chain Λ_n, $n \in N$, if they exist and do not depend upon the initial state of the chain. Then the limiting probability distribution of the process $\Lambda(t)$, $t \geqslant 0$, is of the form

$$q_j = \sum \sigma_k \int_0^\infty P_{kj}^{(k)}(u) \big(1 - F_k(u)\big) du \Big/ \sum \mu_k \sigma_k, \qquad j \in J.$$

Suppose that $\Lambda(t)$, $t \geqslant 0$, is the failure rate function of the element and that Z is its working time. Taking into account (1), analogously to Theorems 1 and 2, we have

THEOREM 3. *If the failure rate function of an element is a piecewise Markov process $\Lambda(t)$, $t \geqslant 0$, then the functions P_i, $i \in J$, satisfy the system of equalities*

$$P_i(x) = G_i(x)\big(1 - F_i(x)\big) + \int_0^x \sum_k \sum_j P_{ij}^{(i)}(u) p_{jk} P_k(x-u) dF_i(u), \qquad i \in J,$$

where $G_i(x) = \sum P_{ij}^{(k)}(x)$ is defined in Corollary 3 under the assumption of the transition rate matrix $(\Theta_{ij}^{(k)})$.

THEOREM 4. *If the limiting probability distribution $\{q_j\}$ of the process $\Lambda(t)$, $t \geqslant 0$, exists and does not depend upon the initial state $\Lambda(0) = \lambda_i$, $\lambda_j = a_j \lambda$, $\lambda \to 0$, then*

$$\lim_{\lambda \to 0} P_i(x/\lambda) = \exp\Big[-\sum a_j q_j x\Big].$$

4. Systems with random load of elements. Assume that the working time of an element depends upon its load. Experimentally, one can find the distribution functions of the virtual working time of an element under constant load or, by a suitable extension of the experiment, one can find the family of distributions of the working times of the elements under load varying in some way. Consider a system of elements being under random load. Now we analyze the distribution function of the working time of elements of a system working in a random environment and of a system in which the element loads depend upon the number of working elements in the system.

4.1. Let $\Lambda(t)$, $t \geqslant 0$, be the semi-Markov process which characterizes the random environment defined in Section 1. Denote by $\boldsymbol{\Lambda}(t) = \{\Lambda_j(\Lambda(t)): j \in J\}$ the vector of failure rates, and by $\boldsymbol{Z} = \{Z_j: j \in J\}$ the vector of working times of the elements of the system. The distribution function of the vector $\boldsymbol{Z}$ can be expressed explicitly but at the same time it is rather difficult to find a practically useful form of this formula. Note, however, that it is a distribution function of by mixture positively dependent random variables (see [10]).

Indeed, if $\Lambda(t, \omega)$, $t \geqslant 0$, is the realization of the process $\Lambda(t)$, $t \geqslant 0$, for fixed $\omega \in \Omega$, then the components of the vector $\boldsymbol{Z} = \boldsymbol{Z}(\omega) = \{Z_j(\omega): j \in J\}$ are independent random variables with marginal distribution functions

$$P_i(Z_i, \omega) = \Pr\big(Z_i(\omega) > z_i\big) = \exp\Big[-\int_0^{z_i} \Lambda_i\big(\Lambda(t, \omega)\big)\,dt\Big], \qquad i \in J, \ \omega \in \Omega,$$

and

$$\Pr(\boldsymbol{Z} > \boldsymbol{z}) = \int_\Omega \prod_{i=1}^m P_i(z_i, \omega)\,d\mu(\omega),$$

where μ is a probabilistic measure on Ω.

4.2. Assume that the element loads of the system depend upon the number of working elements. The working times of the elements of the system can be described in the following way: At the initial moment the virtual working times of the elements X_j, $j = 1, 2, \ldots, m$, where m is the number of elements in the system, are independent random variables with a common distribution function. At the moment $X_{1,m} = \min(X_1, \ldots, X_m)$ one element is failed and the remaining ones are working under different loads. The renewal of some elements (but not all of them) can be assumed. Hence the residual virtual working times of the elements after the moment $X_{1,m}$ are transformed by $T^{(1)}$. In symbols,

$$T^{(1)}: X_j - X_{1,m} \to X_j^{(1)}, \qquad j \in J,$$

where $X_j^{(1)} \geqslant 0$, and there exists a j such that $X_j^{(1)} = 0$ if $X_j - X_{1,m} = 0$. At the moment of the failure of the second element $T^{(2)}$ the residual virtual working times are transformated by $T^{(2)}$, and so on until the failure of the last element in the system.

Knowing the distributions of virtual working times of the elements at the initial moment of work of the system and the transformations $T^{(j)}$, $j = 1, 2, \ldots$, we can find the joint distribution function of working times of the elements in the system, but this problem leads to rather complicated calculations. The estimation of the system reliability in the case of incomplete information of the mentioned probability distribution may be of interest.

5. Systems with associated working times of elements. We say that the working times X of the system elements are *associated random variables* (see [4]) if for any m-variable functions f and g, being monotone for every variable, the random variables $f(X)$ and $g(X)$ are non-negatively correlated. It is known that associated random variables are positively dependent, i.e. for every division of the set $J = \{1, 2, \ldots, m\}$ on the subsets $J_1, J_2, \ldots$ $\ldots, J_r$ the following inequality is satisfied:

$$\Pr(X_j > x_j, j \in J) \geqslant \prod_{k=1}^{r} \Pr(X_j > x_j, j \in J_k).$$

We prove now that in some class of systems with random loads of elements the working times of the elements are associated random variables.

THEOREM 5. *Let $X = (X_1, X_2, \ldots, X_m)$ denote the vector of associated random variables, let $X_{1,m}, X_{2,m}, \ldots, X_{m,m}$ denote the order statistics in this sequence, and let $Y_{k,n}$, $k \in J$, $n = 1, 2, \ldots$, denote independent random variables, being independent also of X.*

Consider the function $T(x, y)$ which is monotone for $x \geqslant 0$ and $y \geqslant 0$, non-negative and such that the function $x + T(a-x, y)$ is monotone for $x \in [0, a]$, $a > 0$.

Then for every $k \in J$ the random variables $T^{(k)}(X) = \{X_j^{(k)}: j \in J\}$ defined by

$$X_j^{(k)} = X_{k,m} + I_{X_j > X_{k,m}} T(X_j - X_{k,m}, Y_{j,k}), \quad j \in J, \tag{5}$$

where I_A is the indicator of A, are associated.

Proof. Let us consider the vector $x = (x_1, x_2, \ldots, x_m)$ and let $R(x) = (x_{1,m}, x_{2,m}, \ldots, x_{m,m})$ be the order value vector for x. For fixed $i \in J$ let

$$x^{(i)} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

and

$$R(x^{(i)}) = (x_{1,m-1}^{(i)}, x_{2,m-1}^{(i)}, \ldots, x_{m-1,m-1}^{(i)}).$$

Let us fix $i, k \in J$ and consider the order value $x_{k,m}$ as the function of x_i which takes the form (see Fig. 1)

$$x_{k,m} = x_{k,m}(x_i) = \begin{cases} x^{(i)}_{k-1,m-1}, & x_i \leqslant x^{(i)}_{k-1,m-1}, \\ x_i, & x^{(i)}_{k-1,m-1} < x_i \leqslant x^{(i)}_{k,m-1}, \\ x^{(i)}_{k,m-1}, & x_i > x^{(i)}_{k,m-1}. \end{cases}$$

Consider the functions

$$x_j^{(k)} = x_{k,m} + I_{x_j > x_{k,m}} T(x_j - x_{k,m}, y), \qquad j \in J,$$

used in (5) and investigate their monotonicity with respect to every component of x. The inequality $x_j > x_{k,m}(x_i)$ is satisfied iff $x_j > x^{(i)}_{k,m-1}$, whence

$$x_j^{(k)} = x_j^{(k)}(x_i) = \begin{cases} x_{k,m}(x_i), & x_j \leqslant x^{(i)}_{k,m-1}, \\ x_{k,m}(x_i) + T\big(x_j - x_{k,m}(x_i), y\big), & x_j > x^{(i)}_{k,m-1}. \end{cases}$$

It is a monotone function of x_i for every j. Thus, using the characterization of associated random variables (see [4]), we have Theorem 5.

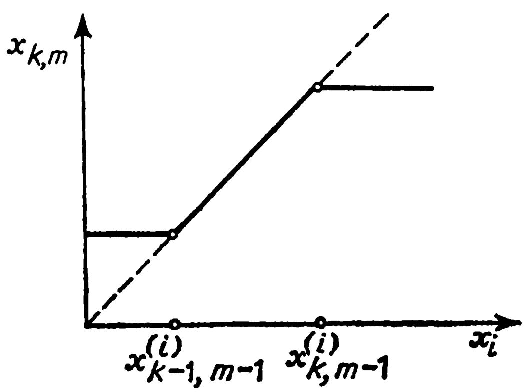

Fig. 1

Now we consider examples of the functions T which may be used in typical situations and failure mechanisms of elements for motivation of their use.

Example 2. (a) The failure of an element causes for the working elements to include in series additional failure mechanisms (see [7]):

$$T(x, y) = \min(x, y).$$

(b) The failure of an element causes for the working elements to include in parallel an additional reserve:

$$T(x, y) = \max(x, y).$$

(c) The moment of the element failure is the regeneration moment of the working elements:

$$T(x, y) = y.$$

Example 3. Let $T(x, y) = ax$, $0 < a < 1$. This function has a simple interpretation if a constant failure rate of elements is assumed. Let the virtual failure rate function of an element be equal to $P(x) = \exp[-\lambda x]$ and let x_0 be the moment of failure of an element in the system. Assuming $X^{(1)} = a(X - x_0)$ as the virtual working time of the element after the moment x_0, we have

$$\Pr\left(a(X-x_0) > x \mid X > x_0\right) = \exp[-x\lambda/a].$$

Hence the failure rate function of an element after the moment x_0 is equal to λ/a, $0 < a < 1$.

COROLLARY 4. *Consider a system with the failure rate function of elements depending upon the number of working elements in the system. Let $m(t)$ denote the number of elements failed and let $\lambda_{m(t)}$ be the failure rate function of an element. Assume that $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_m$. Then the working times of the elements in the system are associated random variables.*

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NIEZAWODNOŚĆ SYSTEMÓW O LOSOWYM OBCIĄŻENIU ELEMENTÓW

STRESZCZENIE

W pracy rozważamy niezawodność elementów o zmieniającej się losowo intensywności awarii oraz systemy złożone z takich elementów. Zakładamy, że intensywność awarii elementu jest procesem półmarkowskim, i przy tym założeniu znajdujemy funkcje niezawodności elementu i jej własności graniczne. Rozważając systemy elementów o zmieniającej się losowo intensywności awarii elementów, dowodzimy, że jeżeli intensywność awarii elementów zależy od losowego otoczenia, to czasy pracy elementów są dodatnio zależne przez mieszanie, natomiast jeżeli intensywność awarii elementów zależy od liczby sprawnych elementów w systemie, to czasy pracy elementów są stowarzyszonymi zmiennymi losowymi.