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APPLICATIONES MATHEMATICAE
XVIII, 1 (1983), p. 73-77

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A MOST BIAS-ROBUST LINEAR ESTIMATE OF THE SCALE PARAMETER OF THE EXPONENTIAL DISTRIBUTION

If the original statistical exponential model is violated in such a way that the random variable under consideration is distributed with pdf $\big(1/\lambda\Gamma(1+1/p)\big)\exp\{-(x/\lambda)^p\}$ rather than $(1/\lambda)\exp\{-x/\lambda\}$, then the sample mean, being MVUE in the original model, is a bias estimate of λ. An estimate which is uniformly most bias-robust in the class of all linear estimates in such an extension of the model is constructed.

Introduction and results. Consider the statistical model

$$M_1 = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,1}, \lambda > 0\}),$$

where R_1^+ is the real half-line, $\mathscr{B}_1^+$ is the family of Borel subsets of R_1^+, and $P_{\lambda,1}$ is the exponential distribution with probability density function (pdf) $f_{\lambda,1}(x) = (1/\lambda)\exp\{-x/\lambda\}$. Consider the extension

$$M_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,p}, \lambda > 0, \ p_1 \leqslant p \leqslant p_2\})$$

of the model M_1, where $0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16$ and $P_{\lambda,p}$ is the exponential power distribution (a special case of the generalized gamma distribution) with pdf

$$f_{\lambda,p}(x) = \big(1/\lambda\Gamma(1+1/p)\big)\exp\{-(x/\lambda)^p\}.$$

The reason for introducing a rather strange-looking number 2.16 will become clear in the sequel.

Some asymptotic problems of the robustness of confidence intervals in the extension $M_{1,\infty}$ of M_1 have been considered by Pollock [1]. We confine ourselves to considerations of the robustness with respect to the bias of estimates of the scale parameter λ.

Let $X_1, X_2, \ldots, X_n$ be a sample from the underlying distribution. It is well known that the sample mean $\bar{X}_n = \sum X_i/n$ is a minimum variance unbiased estimate of λ in the original model M_1. If the "true" distribution is $P_{\lambda,p}$, then the bias of $\bar{X}_n$ is equal to $\mathrm{E}_{\lambda,p}\bar{X}_n - \lambda$, where $\mathrm{E}_{\lambda,p}\bar{X}_n$ is the expected value of $\bar{X}_n$ under the distribution $P_{\lambda,p}$. Following a general concept presented in [2] we define the function

$$b_{\bar{X}_n}(\lambda) = \sup_{p_1 \leqslant p \leqslant p_2} (\mathrm{E}_{\lambda,p}\bar{X}_n - \lambda) - \inf_{p_1 \leqslant p \leqslant p_2} (\mathrm{E}_{\lambda,p}\bar{X}_n - \lambda) \tag{1}$$

which describes how much the bias of $\bar{X}_n$ changes when, given λ, the parameter p runs over the interval $[p_1, p_2]$. Let T be another estimate of λ. The estimate T is more (bias-) robust than $\bar{X}_n$ at $\lambda = \lambda_0$ if $b_T(\lambda_0) < b_{\bar{X}_n}(\lambda_0)$ and is uniformly more robust than $\bar{X}_n$ if $b_T(\lambda) < b_{\bar{X}_n}(\lambda)$ for all $\lambda > 0$.

Given a sample size n and a real-valued vector $a = (a_1, a_2, \ldots, a_n)$, consider the estimate

$$T_n(a) = \sum_{j=1}^{n} a_j X_j^{(n)},$$

where $X_j^{(n)}$ $(j = 1, 2, \ldots, n)$ are order statistics. (Remind that the set of order statistics forms a minimal sufficient statistic in the model M_{p_1,p_2}, $p_1 < p_2$.) Of course, the sample mean $\bar{X}_n$ is a special case of $T_n(a)$. We prove the following

Proposition. *$X_1^{(n)}/\mathrm{E}_{1,1}X_1^{(n)}$ is the uniformly most robust estimate of λ in every extension M_{p_1,p_2} $(0 < p_1 \leqslant 1 \leqslant p_2 < 2.16)$ of the model M_1, in the class of linear estimates $T_n(a)$, $a \geqslant 0$, which are unbiased in the original model M_1.*

Proof. The proof consists in constructing an appropriate $T_n(a)$, $a \geqslant 0$. The bias-robustness of $T_n(a)$ is described by the function

$$b_{T_n(a)}(\lambda) = \lambda \Big(\sup_{p_1 \leqslant p \leqslant p_2} \sum_{j=1}^{n} a_j \mathrm{E}_{1,p} X_j^{(n)} - \inf_{p_1 \leqslant p \leqslant p_2} \sum_{j=1}^{n} a_j \mathrm{E}_{1,p} X_j^{(n)} \Big),$$

where $a = (a_1, a_2, \ldots, a_n)$ is a vector such that $T_n(a)$ is an unbiased estimate of λ in M_1, i.e.,

$$\sum_{j=1}^{n} a_j \mathrm{E}_{1,1} X_j^{(n)} = 1. \tag{2}$$

The problem of constructing the uniformly most bias-robust estimate $T_n(a)$ reduces to finding such an a which minimizes

$$\sup_{p_1 \leqslant p \leqslant p_2} \sum_{j=1}^{n} a_j \mathrm{E}_{1,p} X_j^{(n)} - \inf_{p_1 \leqslant p \leqslant p_2} \sum_{j=1}^{n} a_j \mathrm{E}_{1,p} X_j^{(n)}$$

subject to (2) and to the condition

$$a_j \geqslant 0, \quad j = 1, 2, \ldots, n. \tag{3}$$

Given (j, n), the expectation $\mathrm{E}_{1,p} X_j^{(n)}$ is a decreasing function of $p \in (0, 2.16)$. The upper bound 2.16 is important because $\Gamma(1+1/p)$ is strictly monotone in p in this interval, and so is $F_{1,p}(x)$ for any fixed $x > 0$. A more exact upper bound for the interval of monotonicity is 2.1662276.

Now, we consider the following linear programming problem: minimize

$$\sum_{j=1}^{n} a_j \mathrm{E}_{1,p_1} X_j^{(n)} - \sum_{j=1}^{n} a_j \mathrm{E}_{1,p_2} X_j^{(n)}$$

under conditions (2) and (3).

All vertices of the polyhedron of a's which satisfies (2) and (3) are of the form

$$\left(a_1 = 0, \ldots, a_{j-1} = 0,\ a_j = \frac{1}{\mathrm{E}_{1,1} X_j^{(n)}},\ a_{j+1} = 0, \ldots, a_n = 0\right),$$

$$j = 1, 2, \ldots, n.$$

We conclude that all but one coordinates of the optimal vector a are equal to zero, and the index j_0 of the non-zero coordinate of the optimal vector is that which minimizes

$$\gamma_j^{(n)}(p_1, p_2) = \frac{\mathrm{E}_{1,p_1} X_j^{(n)} - \mathrm{E}_{1,p_2} X_j^{(n)}}{\mathrm{E}_{1,1} X_j^{(n)}}. \tag{4}$$

Using the formula

$$\mathrm{E}_{1,p} X_j^{(n)} = j\binom{n}{j} \int_0^1 F_{1,p}^{-1}(t) t^{j-1} (1-t)^{n-j} dt, \tag{5}$$

where $F_{1,p}(x) = \int_0^x f_{1,p}(u)\, du$, we obtain

$$\gamma_j^{(n)}(p_1, p_2) = \int_0^1 \frac{F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t)}{F_{1,1}^{-1}(t)} g_j^{(n)}(t)\, dt,$$

where

$$g_j^{(n)}(t) = \frac{F_{1,1}^{-1}(t) t^{j-1} (1-t)^{n-j}}{\int_0^1 F_{1,1}^{-1}(t) t^{j-1} (1-t)^{n-j} dt}.$$

It is easy to see that for an appropriate number $t_{n,j} \in (0,1)$ we have $g_j^{(n)}(t) > g_{j+1}^{(n)}(t)$ for $t < t_{n,j}$ and $g_j^{(n)}(t) < g_{j+1}^{(n)}(t)$ for $t > t_{n,j}$. It follows that if $\left(F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t)\right)/F_{1,1}^{-1}(t)$ were an increasing function, we would have $\gamma_j^{(n)} < \gamma_{j+1}^{(n)}$. We show that this is the case. To this end it suffices to prove that $F_{1,p}^{-1}(t)/F_{1,1}^{-1}(t)$ is an increasing function in t for $p < 1$ and a decreasing one for $p > 1$ or, letting $t = F_{1,p}(x)$ and $s_p(x) = F_{1,1}^{-1}\left(F_{1,p}(x)\right)$, to prove that $x/s_p(x)$ increases with x for $p < 1$ and decreases for $p > 1$.

Consider the derivative

$$\frac{d}{dx}\left(\frac{x}{s_p(x)}\right) = s_p^{-2}(x)[s_p(x) - xs_p'(x)].$$

Differentiating the identity

$$\frac{1}{\Gamma(1+1/p)}\int_0^x e^{-u^p}du = \int_0^{s_p(x)} e^{-u}du, \quad x \geqslant 0,$$

we obtain

$$s_p'(x) = \frac{1}{\Gamma(1+1/p)}\exp\{s_p(x) - x^p\}, \quad s_p(0) = 0,$$

and

$$s_p(x) = \frac{1}{\Gamma(1+1/p)}\int_0^x \exp\{s_p(u) - u^p\}du.$$

Hence

$$\frac{d}{dx}\left(\frac{x}{s_p(x)}\right) = \frac{\int_0^x \exp\{s_p(u) - u^p\}du - x\exp\{s_p(x) - x^p\}}{\Gamma(1+1/p)s_p^2(x)}.$$

The integrand in the last formula equals 1 for $u = 0$ and is a decreasing (increasing) function for $p < 1$ ($p > 1$); this follows from the inequalities

$$px^{p-1}\int_x^\infty e^{-u^p}du \gtrless \int_x^\infty d(-e^{-u^p}) = e^{-x^p} \quad \text{for } p \lesseqgtr 1$$

when applied to

$$\frac{d}{dx}\exp\{s_p(x) - x^p\} = \frac{f_{1,p}(x)}{[1 - F_{1,p}(x)]^2}\left(e^{-x^p} - px^{p-1}\int_x^\infty e^{-u^p}du\right).$$

As a consequence we obtain

$$\frac{d}{dx}\left(\frac{x}{s_p(x)}\right) \gtrless 0 \quad \text{for } p \lesseqgtr 1,$$

which proves the monotonicity of $x/s_p(x)$, and hence the monotonicity of $F_{1,p}^{-1}(t)/F_{1,1}^{-1}(t)$. It follows that $T_n = X_1^{(n)}/\mathrm{E}_{1,1}X_1^{(n)}$ is the most bias-robust statistic in the class of all linear estimates $T_n(a)$, $a \geqslant 0$, which are unbiased in the original model M_1.

Some numerical results. The bias-robustness of the estimate T_n in the model M_{p_1,p_2} is described by the function

$$b_{T_n}(\lambda) = \lambda\gamma_1^{(n)}(p_1, p_2).$$

For the exponential distribution with pdf $f_{1,1}(x)$ we have $\mathrm{E}_{1,1}X_1^{(n)} = 1/n$ and by (4) and (5) we obtain

$$\gamma_1^{(n)}(p_1, p_2) = n^2 \int_0^1 \big(F^{-1}_{1,p_1}(t) - F^{-1}_{1,p_2}(t)\big)(1-t)^{n-1}\,dt.$$

Given (p_1, p_2), $0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16$, $F^{-1}_{1,p_1}(t) - F^{-1}_{1,p_2}(t)$ is an increasing function so that $\gamma_1^{(n)}(p_1, p_2)$ is a decreasing function in n. The values of $\gamma_1^{(n)}(p_1, p_2)$ for some small n and some supermodels M_{p_1,p_2} are given in Table 1.

TABLE 1

Values of p_1 and p_2		$p_1 = 0.9$, $p_2 = 1$	$p_1 = 1$, $p_2 = 1.1$	$p_1 = 0.9$, $p_2 = 1.1$	$p_1 = 1$, $p_2 = 2$
$b(p_1, p_2)$		0.178	0.118	0.296	0.436
$\gamma_1^{(n)}(p_1, p_2)$	$n = 2$	0.139	0.093	0.231	0.339
	$n = 3$	0.121	0.081	0.201	0.290
	$n = 4$	0.109	0.074	0.183	0.259
	$n = 5$	0.101	0.070	0.171	0.238
	$n = \infty$	0.052	0.035	0.087	0.114

The bias-robustness function (1) of the sample mean takes the form

$$b_{\bar{X}_n}(\lambda) = \lambda b(p_1, p_2), \qquad \text{where } b(p_1, p_2) = \frac{\Gamma(2/p_1)}{\Gamma(1/p_1)} - \frac{\Gamma(2/p_2)}{\Gamma(1/p_2)}.$$

The values of $b(p_1, p_2)$ are given in Table 1. Note that the robustness of the sample mean does not depend on n.

For large n the random variable $2nF_{1,p}(X_1^{(n)})$ is distributed as χ^2 with 2 degrees of freedom so that $\mathrm{E}[F_{1,p}(X_1^{(n)})] \approx 1/n$. Moreover, $X_1^{(n)}$ is small for n large enough so that

$$F_{1,p}(X_1^{(n)}) \approx X_1^{(n)}/\Gamma(1+1/p).$$

It follows that

$$\gamma_1^{(n)}(p_1, p_2) \approx \Gamma(1+1/p_1) - \Gamma(1+1/p_2)$$

asymptotically. The asymptotic values of this coefficient are given in the last row of Table 1.

References

[1] K. H. Pollock, *Inference robustness vs. criterion robustness: an example,* Amer. Statist. 32 (1978), p. 133-136.

[2] R. Zieliński, *Robustness: a quantitative approach,* Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 25 (1977), p. 1281-1286.

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Received on 13. 3. 1980;
revised version on 24. 3. 1981