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ALGORITHMS 52-53

DETERMINATION OF AN INTERPOLATING QUINTIC SPLINE FUNCTION WITH EQUALLY SPACED AND DOUBLE KNOTS

0. Introduction. The descriptions of two procedures *interp5l02cf* and *interp5l02d13cf*, which determine the interpolating spline functions with equally spaced and double knots, are given. Those functions satisfy conditions (i)-(iv) and (i)-(iii), (iv′), respectively, which are given below.

1. Procedure declaration. Let n be a natural number ($n \geqslant 2$). For given real numbers y_i and y_i'' ($i = 1, 2, \ldots, n$) we determine an interpolating quintic spline function s such that

(i) $s \in C^3[1, n]$;

(ii) in each subinterval $[i, i+1]$ ($i = 1, 2, \ldots, n-1$), s is a quintic polynomial;

(iii) $s(i) = y_i$ and $s''(i) = y_i''$ ($i = 1, 2, \ldots, n$).

Additionally, the functions, determined by conditions (i)-(iii), in the case of procedure *interp5l02cf* satisfy the boundary conditions

(iv) $s'''(1) = y_1'''$ and $s'''(n) = y_n'''$;

and in the case of procedure *interp5l02d13cf* the conditions

(iv′) $s'(1) = y_1'$ and $s'''(1) = y_1'''$,

where y_1''', y_n''', and y_1' are given real numbers.

Remark. In the procedure *interp5l02cf*, n is required to be even.

Data:

- n — number of knots of the function s;
- *y*[*1* : *n*] — array of values of the function s at knots $x_i = i$ ($i = 1, 2, \ldots, n$);
- *y2*[*1* : *n*] — array of the second derivatives of the function s at knots $x_i = i$ ($i = 1, 2, \ldots, n$);
- *y13*, *yn3* — values of the third derivative of the function s at knots number 1 and number n, respectively (in the case of procedure *interp5l02cf*);
- *y11*, *y13* — values of the first and third derivatives of the function s at knot number 1 (in the case of procedure *interp5l02d13cf*).

```
procedure interp5102cf(n,y,y2,y13,yn3,a,b,c,d,e,f);
 value n,y13,yn3;
 integer n;
 real y13,yn3;
 array y,y2,a,b,c,d,e,f;
 begin
  integer i,i1,n1;
  real k1,k2,k3,l1,l2,l3,s,s1,s2,v,v1;
  array al,r[2:n],u[1:n-1];
  if (n÷2)×2≠n
   then go to exit;
  n1:=n-1;
  k1:=y[1];
  k2:=y[2];
  k3:=y[3];
  l1:=y2[1];
  l2:=y2[2];
  l3:=y2[3];
  v:=l1+l3-l2-l2;
  s:=360.0×(k2+k2-k1-k3+l2)+60.0×v;
  s1:=6.0×v;
  v1:=6.0×(l2-l1-y13);
  al[2]:=v1-.66666666667×s+2.6666666667×s1;
  u[2]:=s2:=1.1666666667×s1-.16666666667×s;
  k1:=k2;
  k2:=k3;
  l1:=l2;
  l2:=l3;
  for i:=3 step 1 until n1 do
   begin
```

```
 i1:=i+1;
 k3:=y[i1];
 l3:=y2[i1];
 v:=l1+l3-l2-l2;
 s=360.0×(k2+k2-k1-k3+l2)+60.0×v;
 s1:=6.0×v;
 al[i]:=s2+1.3333333333×s1-.33333333333×s;
 u[i]:=s2:=1.1666666667×s1-.16666666667×s;
 k1:=k2;
 k2:=k3;
 l1:=l2;
 l2:=l3
 end i;
al[n]:=s2-6.0×(yn3+l1-l2);
s:=s1:=.5;
r[2]:=k1:=al[2];
for i:=3 step 1 until n1 do
 begin
  r[i]:=k1:=s×(al[i]-k1);
  s:=1.0/s
 end i;
r[n]:=k1:=-.66666666667×(al[n]-k1);
for i:=n1 step -1 until 2 do
 begin
  r[i]:=k1:=s1×k1+r[i];
  u[i]:=u[i]-k1;
  s1:=1.0/s1
 end i;
u[1]:=.5×(v1-k1);
k1:=y[1];
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```
l1:=y2[1];
s1:=y13;
for i:=1 step 1 until n1 do
 begin
  i1:=i+1;
  k2:=y[i1];
  l2:=y2[i1];
  s2:=y3[i1];
  s:=l1-l2;
  a[i]:=k1;
  b[i]:=k2-k1-.35×l1-.15×l2-.05×s1+.03333333333×s2;
  c[i]:=.5×l1;
  d[i]:=k3:=.16666666667×s1;
  e[i]:=-.25×s-k3-.08333333333×s2;
  f[i]:=.1×s+.05×(s1+s2);
  k1:=k2;
  l1:=l2;
  s1:=s2
 end i
end interp5102d13cf
```

Results:

$a, b, c, d, e, f[1:n-1]$ — arrays of coefficients of the function s.

In each subinterval $[i, i+1]$ $(i = 1, 2, \ldots, n-1)$ the function s is of the form

$$(1) \qquad s(x) = a_i + b_i t + c_i t^2 + d_i t^3 + e_i t^4 + f_i t^5, \quad \text{where } t = x-i.$$

2. Methods used. We begin with the description of the method used for determining the function s described by conditions (i)-(iv).

Let $M_i = s^{(4)}(i+)$ $(i = 1, 2, \ldots, n-1)$, and $N_i = s^{(4)}(i-)$ $(i = 2, 3, \ldots, n)$. In [2], the function s is described by the expression

$$(2) \qquad s(x) = y_i A_0(1-t) + y_{i+1} A_0(t) + y_i'' A_1(1-t) + y_{i+1}'' A_1(t) + \\ + M_i A_2(1-t) + N_{i+1} A_2(t),$$

where $x \in [i, i+1]$ $(i = 1, 2, \ldots, n-1)$, $t = x-i$, and

$$(3) \qquad A_0(t) = t, \quad A_1(t) = (t^3-t)/6, \quad A_2(t) = (t^5-t)/120 - A_1(t)/6.$$

```
procedure interp5102d13cf(n,y,y2,y11,y13,a,b,c,d,e,f);
 value n,y11,y13;
 integer n;
 real y11,y13;
 array y,y2,a,b,c,d,e,f;
 begin
  integer i,i1,n1;
  real k1,k2,k3,l1,l2,l3,s,s1,s2;
  array y3[1:n];
  n1:=n-1;
  y3[1]:=s1:=y13;
  k1:=y[1];
  k2:=y[2];
  l1:=y2[1];
  l2:=y2[2];
  y3[2]:=s2:=30.0×(k1-k2+y11)+10.5×l1+4.5×l2+1.5×y13;
  for i:=2 step 1 until n1 do
   begin
    i1:=i+1;
    k3:=y[i1];
    l3:=y2[i1];
    y3[i1]:=s:=s1+30.0×(k2+k2-k1-k3+l2)+4.5×(l3+l1-l2-l2);
    k1:=k2;
    k2:=k3;
    l1:=l2;
    l2:=l3;
    s1:=s2;
    s2:=s
   end i;
  k1:=y[1];
```

```
l1:=y2[1];
for i:=1 step 1 until n1 do
 begin
  a[i]:=k1;
  i1:=i+1;
  k2:=y[i1];
  l2:=y2[i1];
  s:=r[i1];
  s1:=u[i];
  b[i]:=k2-k1-.16666666667×(l1+l1+l2)+.01944444444×s+
        .02222222222×s1;
  c[i]:=.5×l1;
  d[i]:=.16666666667×(l2-l1)-.02777777778×s-.05555555556×
        s1;
  e[i]:=.04166666667×s1;
  f[i]:=.00833333333×(s-s1);
  k1:=k2;
  l1:=l2
 end i;
exit:
 end interp5102cf
```

The numbers M_i $(i = 1, 2, \ldots, n-1)$ and N_i $(i = 2, 3, \ldots, n)$ satisfy the following system of equations (see [2], equations (4.2)-(4.6)):

$$2M_1 + N_2 = 6(y_2'' - y_1'' - y_1''') \equiv c_1, \tag{4}$$

$$8(M_i + N_i) + 7(M_{i-1} + N_{i+1}) = 360(y_i'' - \Delta^2 y_{i-1}) + 60\Delta^2 y_{i-1} \equiv a_i \quad (i = 2, 3, \ldots, n-1), \tag{5}$$

$$2(M_i + N_i) + (M_{i-1} + N_{i+1}) = 6\Delta^2 y_{i-1} \equiv b_i \quad (i = 2, 3, \ldots, n-1), \tag{6}$$

$$M_{n-1} + 2N_n = 6(y_{n-1}'' - y_n'' + y_n''') \equiv c_n. \tag{7}$$

In [2] it was proved that the system of equations (4)-(7) has exactly one solution (for n even). In [3] a description of the method for solving the system (4)-(7) is given.

Let $u_i = M_i + N_i$, and $v_i = M_{i-1} + N_{i+1}$ $(i = 2, 3, \ldots, n-1)$. Equations (5) and (6) are equivalent to the following ones:

$$8u_i + 7v_i = a_i \quad \text{and} \quad 2u_i + v_i = b_i \quad (i = 2, 3, \ldots, n-1).$$

Hence we obtain $u_i = (-a_i + 7b_i)/6$ and $v_i = (a_i - 4b_i)/3$. The system of equations (4)-(7) can be written in the following way:

$$\begin{aligned} 2M_1 + N_2 &= c_1, \\ M_i + N_i &= u_i \quad (i = 2, 3, \ldots, n-1), \\ M_{i-1} + N_{i+1} &= v_i \quad (i = 2, 3, \ldots, n-1), \\ M_{n-1} + 2N_n &= c_n. \end{aligned}$$

Eliminating the unknowns $M_1, M_2, \ldots, M_{n-1}$ we obtain

$$\begin{aligned} N_2 - 2N_3 &= c_1 - 2v_2, \\ N_{2i-1} - N_{2i+1} &= u_{2i-1} - v_{2i} \quad (i = 2, 3, \ldots, (n-2)/2), \\ N_{2i} - N_{2i+2} &= u_{2i} - v_{2i+1} \quad (i = 1, 2, \ldots, (n-2)/2), \\ N_{n-1} - 2N_n &= u_{n-1} - c_n. \end{aligned}$$

The matrix of this system is tridiagonal. For determining the unknowns $N_2, N_3, \ldots, N_n$, the decomposition LU can be used (see, e.g., [1], equations (2.1.20)). In virtue of (4) and the above notation, the unknowns M_i $(i = 1, 2, \ldots, n-1)$ are determined by

$$M_1 = (c_1 - N_2)/2, \quad M_i = u_i - N_i \quad (i = 2, 3, \ldots, n-1).$$

The coefficients $a_i, b_i, c_i, d_i, e_i, f_i$ $(i = 1, 2, \ldots, n-1)$ of the function (1) are derived from (2) and (3). Hence we obtain

$$\begin{gathered} a_i = y_i, \quad b_i = y_{i+1} - y_i - (2y_i'' + y_{i+1}'')/6 + (8M_i + 7N_{i+1})/360, \\ c_i = y_i''/2, \quad d_i = [y_{i+1}'' - y_i'' - (2M_i + N_{i+1})/6]/6, \\ e_i = M_i/24, \quad f_i = (N_{i+1} - M_i)/120. \end{gathered}$$

Now we present the method used for determining the function s for which conditions (i)-(iii) and (iv′) are satisfied. It is known (see [2]) that for $x \in [i, i+1]$ $(i = 1, 2, \ldots, n-1)$ the function s takes the form

$$s(x) = y_i B_0(1-t) + y_{i+1} B_0(t) + y_i'' B_1(1-t) + y_{i+1}'' B_1(t) - \\ - y_i''' B_2(1-t) + y_{i+1}''' B_2(t), \tag{8}$$

where $t = x - i$ $(i = 1, 2, \ldots, n-1)$, $y_i''' = s'''(i)$ $(i = 1, 2, \ldots, n$; a number y_1''' is given), and

$$\begin{gathered} B_0(t) = t, \quad B_1(t) = (t^4 - t^5)/10 + 3(t^4 - t)/20, \\ B_2(t) = (t^5 - t^4)/20 + (t - t^4)/30. \end{gathered} \tag{9}$$

The numbers y_i''' $(i = 2, 3, \ldots, n)$ are determined as follows (see [2], equations (2.5) and (7.1)):

$$y_2''' = 30(y_1 - y_2 + y_1') + 10.5y_1'' + 4.5y_2'' + 1.5y_1''',$$
$$y_{i+1}''' = y_{i-1}''' + 30(y_i'' - \Delta^2 y_{i-1}) + 4.5\Delta^2 y_i'' \quad (i = 2, 3, \ldots, n-1).$$

Hence and from (8) and (9) we obtain

$$a_i = y_i, \quad b_i = y_{i+1} - y_i - (7y_i'' + 3y_{i+1}'')/20 + (2y_{i+1}''' - 3y_i''')/60,$$
$$c_i = y_i''/2, \quad d_i = y_i'''/6, \quad e_i = (y_{i+1}'' - y_i'')/4 - (2y_i''' + y_{i+1}''')/12,$$
$$f_i = (y_i'' - y_{i+1}'')/10 + (y_i''' + y_{i+1}''')/20.$$

3. Certification. Let the quantities $E1$ and $E2$ be defined by

$$E1 = \max_{1 \leqslant i \leqslant n} \max\left(|y_i - s(i)|,\ |y_i'' - s''(i)|\right),$$
$$E2 = \left\{\sum_{i=1}^{n} \left[(y_i - s(i))^2 + (y_i'' - s''(i))^2\right]\right\}^{1/2}.$$

In the following examples the values of $E1$ and $E2$ are given for various values of n.

Example 3.1. Let

$$y_i = \sin(i), \quad y_i'' = -\sin(i) \quad (i = 1, 2, \ldots, n),$$
$$y_1''' = -\cos(1), \quad y_n''' = -\cos(n), \quad y_1' = \cos(1).$$

The results of using the procedures *interp5l02cf* and *interp5l02d13cf* are given in Tables 3.1 and 3.1′, respectively.

TABLE 3.1

n	$E1$	$E2$
8	.706(−9)	.707(−9)
16	.542(−9)	.543(−9)
32	.728(−10)	.728(−10)
64	.306(−9)	.306(−9)

TABLE 3.1′

n	$E1$	$E2$
8	.209(−10)	.228(−10)
16	.364(−11)	.375(−11)
32	.400(−10)	.401(−10)
64	.500(−11)	.532(−11)

Example 3.2. Let

$$y_i = \exp(10^{-2}i), \quad y_i'' = 10^{-4}\exp(10^{-2}i) \quad (i = 1, 2, \ldots, n),$$
$$y_1''' = 10^{-6}\exp(10^{-2}), \quad y_n''' = 10^{-6}\exp(10^{-2}n), \quad y_1' = 10^{-2}\exp(10^{-2}).$$

The results of using the procedures *interp5l02cf* and *interp5l02d13cf* are given in Tables 3.2 and 3.2′, respectively.

TABLE 3.2

n	$E1$	$E2$
8	.0	.0
16	.178(−14)	.178(−14)
32	.355(−14)	.355(−14)
64	.355(−14)	.355(−14)

TABLE 3.2′

n	$E1$	$E2$
8	.300(−12)	.300(−12)
16	.703(−13)	.703(−13)
32	.125(−12)	.125(−12)
64	.188(−13)	.188(−13)

The execution time on the Odra 1204 computer of the procedure *interp5l02cf* is approximately equal to $32n+984$ msec. and that of *interp5l02d13cf* to $21n+1001$ msec.

References

[1] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The theory of splines and their applications*, New York 1967.
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[3] E. Neuman, *Convex interpolating splines of odd degree*, Utilitas Math. (submitted).

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ALGORYTMY 52-53

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WYZNACZANIE INTERPOLUJĄCEJ FUNKCJI SKLEJANEJ STOPNIA PIĄTEGO Z WĘZŁAMI RÓWNOODLEGŁYMI I PODWÓJNYMI

STRESZCZENIE

Niech n będzie liczbą naturalną ($n \geqslant 2$). Dla danych liczb rzeczywistych y_i and y_i'' ($i = 1, 2, \ldots, n$) wyznacza się współczynniki interpolującej funkcji sklejanej s stopnia piątego, takiej że

(i) $s \in C^3[1, n]$;

(ii) w każdym podprzedziale $[i, i+1]$ ($i = 1, 2, \ldots, n-1$), s jest wielomianem stopnia co najwyżej 5;

(iii) $s(i) = y_i$ oraz $s''(i) = y_i''$ ($i = 1, 2, \ldots, n$).

Funkcja s, opisana przez (i)-(iii), w przypadku procedury *interp5l02cf* spełnia dodatkowo warunki brzegowe

(iv) $s'''(1) = y_1'''$ oraz $s'''(n) = y_n'''$,

a przypadku procedury *interp5l02d13cf* warunki

(iv′) $s'(1) = y_1$ oraz $s'''(1) = y_1'''$,

gdzie y_1''', y_n''', y_1' są danymi liczbami rzeczywistymi.

Uwaga. Dla procedury *interp5l02cf*, n musi być liczbą parzystą.

Dane:

n — liczba węzłów funkcji s;

y [1 : n] — tablica wartości funkcji s w węzłach $x_i = i$ $(i = 1, 2, \ldots, n)$;

y2 [1 : n] — tablica wartości drugich pochodnych funkcji s w węzłach $x_i = i$ $(i = 1, 2, \ldots, n)$;

y13, yn3 — wartości trzecich pochodnych funkcji s w węzłach równych odpowiednio 1 i n (w przypadku procedury *interp5l02cf*);

y11, y13 — wartości pierwszej i trzeciej pochodnej funkcji s w węźle równym 1 (w przypadku procedury *interp5l02d13cf*).

Wyniki:

a, b, c, d, e, f[1 : n − 1] — tablice współczynników funkcji s.

W każdym podprzedziale $[i, i+1]$ $(i = 1, 2, \ldots, n-1)$ funkcja s ma postać określoną wzorem (1).