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A SILENT-SILENT DUEL WITH RANDOMLY DETECTED OPPONENTS

1. Introduction. There are many papers on duels (see [4]) where various kinds of uncertainty were introduced. In this paper we consider a modified model of a silent-silent duel in which the opponents are detected at random moments of time determined by the values of two independent Markov chains. We assume that the accuracy functions of both players are equal. Under some assumptions concerning the transition density function of the Markov chains we prove that it is optimal for both players to fire the bullet at the first detection moment after a fixed moment $x^* \in [0,1]$.

First, we describe the game model. Each of two opponents has the chance to shoot his only bullet at one of the random moments when his opponent is detected in the time interval $[0,1]$. The random detection moments are determined by the values of two independent Markov chains $\{\xi_n^i, n \geqslant 1\}$ $(i=1,2)$ with the same density of the transition probability $p(x,y)$ satisfying the condition $p(x,y) \equiv 0$ for $0 \leqslant y < x$. We also assume that the probability $\int_1^\infty p(x,y)\,dy$ that no detection in the time interval $(x,1]$ occurs is positive and that only a finite number of detection moments may occur in the interval $[0,1]$. Next, the probability of hitting the opponent (accuracy function) depends on time and is equal for both players. We may assume the accuracy function to be $P(t) = t$, $t \in [0,1]$. The random pay-off for the first player is $+1$ if only he survives the duel and -1 if only the second player survives. Otherwise, the pay-off is zero. We see that each time the opponent is detected the player has to decide whether he should shoot or wait for a better opportunity. The game is over when at least one of the players is hit or when $t=1$.

2. Normal form of the game. Let us define a player's strategy in the considered game. A pure behaviour strategy is a measurable function $\pi\colon [0,1]\to[0,1]$, where $\pi(x)$ is the probability that the player shoots his bullet provided the opponent is detected at the moment $x\in[0,1]$.

Let L_1 be the space of all summable functions on $[0,1]$ and L_1' its dual. We introduce the space $\big(L_1', \sigma(L_1', L_1)\big)$, where $\sigma(L_1', L_1)$ is the w^*-topology in L_1'. Now, if Π is the set of all measurable functions $\pi\colon [0,1]\to[0,1]$, then from the well-known Alouglu-Bourbaki theorem we have

LEMMA 1. *The strategy set Π is compact in $\big(L_1', \sigma(L_1', L_1)\big)$.*

Now define the pay-off function for the game. First, we introduce the probability space $(\Omega, \mathscr{F}, P_\pi)$ for every $\pi\in\Pi$. Let $X = [0,\infty)$ and $A = \{0,1\}$. We denote by $\mathfrak{X}$ and $\mathfrak{A}$ the σ-algebras of Borel subsets in X and A, respectively. Now, we put

$$\Omega = X\times A\times X\times A\times\ldots \quad\text{and}\quad \mathscr{F} = \mathfrak{X}\otimes\mathfrak{A}\otimes\mathfrak{X}\otimes\mathfrak{A}\otimes\ldots$$

and we denote by P_π the unique probability measure on $(\Omega,\mathscr{F})$ determined by the transition density function p and a strategy $\pi\in\Pi$ (see [2]).

Next, we consider the projections ξ_n and a_n from Ω onto the n-th state space X and the action space A, respectively. We also notice that every strategy $\pi\in\Pi$ determines a randomized Markov stopping time

$$\tau = \inf\{n\geqslant 1 \mid a_n(\omega) = 0\}$$

and the Markov chain $(\xi_n, \mathscr{F}_n^\pi, P_x^\pi)$, where $\mathscr{F}_n^\pi = \sigma\{\xi_k, a_k;\ k\leqslant n\}$ and for $x\in X$, $B\in\mathfrak{X}$ we have

$$P_x^\pi\{\xi_1\in B\} = [1-\pi(x)]\int_B p(x,y)\,dy,\qquad P_x^\pi\{\xi_1\notin X\} = \pi(x).$$

Evidently, we put $\pi(x)\equiv 1$ for $x\geqslant 1$.

Now, let ξ_τ be the random moment of time at which the player shoots his bullet. The event $E = \{\omega\in\Omega \mid \xi_\tau(\omega) > 1\}$ means that the player did not shoot in the time interval $[0,1]$. The above considerations are valid for both players and under the assumption of independence we have the random pay-off $\varphi\colon \Omega\times\Omega\to\{-1,0,1\}$ for the first player defined by

$$\varphi = I_{\bar{E}\times\bar{E}}\,k(\xi^1_{\tau_1}, \xi^2_{\tau_2}) + I_{\bar{E}\times E}\,\xi^1_{\tau_1} - I_{E\times\bar{E}}\,\xi^2_{\tau_2},$$

where

$$k(x,y) = \begin{cases} x-y+xy & \text{if } 0\leqslant x<y\leqslant 1,\\ 0 & \text{if } 0\leqslant x=y\leqslant 1,\\ -y+x-xy & \text{if } 0\leqslant y<x\leqslant 1,\end{cases}$$

and I_B is the indicator function of the event $B \subset \Omega \times \Omega$. Now, the pay-off function of the game is given by

$$K(\pi_1, \pi_2) = \int_{\Omega \times \Omega} \varphi \, dP_{\pi_1} \times P_{\pi_2}, \qquad (\pi_1, \pi_2) \in \Pi \times \Pi.$$

We also observe that if F_{π_i} is the distribution function of $\xi^i_{\tau_i}$ $(i = 1, 2)$, then

$$(1) \quad K(\pi_1, \pi_2) = \int_0^1 x dF_{\pi_1}(x) - \int_0^1 y dF_{\pi_2}(y) + \\ + \int_0^1 x dF_{\pi_1}(x) \int_0^1 y dF_{\pi_2}(y) - 2 \int_0^1 x \int_0^x y dF_{\pi_2}(y) dF_{\pi_1}(x).$$

The distribution function F_π is determined by

$$F_\pi(x) = \int_0^x \pi(t) f(t) dt, \qquad x \in [0, \infty),$$

where $f(t)$ satisfies the following integral equation:

$$(2) \qquad f(x) = p(0, x) + \int_0^x f(t)[1 - \pi(t)] p(t, x) dt$$

if $x \in [0, 1]$, and

$$f(x) = p(0, x) + \int_0^x f(t)[1 - \pi(t)] p(t, x) dt$$

if $x \in [1, \infty)$.

Assuming that p is bounded on $[0, 1] \times [0, 1]$ and that $p(0, \cdot)$ is square integrable on $[0, 1]$, we prove the existence and uniqueness of the solution of equation (2).

Let us introduce the normal form $\Gamma = (\Pi, \Pi, K)$ for the considered game, where K is given in (1). In an analogous way as in [1] we can prove that the pay-off function K is continuous on $\Pi \times \Pi$ if $p(\cdot, x) p(0, \cdot)$ is an element of L_1 continuously depending on $x \in [0, 1]$. Similarly, we can prove that for every probability measure μ on Π there is an equivalent strategy $p_\mu \in \Pi$ resulting in the same distribution function for ξ_τ as in the case of the mixed strategy μ. Now, applying the Glickberg theorem (see [1]), we obtain

LEMMA 2. *If the transition density p satisfies the above-stated assumptions, then the game $\Gamma = (\Pi, \Pi, K)$ has a solution.*

3. Associated optimal stopping problem. First, using equality (1) we find that, for every $(\pi_1, \pi_2) \in \Pi \times \Pi$,

$$K(\pi_1, \pi_2) = \int_0^1 g(x \mid \pi_2) dF_{\pi_1}(x) - \int_0^1 y dF_{\pi_2}(y),$$

where

$$(3)\qquad g(x \mid \pi_2) = \begin{cases} x\Big[1+\int\limits_0^1 y\,dF_{\pi_2}(y) - 2\int\limits_0^x y\,dF_{\pi_2}(y)\Big] & \text{if } x\in[0,1],\\ 0 & \text{if } x>1. \end{cases}$$

Assume that the second player's strategy $\pi_2 \in \Pi$ is fixed. We notice that the problem

$$(4)\qquad \sup_{\pi_1\in\Pi} K(\pi_1, \pi_2)$$

is equivalent to the problem of determining the optimal randomized stopping time τ_1 in the optimal stopping problem of the Markov chain $\{\xi_n^1, n \geqslant 1\}$ with the gain function (3) and the operator T defined by

$$Tu(x) = \int\limits_x^1 u(y)p(x,y)\,dy, \qquad x\in[0,1],$$

where u is a bounded measurable non-negative function.

We use Theorem 21 from [3] which implies that it is sufficient to consider only the class of non-randomized Markov stopping times. To solve problem (4) we discuss the optimality equation

$$(5)\qquad v(x) = \max\{g(x \mid \pi_2), Tv(x)\}, \qquad x\in[0,1].$$

We introduce the function

$$(6)\qquad G(x \mid \pi_2) = \frac{Tg(x \mid \pi_2)}{g(x \mid \pi_2)}, \qquad x\in(0,1),$$

and we find that

$$\lim_{x\to 0+} G(x \mid \pi_2) = +\infty \quad \text{and} \quad \lim_{x\to 1^-} G(x \mid \pi_2) = 0.$$

Moreover, we see that

$$(7)\qquad G(x \mid \pi_2) = W(x) - 2\int\limits_x^1 \frac{y\int\limits_x^y u\,dF_{\pi_2}(u)}{x a(x \mid \pi_2)}\, p(x,y)\,dy,$$

where

$$(8)\qquad W(x) = \int\limits_x^1 \frac{y}{x}\, p(x,y)\,dy$$

and

$$a(x \mid \pi_2) = 1+\int\limits_0^1 y\,dF_{\pi_2}(y) - 2\int\limits_0^x y\,dF_{\pi_2}(y).$$

Now, assume that for $0 \leqslant x \leqslant y \leqslant 1$

$$\frac{\partial p(x, y)}{\partial x} \leqslant p(x, x)p(x, y). \tag{9}$$

From (9) it follows that the equation $W(x) = 1$ has a unique root w^* in $(0, 1)$, and by (7) we have $G(x \mid \pi_2) < 1$ for every $x \in [w^*, 1]$ and $\pi_1 \in \Pi$.

Let $f_0(x)$ be the solution of equation (2) for $\pi(x) \equiv 0$. We find easily that, for every $\pi_2 \in \Pi$,

$$f_0(x) \geqslant f(x), \qquad x \in (0, 1). \tag{10}$$

We assume that the following condition is valid:

$$1 - \int_0^x t f_0(t)\,dt - 2x^2 f_0(x) > 0, \qquad x \in (0, w^*). \tag{11}$$

Next, we evaluate

$$\begin{aligned} g'(x \mid \pi_2) &= a(x \mid \pi_2) + x[-2x\pi_2(x) f_{\pi_2}(x)] \\ &= \int_x^1 y\pi_2(y) f_{\pi_2}(y)\,dy + 1 - \int_0^x y\pi_2(y) f_{\pi_2}(y)\,dy - 2x^2\pi_2(x) f_{\pi_2}(x) \\ &\geqslant 1 - \int_0^x y f_{\pi_2}(y)\,dy - 2x^2 f_{\pi_2}(x). \end{aligned} \tag{12}$$

Now, if condition (11) is satisfied, then from (10) and (12) we obtain

$$g'(x \mid \pi_2) > 0, \qquad x \in (0, w^*), \tag{13}$$

for every $\pi_2 \in \Pi$. Using (6) we get

$$G'(x \mid \pi_2) = -p(x, x) + \int_x^1 \frac{g(y \mid \pi_2)}{g(x \mid \pi_2)} \frac{\partial p(x,y)}{\partial x}\,dy - \\ - \frac{g'(x \mid \pi_2)}{g(x \mid \pi_2)} \int_x^1 \frac{g(y \mid \pi_2)}{g(x \mid \pi_2)} p(x, y)\,dy.$$

We infer from (9) and (13) that for every $\pi_2 \in \Pi$ the inequality

$$G'(x \mid \pi_2) < p(x, x)[G(x \mid \pi_2) - 1], \qquad x \in (0, w^*),$$

holds. Hence, we have

LEMMA 3. *If conditions* (9) *and* (11) *are satisfied, then for every* $\pi_2 \in \Pi$ *there exists a unique point* $z_{\pi_2} \in (0, w^*)$ *such that*

$$G(x \mid \pi_2) < 1 \quad \text{if } x \in (z_{\pi_2}, 1]$$

and

$$G(x \mid \pi_2) > 1 \quad \text{if } x \in [0, z_{\pi_2}).$$

We also notice that the solution of the functional equation (5) is given by

$$v_{\pi_2}(x) = \begin{cases} g(x \mid \pi_2) & \text{if } x \in (z_{\pi_2}, 1], \\ Tv_{\pi_2}(x) & \text{if } x \in [0, z_{\pi_2}]. \end{cases}$$

Using Lemmas (P1) and (P2) from [2] as well as our Lemma 2, we see that against any fixed strategy $\pi_2 \in \Pi$ the best the first player can do is to apply the strategy $\pi(x) = I_{[z_{\pi_2},1]}(x)$, where I is the indicator function of the subset $[z_{\pi_2}, 1]$. Thus, the associated optimal stopping problem helped to distinguish a simple subclass of behaviour strategies in the game considered. In the sequel, we seek the optimal strategies for both players in that subclass of strategies.

4. Reduced game over the unit square and its solution. The simple class of behaviour strategies established in the previous section is described by a parameter in $[0, 1]$. Therefore, we may consider the reduced game $\bar{\Gamma} = ([0,1], [0,1], \bar{K})$, where $\bar{K}$ is a restriction of K, e.g. $K(I_{[x_0,1]}, I_{[y_0,1]}) = \bar{K}(x_0, y_0)$, $x_0, y_0 \in [0, 1]$. Let us consider the case $0 \leqslant y_0 \leqslant x_0 \leqslant 1$.

From (1) we get

$$\bar{K}(x_0, y_0) = \int_{x_0}^{1} x f_{x_0}(x)\,dx - \int_{y_0}^{1} y f_{y_0}(y)\,dy + \int_{x_0}^{1} x f_{x_0}(x)\,dx \int_{y_0}^{1} y f_{y_0}(y)\,dy - 2\int_{x_0}^{1} x \int_{y_0}^{x} y f_{y_0}(y)\,dy f_{x_0}(x)\,dx. \tag{14}$$

Since the game is symmetric, the necessary condition for the existence of a pure optimal strategy for both players takes the form

$$\frac{\partial \bar{K}}{\partial x_0}(x_0, x_0) = 0, \qquad \frac{\partial \bar{K}}{\partial y_0}(x_0, x_0) = 0. \tag{15}$$

For $0 \leqslant y_0 \leqslant x_0 \leqslant 1$, from (14) we obtain

$$\frac{\partial \bar{K}}{\partial x_0}(x_0, y_0) = f_{x_0}(x_0)\Big\{\Big[1 + \int_{y_0}^{1} y f_{y_0}(y)\,dy\Big]\Big[-x_0 + \int_{x_0}^{1} x p(x_0, x)\,dx\Big] + 2x_0 \int_{y_0}^{x_0} y f_{y_0}(y)\,dy - 2\int_{x_0}^{1} x \int_{y_0}^{x} y f_{y_0}(y)\,dy\, p(x_0, x)\,dx\Big\}$$

and

$$\frac{\partial \bar{K}}{\partial y_0}(x_0, y_0) = f_{y_0}(y_0)\Big\{-\Big[1 + \int_{x_0}^{1} x f_{x_0}(x)\,dx\Big]\Big[-y_0 + \int_{y_0}^{1} y p(y_0, y)\,dy\Big] + 2\int_{x_0}^{1} y \int_{x_0}^{y} x f_{x_0}(x)\,dx\, p(y_0, y)\,dy\Big\}.$$

Conditions (15) imply

$$(16)\qquad x_0\Big[1+\int_{x_0}^{1} xf_{x_0}(x)\,dx\Big]-\int_{x_0}^{1} x\Big[1+\int_{x_0}^{1} xf_{x_0}(x)\,dx-$$
$$-2\int_{x_0}^{x} yf_{x_0}(y)\,dy\Big]\,p(x_0,x)\,dx=0,$$

where

$$f_{x_0}(x)=p(0,x)+\int_{0}^{x_0} f(t)p(t,x)\,dt,\qquad x\in[x_0,1],$$

and

$$f(x)=p(0,x)+\int_{0}^{x} f(t)p(t,x)\,dt,\qquad x\in[0,x_0].$$

We discuss now the existence and uniqueness of the solution of equation (16). Denote by $S(x_0)$ the left-hand side of (16). We check easily that $S(w^*)>0$, where w^* is the root of the equation $W(x)=1$ and the function W is given by (8). We also see that

$$S(0)=-\int_{0}^{1} xp(0,x)\,dx\Big[1+\int_{0}^{1} xp(0,x)\,dx\Big]+2\int_{0}^{1} x\int_{0}^{x} yp(0,y)\,dy\,p(0,x)\,dx$$
$$<-\int_{0}^{1} xp(0,x)\,dx\Big[1-\int_{0}^{1} xp(0,x)\,dx\Big]<0.$$

Thus, it remains to examine the monotonicity property of the function $S(x_0)$ on $(0,w^*)$. One could find various sufficient conditions for the uniqueness of the solution x^* of equation (16). Let us study the derivative

$$(17)\qquad S'(x_0)=1+\int_{x_0}^{1} xf_{x_0}(x)\,dx+x_0f_{x_0}(x_0)\Big[-x_0+\int_{x_0}^{1} xp(x_0,x)\,dx\Big]+$$
$$+x_0\Big[1+\int_{x_0}^{1} yf_{x_0}(y)\,dy\Big]p(x_0,x_0)-f_{x_0}(x_0)\int_{x_0}^{1} x\Big[-x_0+$$
$$+\int_{x_0}^{1} yp(x_0,y)\,dy+2x_0-2\int_{x_0}^{x} yp(x_0,y)\,dy\Big]p(x_0,x)\,dx-$$
$$-\int_{x_0}^{1} x\Big[1+\int_{x_0}^{1} yf_{x_0}(y)\,dy-2\int_{x_0}^{x} yf_{x_0}(y)\,dy\Big]\frac{\partial p(x_0,x)}{\partial x_0}\,dx.$$

Using once again condition (9) in (17) we see that

$$(18)\qquad S'(x_0) \geqslant p(x_0, x_0)S(x_0)+1+\int_{x_0}^{1} xf_{x_0}(x)\,dx+ \\ +f_{x_0}(x_0)\Big[-x_0^2-\Big(\int_{x_0}^{1} xp(x_0, x)\,dx\Big)^2\Big]+ \\ +2\int_{x_0}^{1} x\int_{x}^{x} yp(x_0, y)\,dy\,p(x_0, x)\,dx.$$

From (11) we obtain the inequality

$$1+\int_{x_0}^{1} xf_{x_0}(x)\,dx-2x_0^2 f_{x_0}(x_0)>0, \qquad x_0\in(0, w^*).$$

Hence it is sufficient to assume that, for $x_0\in(0, w^*)$,

$$(19)\qquad x_0^2-\Big(\int_{x_0}^{1} xp(x_0, x)\,dx\Big)^2+2\int_{x_0}^{1} x\int_{x_0}^{x} yp(x_0, y)\,dy\,p(x_0, x)\,dx>0.$$

From (18) we have $S'(x_0)>p(x_0', x_0)S(x_0)$ in $(0, w^*)$, and the uniqueness of x^* follows.

To prove the optimality of the strategy x^* it is sufficient to show that for $x^*\leqslant x_0$ and $y_0\leqslant x^*$ the conditions

$$(20)\qquad \frac{\partial\overline{K}}{\partial x_0}(x_0, x^*)\leqslant 0, \qquad \frac{\partial\overline{K}}{\partial y_0}(x^*, y_0)\leqslant 0$$

are satisfied. First, let us introduce some auxiliary functions:

$$(21)\qquad \begin{aligned} R(x_0) &= -x_0+\int_{x_0}^{1} xp(x_0, x)\,dx, \qquad x_0\in[0, 1],\\ \Phi^*(x) &= \int_{x^*}^{1} yf_{x^*}(y)\,dy, \qquad x\in[x^*, 1],\\ N(x_0) &= \int_{x_0}^{1} x\Phi^*(x)p(x_0, x)\,dx-x_0\Phi^*(x_0), \qquad x_0\in[x^*, 1],\\ M(x_0) &= N(x_0)R^{-1}(x_0), \qquad x_0\in[x^*, 1]. \end{aligned}$$

Using the notation from (21) and relation (16) we obtain

$$\frac{\partial\overline{K}}{\partial x_0}(x_0, x^*) = 2f_{x_0}(x_0)R^{-1}(x^*)[R(x_0)N(x^*)-R(x^*)N(x_0)], \qquad x_0\in[x^*, 1],$$

and

$$\frac{\partial \bar{K}}{\partial y_0}(x^*, y_0) = 2f_{y_0}(y_0)R^{-1}(x^*)\left\{\int_{x^*}^{1} y\Phi^*(y)[R(x^*)p(y_0, y) - \right.$$

$$\left. -R(y_0)p(x^*, y)]dy\right\}, \quad y_0 \in [0, x^*].$$

Now, let us study the following properties of the function $M(x_0)$, $x_0 \in [x^*, 1]$:

(i) $$\lim_{x_0 \to w^*+} M(x_0) = -\infty,$$

(ii) $$\lim_{x_0 \to w^*-} M(x_0) = +\infty,$$

(iii) $$M(1) = \Phi^*(1),$$

(iv) $$M(x_0) \leqslant \Phi^*(x_0) \leqslant \Phi^*(1) < M(x^*), \quad x_0 \in (w^*, 1],$$

(v) $$M(x_0) \geqslant M(x^*), \quad x_0 \in [x^*, w^*).$$

We prove properties (iv) and (v). Let $x_0 \in (w^*, 1]$. Then $R(x_0) < 0$ and $M(x_0) \leqslant \Phi^*(x_0)$ by the definition of $M(x_0)$. Since $\Phi^*(1) < 1$ and $M(x^*) = 0.5[1+\Phi^*(1)]$ by (16), we have $0.5\ [1+\Phi^*(1)] \geqslant \Phi^*(1)$ and (iv) holds. Now, let $x_0 \in [x^*, w^*)$ and $C(x_0) = 2R(x_0)[M(x_0) - M(x^*)]$. We obtain

$$C(x_0) = 2N(x_0) - R(x_0)[1+\Phi^*(1)], \quad x_0 \in [x^*, w^*],$$

where $C(x^*) = 0$ and $C(w^*) > 0$. Next, we evaluate

$$C'(x_0) = [1 + x_0 p(x_0, x_0)][1+\Phi^*(1) - 2\Phi^*(x_0)] -$$

$$-2x_0^2 f_{x^*}(x_0) + \int_{x_0}^{1} x[2\Phi^*(x) - 1 - \Phi^*(1)]\frac{\partial p(x_0, x)}{\partial x_0}dx.$$

Using assumption (9) we get

$$C'(x_0) \geqslant 1 + \Phi^*(1) - 2\Phi^*(x_0) - 2x_0^2 f_{x^*}(x_0) + p(x_0, x_0)C(x_0).$$

Finally, from (11) we have $C'(x_0) > p(x_0, x_0)C(x_0)$, and since $R(x_0) > 0$, we obtain property (v).

It is easy to notice that properties (iv) and (v) imply the first inequality in (20). In order to obtain the second inequality in (20) it is sufficient to assume that, for $x \in [0, w^*)$,

(22) $$\frac{\partial}{\partial x}\frac{p(x, y)}{R(x)} \geqslant 0, \quad 0 \leqslant x \leqslant y \leqslant 1,$$

or

$$\frac{\partial}{\partial x} p(x, y) \geqslant 0, \quad 0 \leqslant x \leqslant y \leqslant 1.$$

Thus we have proved the following

THEOREM. *If the assumptions of Lemmas* 1-3 *and the conditions* (19) *and* (22) *are satisfied, then the optimal strategy for both players in the considered game is uniquely determined by the solution of equation* (16).

We give a simple example of the game. Let $p(x, y) = I(y-x)$, where I is the indicator function of $[0, 1]$. Then $f(x) = e^x, f_{x_0}(x) = e^{x_0}$ for $x \in [x_0, 1]$ and $R(x_0) = 0.5[1-2x_0-x_0^2]$. We also find that $w^* = \sqrt{2}-1$ and equation (16) takes the form

$$[1-2x_0-x_0^2]x_0^{-1}(1-x_0^2)^{-1} = e^{x_0}. \tag{24}$$

The solution of equation (24) is $x^* = 0.2851$.

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