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R. SMARZEWSKI (Lublin)

ON WEAK CHEBYSHEV SUBSPACES AND CHEBYSHEV APPROXIMATION BY THEIR ELEMENTS

1. Introduction. Let $C[a, b]$ be a space of real-valued functions defined and continuous on the closed interval $[a, b]$ normed by

$$\|f\| = \max\{|f(x)|: x \in [a, b]\},$$

and let M be an n-dimensional subspace of $C[a, b]$.

Definition 1. M is said to be a *Haar subspace* on $[a, b]$ if each non-trivial function from M has no more than $n-1$ zeroes in $[a, b]$.

Definition 2. M is said to be a *weak Chebyshev subspace* on $[a, b]$ if each function from M changes its sign in $[a, b]$ at most $n-1$ times.

If $[c, d] \subset [a, b]$ and $f \in C[a, b]$, then we denote by $f|[c, d]$ the function f restricted to $[c, d]$. Let x_i, $a = x_0 < x_1 < \ldots < x_{s+1} = b$ $(s \geqslant 0)$ be fixed knots and let P_i $(i = 0, 1, \ldots, s)$ be n_i-dimensional Haar subspaces on intervals $[x_i, x_{i+1}]$. Let us write

$$(1)\qquad P[x_1, \ldots, x_s] = \{p \in C[a, b]: p|[x_i, x_{i+1}] \in P_i,\ i = 0, 1, \ldots, s\}.$$

In the sequel, for the convenience of notation, we denote $P[x_1, \ldots, x_s]$ shortly by P. Obviously, P is a non-empty linear subspace of $C[a, b]$. In Section 2 we shall establish some important properties of the subspace P. Applications of these properties to the linear Chebyshev approximation by elements of P will be discussed in Section 3. In Section 4, a generalization of the non-linear Chebyshev approximation from [3] will be given.

2. Some properties of the subspace P. The following theorem generalizes Theorem 4 from [1] (see also Remark 1, ibidem, p. 36).

THEOREM 1. *P is a weak Chebyshev subspace on $[a, b]$ of dimension*

$$\sum_{i=0}^{s} n_i - s.$$

Proof. First, consider the case $s = 1$. For each function $p \in P = P[x_1]$ we have $p = q*r$, where the operation $*$ indicates that $p|[a, x_1] = q \in P_0$ and $p|[x_1, b] = r \in P_1$. Let p_1 be an arbitrary fixed positive function in P

and let p_i $(i = 1, \ldots, n_0)$ be a basis for P_0 and let p_i $(i = n_0, \ldots, n_0+n_1-1)$, where $p_{n_0} = p_1$, be a basis for P_1. Such a function p_1 exists, since in every Haar subspace there exists a positive function. Now, if μ_i are non-zero real numbers such that $p_i(x_1) = \mu_i p_1(x_1)$ for $i = 1, \ldots, n_0+n_1-1$, then we may prove — analogously as in [1] — that the functions $p_i * \mu_i p_1$ for $i = 1, \ldots, n_0$ and $\mu_i p_1 * p$ for $i = n_0+1, \ldots, n_0+n_1-1$ are linearly independent.

Since for an arbitrary function $p = q*r \in P$ and a constant μ such that $p(x_1) = \mu p_1(x_1)$ we have

$$p = (q*\mu p_1)+(\mu p_1 * r)-(\mu p_1 * \mu p_1),$$

P is an (n_0+n_1-1)-dimensional subspace. Hence we can prove by induction, as in [1], that P has dimension

$$\sum_{i=0}^{s} n_i - s$$

and that P is a weak Chebyshev subspace.

Remark. Bartelt noted in [1] that neither Theorems 1 and 2 nor Theorem 3 are valid without the assumption $1 \in M$. But this is not true. The subspace spanned by the system of functions $\{x, x^2, \ldots, x^n\}$, where $x \in [a, b]$ and $0 \notin [a, b]$, is a simple counterexample. It is known that each Haar subspace contains a positive function. Conversely, the assumption $1 \in M$ in Theorems 1, 2 and 3 from [1] may be replaced by the weaker assumption that M contains a positive function. We omit the proofs of these generalized theorems, since they are essentially the same as in [1].

Moreover, Bartelt has proved that in $C[a, b]$ there exist no weak Chebyshev subspaces, practically applicable, other than those defined by (1).

The following lemma in a sense characterizing Haar subspaces will be useful in the sequel.

LEMMA 1. *Let H be an n-dimensional Haar subspace on $[a, b]$ and let an arbitrary number $\lambda \neq 0$ and knots x_i $(i = 1, \ldots, n-2)$, $a < x_1 < \cdots \ldots < x_{n-2} < b$, be given. Then there exist a number ε_1,*

$$0 < \varepsilon_1 < \min_{i=0,\ldots,n-2} (x_{i+1}-x_i)/2, \quad \textit{where } x_0 = a,\ x_{n-1} = b,$$

and a function $h \in H$ such that $h(a) = \lambda$, h changes its sign exactly at $n-2$ points $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$, where $0 < \varepsilon < \varepsilon_1$ and $h(x) \neq 0$ for all $x \in [a, b] \setminus \{y_1, \ldots, y_{n-2}\}$.

Proof. It is known that there exists a function $g \in H$ such that $g(a) = \lambda$, $g(b) = 0$, and g changes its sign exactly at $n-2$ points x_i

$(i = 1, \ldots, n-2)$. Additionally, $g(x) \neq 0$ for all $x \in [a, b] \setminus \{x_1, \ldots, x_{n-2}, b\}$. Let p be an arbitrary fixed positive function in H and let

$$\sigma = \operatorname{sgn}[g(x)\colon x \in (x_{n-2}, b)],$$

where

$$\operatorname{sgn} f(x) = \begin{cases} 1, & f(x) > 0, \\ 0, & f(x) = 0, \\ -1, & f(x) < 0. \end{cases}$$

Write

$$h_\mu(x) = g(x) + \mu\sigma p(x), \qquad \mu > 0.$$

We have $h_\mu(x) > 0$ for all $\mu > 0$ and $x \in [x_{n-2}, b]$. Let a positive real number ε_1 be so small that the function $g(x)$ is strictly monotone in every interval $(x_i - \varepsilon, x_i + \varepsilon)$ for $0 < \varepsilon < \varepsilon_1$ and $i = 1, \ldots, n-2$. Since g is continuous and H is an n-dimensional Haar subspace, then there exists a $\mu_1 = \mu_1(\varepsilon) > 0$ such that the function $h_\mu(x)$, $0 < \mu < \mu_1$, changes its sign exactly at $n-2$ points $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$ for $i = 1, \ldots, n-2$. A function h defined by

$$h(x) = \lambda h_\mu(x) / \big(\lambda + \mu\sigma p(a)\big)$$

has the same properties and, additionally, $h(a) = \lambda$. Hence the proof of the lemma is complete.

Definition 3. A function $f \in C[a, b]$ is said to *alternate* $n-1$ *times* on a subset D of $[a, b]$ if there are n points $x_1 < x_2 < \ldots < x_n$ in D such that $f(x_i) = -f(x_{i+1})$ for $i = 1, \ldots, n-1$. The points x_i are called *alternation points*.

We denote by $\mu(f, D)$ the maximal number of alternation points of the function f in D. Obviously, it is possible that $\mu(f, D) = \infty$, since

$$f(x) = \begin{cases} x \sin \pi/x, & 0 < x \leqslant 2, \\ 0, & x = 0, \end{cases}$$

and $D = \{2, 2/3, 2/5, 2/7, \ldots\}$. If $D = \emptyset$, then we set $\mu(f, D) = 0$.

THEOREM 2. *Let $f \in C[a, b]$ and let D be a closed subset in $[a, b]$ such that $|f(x)| > 0$ for all $x \in D$. Let $D_i = [x_i, x_{i+1}] \cap D$ for $i = 0, 1, \ldots, s$. Then there exists a function $g \in P = P[x_1, \ldots, x_s]$ such that $f(x)g(x) > 0$ for all $x \in D$ if and only if the following conditions are satisfied:*

(2) $$\mu(f, D_i) \leqslant n_i, \qquad i = 0, 1, \ldots, s;$$

(3) $$\mu(f, D_{j-1} \cup D_j) \leqslant n_{j-1} + n_j - 1$$

for each j such that $1 \leqslant j \leqslant s$ and $D_{j-1}, D_j \neq \emptyset$;

(4) $$\mu(f, D_u \cup D_v) \leqslant n_u + n_v - 1$$

for every u, v *such that* $0 \leqslant u < v-1 \leqslant s-1$, $D_u, D_v \neq \emptyset$, $D_i = \emptyset$, *and* $n_i = 1$ *for* $i = u+1, \ldots, v-1$.

Proof. Necessity. If there exists a set D_i containing at least n_i+1 alternation points, then the proof of the necessity is obvious, since P_i is an n_i-dimensional Haar subspace. Now, assume that

$$\mu(f, D_{j-1} \cup D_j) > n_{j-1}+n_j-1 \quad \text{for some } 1 \leqslant j \leqslant s.$$

Then by elementary considerations we may prove that the inequality

$$f(x)g(x) > 0 \quad \text{for } x \in D_{j-1} \cup D_j \text{ and } g|[x_{j-1}, x_{j+1}] \in C[x_{j-1}, x_{j+1}]$$

holds if the function g has either n_{j-1} zeroes in $[x_{j-1}, x_j]$ or n_j zeroes in $[x_j, x_{j+1}]$. Hence we obtain a contradiction.

If condition (4) is not satisfied for some u, v, then from $n_i = 1$ $(i = u+1, \ldots, v-1)$ it follows that the inequality

$$f(x)g(x) > 0 \quad \text{for } x \in D_u \cup \ldots \cup D_v \text{ and } g|[x_u, x_{v+1}] \in C[x_u, x_{v+1}]$$

holds if the function g has either n_u zeroes in $[x_u, x_{u+1}]$ or n_v zeroes in $[x_v, x_{v+1}]$. This gives a contradiction. The proof of the necessity is now completed.

Sufficiency. Let x_{ik}, $x_i \leqslant x_{i1} < \ldots < x_{ik_i} \leqslant x_{i+1}$, be alternation points of the function f in D_i for $i = 0, 1, \ldots, s$.

Case 1. First consider the case where the following conditions are satisfied:

(a) $1 \leqslant k_i \leqslant n_i$ for $i = 0, 1, \ldots, s$;

(b) $\operatorname{sgn} f(x_{ik_i}) = \operatorname{sgn} f(x_{i+1,1})$ for $i = 0, 1, \ldots, s-1$;

(c) conditions (2) and (3) hold.

For $k = 1, \ldots, k_i$ we put

$$a_{ik} = \inf\{x\colon x \in [x_{i,k-1}, x_{ik}] \cap D \text{ and } \operatorname{sgn} f(x) = \operatorname{sgn} f(x_k)\},$$

$$b_{ik} = \sup\{x\colon x \in [x_{ik}, x_{i,k+1}] \cap D \text{ and } \operatorname{sgn} f(x) = \operatorname{sgn} f(x_k)\}.$$

From the continuity of the function f on $[a, b]$ it follows that $b_{ik} < a_{i,k+1}$ for $k = 1, \ldots, k_i-1$, since the set D is closed. If $k_0 = 1$, we choose a function h_0 in P_0 such that $h_0(a) = f(x_1)$. This function exists, since a positive function exists in P_0. If $2 \leqslant k_0 \leqslant n_0$, we choose a function h_0 in P_0 such that $h_0(a) = f(x_1)$, h_0 changes its sign at r points z_{0k}, where $z_{0k} \in (b_{0k}, a_{0,k+1})$ for $k = 1, \ldots, k_0-1$, $z_{0k} \in (b_{01}, a_{02})$ for $k = k_0, k_0+1, \ldots, r$, and

$$r = \begin{cases} n_0-1 & \text{if } n_0-k_0 \text{ is even,} \\ n_0-2 & \text{otherwise.} \end{cases}$$

In the case $r = n_0-1$ the existence of h_0 follows directly from the fact that P_0 is a Haar subspace, and in the case $r = n_0-2$ from Lemma 1. We have $f(x)h_0(x) > 0$ for all $x \in D_0$ and the function h_0 defined above. By using (a) and (b) we may choose, in the analogous way as h_0, a function

$h_1 \in P_1$ such that $h_1(x_1) = h_0(x_1)$ and $f(x)h_1(x) > 0$ for all $x \in D_1$. Consequently, we may prove by induction that there exist functions $h_i \in P_i$ ($i = 2, \ldots, s$) defined and continuous on intervals $[x_i, x_{i+1}]$ and such that $h_i(x_i) = h_{i-1}(x_i)$ and $f(x)h_i(x) > 0$ for all $x \in D_i$. Let us define a function h by

$$h\,|\,[x_i, x_{i+1}] = h_i, \quad i = 0, 1, \ldots, s.$$

Clearly, $h \in P$ and $f(x)h(x) > 0$ for all $x \in D$.

Case 2 (general). Now, it is sufficient to prove that there exist a function F defined and continuous on $[a, b]$ and a closed set B, $D \subset B \subset [a, b]$, such that $F(x) = f(x)$ for all $x \in D$, $|F(x)| > 0$ for all $x \in B$ and that for alternation points of F in B conditions (a), (b) and (c) from Case 1 are satisfied.

We may assume without loss of generality that

$$\operatorname{sgn} f(x_{j-1,k_{j-1}}) = \operatorname{sgn} f(x_{j1}) \tag{5}$$

and

$$\operatorname{sgn} f(x_{u,k_u}) = \operatorname{sgn} f(x_{v1}) \tag{6}$$

for arbitrary j, u, and v as in (3) and (4).

In fact, if (5) is not true, e.g.

$$\operatorname{sgn} f(x_{j-1,k_{j-1}}) = -\operatorname{sgn} f(x_{j1}),$$

then it follows from (2) and (3) that either $k_j < n_j$ or $k_{j+1} < n_{j+1}$. In the first case let us denote by F a function defined and continuous on $[a, b]$ such that $F(x) = f(x)$ for all $x \in D$, and $F(z) = -\operatorname{sgn} f(x_{j1})$, where $z \in (x_{j-1,k_{j-1}}, x_j)$ is arbitrarily fixed. In the second case we choose $z \in (x_j, x_{j1})$ and $F(z) = -\operatorname{sgn} f(x_{j-1,k_{j-1}})$. Since the set $B = \{z\} \cup D$ is closed and f is continuous on this set, such a function F exists by the well-known Tietze theorem. The point z is a new alternation point of F in the set $D_{j-1} \cup D_j \cup \{z\}$.

We may consider (6) in the same way as (5). For the convenience of notation, we denote the obtained function F and the set B by f and D, respectively. Obviously, (2)-(6) are satisfied for the function f and the set D.

Now, for all u and v as in (4) we define a continuous function F on $[a, b]$ by $F(x) = f(x)$ for all $x \in D$ and $F(z_i) = \operatorname{sgn} f(x_{v1})$, where $z_i \in (x_i, x_{i+1})$ for $i = u+1, \ldots, v-1$. Also, set $B = D \cup \{z_{u+1}, \ldots, z_{v-1}\}$. Let the obtained function F and the closed set B also be denoted by f and D, respectively. Conditions (2)-(6) are satisfied for the function f and the set D. Additionally, we have $D_i \neq \emptyset$ for $i = u+1, \ldots, v-1$ and all u and v as in (4). Obviously, the last construction is valid if $n_i \geq 1$ for $i = u+1, \ldots, v-1$ in (4) and with trivial modifications also if $u = -1$ or $v = s+1$ in (4).

For the completeness of the proof we must consider the case where (2) holds for u and v,

$$\mu(f, D_u \cup D_v) = n_u + n_v \quad \text{and} \quad \operatorname{sgn} f(x_{u,k_u}) = -\operatorname{sgn} f(x_{v1}),$$

where $0 \leqslant u < v-1 \leqslant s-1$, D_u, $D_v \neq \emptyset$, $D_i = \emptyset$ for $i = u+1, \ldots, v-1$, and there exists t, $u < t < v$, such that $n_t > 1$. In this case, while defining on $[a, b]$ a continuous function F such that $F(x) = f(x)$ for $x \in D$, $F(z_1) = \operatorname{sgn} f(x_{u,k_u})$ and $F(z_2) = \operatorname{sgn} f(x_{v1})$, where $z_1, z_2 \in (x_t, x_{t+1})$, and a closed set B equal to $\{z_1, z_2\} \cup D$, we obtain the case where (4) and (6) are satisfied, which we considered above. Let us again denote the obtained function F and the closed set B by f and D, respectively. Since (a), (b) and (c) are satisfied for these f and D, we may use the method from Case 1. Thus the proof of Theorem 2 is completed.

The following two examples illustrate the role of conditions (3) and (4) in Theorem 2.

Example 1. Let the subspace $P[9/24, 11/24]$, where $P_0 = P_2 = \operatorname{span}\{1, x\}$ and $P_1 = \operatorname{span}\{1\}$, be defined on the interval $[9/40, 2]$, et $f(x) = \cos \pi/x$, and $D = \{1/4, 1/3, 1/2, 1\}$. In this case, condition (4) ls not satisfied and Theorem 2 is not true.

Example 2. Let the subspace $P[5/12]$, where $P_0 = P_1 = \operatorname{span}\{1, x\}$, be defined on the interval $[9/40, 2]$ and let f and D be defined as in Example 1. In this case, condition (3) is not satisfied and Theorem 2 is not true.

3. Linear Chebyshev approximation by elements of P. Let the function $f \in C[a, b]$ and the subspace $P = P[x_1, \ldots, x_s]$ defined by (1) be given. A function $g \in P$ is the *best Chebyshev approximation* for f if

$$\|f-g\| \leqslant \|f-h\| \quad \text{for all } h \in P.$$

From the general theory of linear approximation (see, e.g., [2] or [5]) it follows that the best approximation $g \in P[x_1, \ldots, x_s]$ exists for all functions $f \in C[a, b]$ and that the following theorem holds:

THEOREM 3. *An element $g \in P$ is the best approximation for $f \in C[a, b] \setminus P$ if and only if on the set $D = \{x: |e(x)| = \|e\|\}$ there exists no function $h \in P$ of the same sign as the error function $e = f-g$.*

Obviously, this approximation is not unique for every function $f \in C[a, b]$.

Definition 4. An alternans of the function f on the closed subset D of $[a, b]$ has *Property* A if at least one of the following three conditions is satisfied:

(i) there exists i such that $0 \leqslant i \leqslant s$ and $\mu(f, D_i) > n_i$;

(ii) there exists j such that $1 \leqslant j \leqslant s$ and $\mu(f, D_{j-1} \cup D_j) > n_{j-1} + n_j - 1$;

(iii) there exist u and v such that $0 \leqslant u < v-1 \leqslant s-1$, $D_u, D_v \neq \emptyset$, $D_i = \emptyset$ and $n_i = 1$ for $i = u+1, \ldots, v-1$, and $\mu(f, D_u \cup D_v) > n_u + n_v - 1$.

From Theorems 2 and 3 we obtain directly

ALTERNATION THEOREM. *An element $g \in P$ is the best approximation in P for $f \in C[a, b] \setminus P$ if and only if the error function $e = f - g$ has the alternans with Property A on the subset $D = \{x\colon |e(x)| = \|e\|\}$ of $[a, b]$.*

COROLLARY. *If the error function $e = f - g$ has at least*

$$\sum_{i=0}^{s} n_i - s + 1$$

alternation points in the subset $D = \{x\colon |e(x)| = \|e\|\}$ of $[a, b]$, then the function g is the best approximation in P for f.

4. Non-linear approximation by elements of P. Let H be an n-dimensional Haar subspace on $[a, b]$ and let $[c, d] \subset [a, b]$. We put

$$E_n(f, c; d) = \max_{x \in [c,d]} |f(x) - g(x)| = \inf_{h \in H} \max_{x \in [c,d]} |f(x) - h(x)|.$$

Moreover, let $f \oplus H$ denote a subspace spanned by the function f and the subspace H.

LEMMA 2. *If $f \oplus H$, where $f \in C[a, b] \setminus H$, is an $(n+1)$-dimensional Haar subspace, then*

(i) *The set $D = \{x\colon |r(x)| = \|r\|\} \cap [c, d]$ contains exactly n alternation points of the error function $r = f - g$ and $c, d \in D$.*

(ii) *$E_n(f; c, d)$ is a non-negative continuous function of variables c and d, which is a strictly increasing function of d for a fixed c and a strictly decreasing function of c for a fixed d.*

Proof. For the continuity of $E_n(f; c, d)$ see [4] or [5]. From the Alternation Theorem it follows that D contains k points, $k \geqslant n$. If $k > n$, then the function $r - \lambda p$, where $p(x) > 0$ for $x \in [a, b]$ and λ is a sufficiently small number, has at least $n+1$ zeroes. Thus we obtain a contradiction. We may complete the proof of parts (i) and (ii) by similar arguments.

In this section we assume that the subspaces P_i in the definition of $P[x_1, \ldots, x_s]$ in (1) are n_i-dimensional Haar subspaces on the interval $[a, b]$. Now, consider the following non-linear Chebyshev problem:

For an arbitrary fixed function $f \in C[a, b] \setminus P$ determine knots z_i, $a \leqslant z_1 \leqslant \ldots \leqslant z_s \leqslant b$, and a function $g \in P[z_1, \ldots, z_s]$ such that

$$\|f - g\| = \min_{a \leqslant x_1 \leqslant \ldots \leqslant x_s \leqslant b} \; \min_{h \in P[x_1, \ldots, x_s]} \|f - h\|. \tag{7}$$

We set $F_s(f; a, b) = \|f - g\|$.

Gavrilović [3] has solved problem (7) with the assumption that $P_i = \operatorname{span}\{1, x\}$ and that a subspace spanned by functions $1, x$ and $f(x)$

is a Haar subspace on $[a, b]$. Here we solve this problem only under the assumption that $f \oplus P_i$ for $i = 0, 1, \ldots, s$ are (n_i+1)-dimensional Haar subspaces on $[a, b]$. This generalizes the fact that f is a strictly convex or strictly concave function on $[a, b]$ in the sense of definitions from [6].

THEOREM 4. *If $f \oplus P_i$ for $i = 0, 1, \ldots, s$ are (n_i+1)-dimensional Haar subspaces on $[a, b]$, then there exist a unique sequence of knots z_i, $a < z_1 < \ldots < z_s < b$, and a unique function $g \in P[z_1, \ldots, z_s]$ such that* (7) *holds. For this function we have*

$$\|f-g\| = E_{n_i}(f; z_i, z_{i+1}), \qquad i = 0, 1, \ldots, s,$$

where $z_0 = a$ and $z_{s+1} = b$.

Proof. If $s = 1$, then the theorem follows from Lemma 2. For $s > 1$, the proof is analogous as in [3].

The set $D = \{x\colon |e(x)| = \|e\|\}$, where $e = f-g$ and g is defined by (7), contains exactly

$$\sum_{i=0}^{s} n_i - s + 1$$

points, and $z_i \in D$ for $i = 0, 1, \ldots, s+1$. Additionally, the error function e alternates at these points. We note that a method analogous to that from [3] may be applicable to determine the knots z_i and the function g in (7).

Example 3. Let $f(x) = 1/(x-c)$, where $x \in [a, b]$ and $c \notin [a, b]$. Moreover, let $P_i = \operatorname{span}\{1, x\}$ for $i = 0, 1, \ldots, s$ and $\sigma = \operatorname{sgn}(c-a)$. The knots z_i for $i = 1, \ldots, s$ and $F_s(1/(x-c); a, b)$ are determined [3] by the following non-linear system of equations:

$$|c-z_i|^{-1/2} - |c-z_{i-1}|^{-1/2} = \sigma \sqrt{2F_s\left(\frac{1}{x-c}; a, b\right)}, \qquad i = 1, \ldots, s+1.$$

Solving this system of equations we obtain

$$F_s\left(\frac{1}{x-c}; a, b\right) = \frac{1}{2}\left(\frac{v-u}{s+1}\right)^2, \qquad z_i = c - \sigma\left(\frac{s+1}{iv+(s-i+1)u}\right)^2,$$
$$i = 1, \ldots, s,$$

where $u = |c-a|^{-1/2}$ and $v = |c-b|^{-1/2}$.

The polygonal line g in (7) is uniquely determined by the vertex (z_i, y_i), where

$$y_i = \frac{1}{z_i - c} + \sigma F_s\left(\frac{1}{x-c}; a, b\right), \qquad i = 0, 1, \ldots, s+1.$$

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DEPARTMENT OF NUMERICAL METHODS
M. CURIE-SKŁODOWSKA UNIVERSITY
20-031 LUBLIN

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R. SMARZEWSKI (Lublin)

O SŁABYCH PODPRZESTRZENIACH CZEBYSZEWA ORAZ JEDNOSTAJNEJ APROKSYMACJI PRZEZ ICH ELEMENTY

STRESZCZENIE

W pracy omówiliśmy słabe podprzestrzenie Czebyszewa $P = P[x_1, \ldots, x_s]$, zdefiniowane przez wzór (1), oraz jednostajną, liniową i nieliniową aproksymację przez ich elementy. Tego typu podprzestrzenie badał Bartelt [1]. W rozdziale 2 zauważyliśmy, że wszystkie twierdzenia z pracy [1] pozostają słuszne dla nieco ogólniejszych założeń. W szczególności, twierdzenie 1 z tej pracy jest uogólnieniem twierdzenia 4 z [1]. W twierdzeniu 2 ujęliśmy pewną własność podprzestrzeni P, grającą istotną rolę w teorii liniowej jednostajnej aproksymacji funkcji ciągłych przez elementy z P. Aproksymacja ta została omówiona w rozdziale 3. W rozdziale 4 uogólniliśmy wyniki pracy [3], dotyczące nieliniowej aproksymacji funkcji ciągłych przez elementy P, oraz skonstruowaliśmy najlepszą aproksymację przez łamane dla funkcji $f(x) = 1/(x-c)$.
