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NUMERICAL SOLUTION OF GENERALIZED ABEL INTEGRAL EQUATIONS BY SPLINE FUNCTIONS

1. Introduction. The purpose of this paper is to present numerical methods for the solution of singular integral equations of the following two types:

$$g(t) = \int_0^t \frac{f(s)}{[h(t)-h(s)]^a}\,ds, \qquad t \in [0, R], \tag{1.1}$$

and

$$g(t) = \int_t^R \frac{f(s)}{[h(s)-h(t)]^a}\,ds, \qquad t \in [0, R], \tag{1.2}$$

where f is an unknown function, $0 < a < 1$, and h is a strictly increasing and continuously differentiable function in $[0, R]$. Since

$$\int \frac{f(s)}{[h(t)-h(s)]^a}\,ds = \int \frac{f(s)}{[(h(t)-h(0))-(h(s)-h(0))]^a}\,ds,$$

we may assume without loss of generality that $h(0) = 0$.

It is well known (see, e.g., [10]) that for equations (1.1) and (1.2) the following inversion formulae hold:

$$f(s) = \frac{\sin a\pi}{\pi}\frac{d}{ds}\int_0^s \frac{h'(t)g(t)}{[h(s)-h(t)]^{1-a}}\,dt, \qquad s \in [0, R], \tag{1.3}$$

and

$$f(s) = -\frac{\sin a\pi}{\pi}\frac{d}{ds}\int_s^R \frac{h'(t)g(t)}{[h(t)-h(s)]^{1-a}}\,dt, \qquad s \in [0, R]. \tag{1.4}$$

The numerical methods of solving equations (1.1) and (1.2) presented in this paper are based on inversion formulae (1.3) and (1.4) and generalize the high accuracy numerical methods described for the special selection $h(t) = t^p$, $p = 1/i$, and $i = 1/2, 1, 2, \ldots$, of the function h as given in our previous papers [5], [8], [9]. Note that the solutions of these equations are often necessary in the theory of mixed boundary value problems [10] as well as in other physical and mathematical problems (see, e.g., [2], [4], [6]). In particular, the numerical methods given in this paper can be used to solving the important equations (see [10]) of types (1.1) and (1.2) with h equal to $1-\cos(t)$ and $R = \pi$.

Now, let the function $S(x)$ be defined on the interval $[0, a]$. Then we denote by S_Δ, where $\Delta = \{x_1, x_2, \ldots, x_n\}$ and $0 = x_1 < x_2 < \ldots \ldots < x_n = a$, a spline function of degree $m = 2r-1$ $(1 \leqslant r \leqslant n)$ which either interpolates $S(x)$ at nodes x_i with arbitrary boundary conditions or approximates smoothly the values $\tilde{S}(x_i)$ of $S(x_i)$ in the sense of Reinsch [7]. Moreover, we require that the spline function S_Δ is given in the form

$$S_\Delta(x) = \sum_{i=0}^{m} a_i x^i + \sum_{j=1}^{n-1} \beta_j \theta(x, x_j)(x-x_j)^m, \tag{1.5}$$

where

$$\theta(x, x_j) = \begin{cases} 0 & \text{if } x < x_j, \\ 1 & \text{if } x \geqslant x_j. \end{cases}$$

A numerically stable method for determining a_i and β_j in (1.5) is given in [5] and [9].

2. Numerical solution of equation (1.1). In this section we assume that the function g is continuously differentiable on the interval $[0, R]$ and that $m > 1$, i.e. spline functions are of degree greater than or equal to 3. Interchanging the order of differentiation and integration in (1.3) we obtain

$$f(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(0)}{h^{1-a}(s)} + \int_0^s \frac{g'(t)}{[h(s)-h(t)]^{1-a}}\,dt\right) \tag{2.1}$$

for all $s \in (0, R]$.

Now, let a network Δ of the interval $[0, R]$ be such that $0 = t_1 < t_2 < \ldots < t_n = R$. Since $h(0) = 0$ and the function h is strictly increasing on $[0, R]$, the induced network Δ_x, $x = h(t)$, of the interval $[0, h(R)]$ satisfies $0 = h(t_1) < h(t_2) < \ldots < h(t_n) = h(R)$.

Let us denote by $(g\circ h^{-1})_{\Delta_x}(x)$, $x \in [0, h(R)]$, a spline function interpolating or smoothing the function $g\circ h^{-1}$ at the nodes $h(t_i)$, $i = 1, 2, \ldots \ldots, n$. We assume that the spline function $(g\circ h^{-1})_{\Delta_x}$ is given in the form

(1.5), i.e. that coefficients a_i and β_j are known. By (2.1) we define the approximate solution f_Δ of equation (1.1) in the form

$$(2.2)\qquad f_\Delta(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(0)}{h^{1-a}(s)} + \int_0^s \frac{g'_\Delta(t)}{[h(s)-h(t)]^{1-a}}\,dt\right),$$

where

$$(2.3)\qquad g_\Delta(t) = \sum_{i=0}^{m} a_i h^i(t) + \sum_{j=1}^{n} \beta_j \theta[h(t), h(t_j)][h(t)-h(t_j)]^m,$$

and a_i and β_j are coefficients of the spline function $(g\circ h^{-1})_{\Delta_x}$. Differentiating the function g_Δ given by (2.3) and substituting g'_Δ to (2.2) we get

$$(2.4)\qquad f_\Delta(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(0)}{h^{1-a}(s)} + \sum_{i=1}^{m} a_i a_i(s) + \sum_{j=1}^{n} \beta_j b_j(s)\right),$$

where

$$a_i(s) = i\int_0^s \frac{h'(t)h^{i-1}(t)}{[h(s)-h(t)]^{1-a}}\,dt, \qquad i = 1, 2, \ldots, m,$$

and

$$b_j(s) = m\int_0^s \frac{\theta[h(t), h(t_j)][h(t)-h(t_j)]^{m-1}h'(t)}{[h(s)-h(t)]^{1-a}}\,dt, \qquad j = 1, 2, \ldots, n.$$

Now, we derive formulae for $a_i(s)$ and $b_j(s)$ which do not require a numerical integration. This process is possible because the presence of $h'(t)$ in the formulas for $a_i(s)$ and $b_j(s)$ justifies a change of variables. For this purpose we need the formula

$$(2.5)\qquad \int \frac{x^{i-1}}{[h(s)-x]^{1-a}}\,dx = -\frac{[h(s)-x]^a}{a+i-1}\sum_{v=0}^{i-1} e_v(i-1)x^{i-v-1}h^v(s),$$

where

$$(2.6)\qquad e_v(i) = \begin{cases} 1, & v = 0, \\ e_{v-1}(i)\,\dfrac{i-v+1}{i-v+a}, & v = 1, 2, \ldots \end{cases}$$

Formula (2.5) can be easily deduced from (2.8) and (2.9) given in [9]. Substituting $x = h(t)$ in the formula for $a_i(s)$ and using (2.5) we get

$$(2.7)\qquad a_i(s) = \frac{i!\,h^{a+i-1}(s)}{\prod_{v=0}^{i-1}(a+v)}, \qquad i = 1, 2, \ldots, m.$$

Analogously, from the formula for $b_j(s)$ and (2.5) we obtain

(2.8)
$$b_j(s) = m[h(s)-h(t_j)]^a\,\theta[h(s), h(t_j)]\sum_{i=0}^{m-1}\binom{m-1}{i}[-h(t_j)]^{m-i-1}c_i(t_j, s)$$

for $j = 1, 2, \ldots, n$, where the functions $c_i(u, s)$ are defined by

$$c_i(u, s) = \frac{1}{a+i}\sum_{v=0}^{i} e_v(i)h^{i-v}(u)h^v(s). \tag{2.9}$$

For computer evaluations of the functions $a_i(s)$ and $c_i(u, s)$, which are essential in the numerical method presented above, we propose the following formulae:

$$a_1(s) = \frac{h^a(s)}{a},$$

$$a_{i+1}(s) = \frac{i+1}{a+i}a_i(s)h(s), \quad i = 1, 2, \ldots, m-1,$$

and

$$c_0(u, s) = \frac{1}{a},$$

$$c_i(u, s) = \frac{1}{a+i}[h^i(u)+ih(s)c_{i-1}(u, s)], \quad i = 1, 2, \ldots, m-1.$$

These formulae for $a_i(s)$ and $c_i(u, s)$ can be proved inductively by using (2.6), (2.7), and (2.9).

Let us define the norm in the space of all continuous functions determined on the interval $[0, a]$ by

$$\|f\| = \max\{|f(x)|: x \in [0, a]\}.$$

We can now establish a uniform convergence of the approximate solution to the exact solution of equation (1.1).

THEOREM 1. *Let g and h be continuously differentiable functions defined on $[0, R]$ and let h be a strictly increasing function on $[0, R]$ such that $h(0) = 0$. Then for every $s \in [0, R]$ we have*

$$|f(s)-f_\Delta(s)| \leqslant \frac{h^a(s)h'(s)\sin a\pi}{a\pi}\, e\big((g\circ h^{-1})', \Delta_x\big),$$

where f is the solution of equation (1.1), $a \in (0, 1)$, *$f_\Delta(s)$ is given by* (2.4) *and* (2.6)-(2.9), *and*

$$e\big((g\circ h^{-1})', \Delta_x\big) = \|(g\circ h^{-1})'-(g\circ h^{-1})'_{\Delta_x}\|.$$

Proof. By (2.1) and (2.2) we get

$$|f(s)-f_\Delta(s)| = \frac{h'(s)\sin a\pi}{\pi}\left|\int_0^s \frac{g'(t)-g'_\Delta(t)}{[h(s)-h(t)]^{1-a}}\,dt\right|.$$

Consequently, from (2.3) and (2.5) it follows that

$$|f(s)-f_\Delta(s)| \leqslant \frac{h'(s)e\big((g\circ h^{-1})', \Delta_x\big)\sin a\pi}{\pi}\left|\int_0^{h(s)} \frac{dx}{[h(s)-x]^{1-a}}\right|$$

$$= \frac{h^a(s)h'(s)\sin a\pi}{a\pi}\, e\big((g\circ h^{-1})', \Delta_x\big).$$

This completes the proof.

The significance of Theorem 1 lies in the fact that we can replace the investigation of the uniform convergence f_Δ to f, under the assumption that

$$|\Delta| = \max\{t_i-t_{i-1}\colon\ i = 2, 3, \ldots, n\}\to 0,$$

by the investigation of the uniform convergence of the derivative of the spline function $(g\circ h^{-1})_{\Delta_x}$ to the derivative of the function $g\circ h^{-1}$ as

$$|\Delta_x| = \max\{h(t_i)-h(t_{i-1})\colon\ i = 2, 3, \ldots, n\}\to 0.$$

Recently, many theorems in the theory of spline functions on this last subject have been formulated. In particular, if $g, h\in C^{(4)}[0, R]$ and $(g\circ h^{-1})_{\Delta_x}$ is an interpolating spline function satisfying the boundary conditions

$$(g\circ h^{-1})^{(i)}_{\Delta_x}(a) = (g\circ h^{-1})^{(i)}(a),$$

where $a = 0$ and $a = h(R)$ and $i = 1$ or $i = 2$, then from [3] it follows that we can put

$$e\big((g\circ h^{-1})', \Delta_x\big) = \frac{|\Delta_x|^3}{24}\,\|(g\circ h^{-1})^{(4)}\|$$

in Theorem 1.

Examples. Now we use the method presented above to the numerical solution of the integral equations

$$t^5 = \int_0^t \frac{f(s)}{\sqrt{t^2-s^2}}\,ds, \qquad t\in[0, 1],$$

and

$$\cos^5(t) = \int_0^t \frac{f(s)}{\sqrt{\cos(s)-\cos(t)}}\,ds, \qquad t\in[0, \pi].$$

By virtue of (2.1) the exact solutions of these equations are equal to

$$f(s) = \frac{15}{8}s^5 \tag{2.10}$$

and

(2.11)

$$f(s) = \frac{\sin(s)\sqrt{1-\cos(s)}}{\pi}\sum_{i=0}^{5}\binom{5}{i}\frac{\big(1-\cos(s)\big)^{4-i}\cos^{i-1}(s)\big(11\cos(s)-2i\big)}{10-2i+1},$$

respectively.

In Table 1 we list the absolute errors $|f(s_i)-f_\Delta(s_i)|$ for the first example.

TABLE 1

s	Absolute errors		
	$n = 51$	$n = 101$	$n = 201$
0.2	$1.1_{10}-7$	$9.2_{10}-9$	$1.2_{10}-9$
0.4	$3.9_{10}-8$	$3.4_{10}-9$	$2.3_{10}-10$
0.6	$1.7_{10}-8$	$1.5_{10}-9$	$1.3_{10}-10$
0.8	$9.6_{10}-9$	$7.7_{10}-9$	$4.1_{10}-10$
1	$6.4_{10}-9$	$1.3_{10}-9$	$4.1_{10}-10$

In this example, $s = s_i = i/5$ $(i = 1, 2, \ldots, 5)$, each $f(s_i)$ has been calculated from (2.10), and each $f_\Delta(s_i)$ from (2.4) and (2.6)-(2.9). Using the method proposed in [5] and [9] we have calculated the coefficients α_i and β_j of the function g_Δ under the assumption that

$$t_i = \sqrt{\frac{i-1}{n-1}}, \quad i = 1, 2, \ldots, n, \quad m = 3,$$

$$S(0) = S'(0) = 0, \quad S(1) = 1, \quad S'(1) = \frac{5}{2},$$

where $S(x) = (g\circ h^{-1})_{\Delta_x}(x)$, $g(t) = t^5$, and $h(t) = t^2$.

We note that $g\circ h^{-1} \in C^{(2)}[0, 1]$ in this example. Therefore, using Theorem 1 from this paper and Theorem 2.3.3 from [1], p. 31, we can derive the following estimations of absolute errors $|f(s)-f_\Delta(s)|$ $(s \in [0, 1]$, $n = 51$, $n = 101$, $n = 201)$:

$$|f(s)-f_\Delta(s)| \leqslant Cs^2,$$

where $C = 0.02$, 0.0048, and 0.0012, respectively.

Moreover, for the second example, in Table 2 we give the absolute errors $|f(s_i)-f_\Delta(s_i)|$ for $s = s_i = \pi i/5$ and $i = 1, 2, \ldots, 5$, where each

$f(s_i)$ has been determined by (2.11) and each $f_\Delta(s_i)$ has been calculated from (2.4) and (2.6)-(2.9) under the assumption that

$$t_i = \arccos\left(\frac{n-2i-1}{n-1}\right), \quad i = 1, 2, \ldots, n, \quad m = 3,$$

$$S(0) = 1, \quad S(2) = -1, \quad S'(0) = S'(2) = -5,$$

where $S(x) = (g \circ h^{-1})_{\Delta_x}(x)$, $g(t) = \cos^5(t)$, and $h(t) = 1-\cos(t)$.

TABLE 2

s	Absolute errors		
	$n = 51$	$n = 101$	$n = 201$
0.6283	$8.6_{10}-7$	$6.2_{10}-8$	$7.6_{10}-10$
1.2566	$5.9_{10}-7$	$3.1_{10}-8$	$1.0_{10}-9$
1.8850	$2.8_{10}-7$	$7.0_{10}-8$	$7.1_{10}-9$
2.5133	$1.1_{10}-6$	$1.1_{10}-7$	$9.6_{10}-9$
3.1416	$6.8_{10}-15$	$6.2_{10}-16$	$5.7_{10}-17$

All calculations were performed on the ODRA 1204 computer in single precision (37-bit mantissa).

3. Numerical solution of equation (1.2). Assume that the functions g and h have the same properties as those in the previous section. Moreover, let g_Δ be also defined as in Section 2. Differentiating (1.4), we obtain the solution of equation (1.2) in the form

$$(3.1) \qquad f(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(R)}{[h(R)-h(s)]^{1-a}} - \int_s^R \frac{g'(t)}{[h(t)-h(s)]^{1-a}}dt\right)$$

for all $s \in [0, R)$. Next, let us define the approximate solution f_Δ of equation (1.2) by

$$(3.2) \qquad f_\Delta(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(R)}{[h(R)-h(s)]^{1-a}} - \int_s^R \frac{g'_\Delta(t)}{[h(t)-h(s)]^{1-a}}dt\right).$$

If we substitute $g_\Delta(t)$ in (3.2) by the right-hand side of (2.3), then, as in Section 2, we obtain the following numerical method to determine $f_\Delta(s)$:

$$(3.3) \qquad f_\Delta(s) = \frac{h'(s)\sin a\pi}{\pi}\left(\frac{g(R)}{[h(R)-h(s)]^{1-a}} - \sum_{i=1}^{m} a_i a_i(s) - \sum_{j=1}^{n} \beta_j b_j(s)\right),$$

where

$$a_i(s) = i[h(R)-h(s)]^a c_{i-1}(R, s), \qquad i = 1, 2, \ldots, m,$$

$$b_j(s) = m\sum_{i=0}^{m-1}\binom{m-1}{i}[-h(t_j)]^{m-i-1}C_{ij}(s), \qquad j = 1, 2, \ldots, n,$$

(3.4)

$$C_{ij}(s) = \begin{cases}[h(R)-h(s)]^a c_i(R, s) & \text{if } h(s) \geqslant h(t_j),\\ [h(R)-h(s)]^a c_i(R, s)-[h(t_j)-h(s)]^a c_i(t_j, s) & \text{if } h(s) < h(t_j),\end{cases}$$

and the functions $c_i(u, s)$, $i = 0, 1, \ldots, m-1$, are defined by (2.9). Using similar arguments as in the proof of Theorem 1 we obtain the following

THEOREM 2. *Let g, h, and $e\big((g\circ h^{-1})', \Delta_x\big)$ be such as in Theorem 1 and let $a \in (0, 1)$. Then for every $s \in [0, R]$ we have*

$$|f(s)-f_\Delta(s)| \leqslant \frac{[h(R)-h(s)]^a h'(s)\sin a\pi}{a\pi} e\big((g\circ h^{-1})', \Delta_x\big),$$

where f is the solution of equation (1.2) *and $f_\Delta(s)$ is defined by* (3.3), (3.4), (2.6), *and* (2.9).

Example. Now we solve numerically the following equation:

$$t^6+t^3 = \int_t^{10}\frac{f(s)}{\sqrt{s^2-t^2}}ds, \qquad t \in [0, 10].$$

By (3.1) this equation has a solution equal to

$$(3.5)\qquad f(s) = \frac{2s}{\pi}\left\{\frac{s^6+(15/2)s^2+250}{\sqrt{100-s^2}}-\sqrt{100-s^2}\left[\frac{11}{5}s^4+60s^2+\frac{4015}{2}\right]- -\frac{3}{2}s^2\ln\frac{10+\sqrt{100-s^2}}{s}\right\}.$$

In Table 3 we list the relative errors

$$\frac{|f(s_i)-f_\Delta(s_i)|}{|f(s_i)|}.$$

TABLE 3

s	Relative errors	s	Relative errors
1	$2.7_{10}-9$	6	$2.1_{10}-9$
2	$4.8_{10}-9$	7	$1.5_{10}-8$
3	$1.9_{10}-8$	8	$3.1_{10}-9$
4	$4.2_{10}-9$	9	$2.3_{10}-8$
5	$4.9_{10}-10$		

In this example, $s = s_i = i$ $(i = 1, 2, \ldots, 9)$, each $f(s_i)$ has been determined from (3.5), and each $f_\Delta(s_i)$ has been calculated from (3.3), (3.4), (2.6), and (2.9) under the assumptions

$$t_i = \frac{10(i-1)}{50}, \quad i = 1, 2, \ldots, 51, \quad m = 3,$$

$$S(0) = S'(0) = 0, \quad S(100) = 1{,}001{,}000, \quad S'(100) = 30{,}015,$$

where $S(x) = (g \circ h^{-1})_{\Delta_x}(x)$, $g(t) = t^6 + t^3$, and $h(t) = t^2$.

Using our methods from Sections 2 and 3 to calculate the k values of $f_\Delta(s_i)$, $0 \leqslant s_1 < s_2 < \ldots < s_k \leqslant R$, we propose, in view of [3] and (2.3), to choose the knots t_i so that the quantity

$$\frac{\max\{h(t_i) - h(t_{i-1}):\ i = 2, \ldots, n\}}{\min\{h(t_i) - h(t_{i-1}):\ i = 2, \ldots, n\}}$$

is as small as possible and that the number of knots $h(t_i)$, $i = 1, 2, \ldots, n$, lying in each interval (s_{j-1}, s_j) $(j = 1, 2, \ldots, k+1$, $s_0 = 0$, and $s_{k+1} = R)$ is proportional to the length $s_j - s_{j-1}$ of (s_{j-1}, s_j).

The method proposed in this paper may give worse results than the method described in [9]. For example, using smoothing spline functions in the sense of Reinsch we have obtained, for the experimental data considered in [9], Section 4, the approximate solution f_Δ such that the quantity

$$\max\{|f(i/30) - f(i/30)| : i = 0, 1, \ldots, 30\}$$

was equal to 0.0008 (see [9], Table 4). For the method from this section this quantity is equal to 0.03. Therefore, to solve equations (1.1) and (1.2) in the case $h(t) = t^p$, $p = 1/i$, and $i = 1/2, 1, 2, \ldots$, we prefer to use the numerical methods from [9].

4. Final remarks. The crucial assumption in Sections 2 and 3 lies in the differentiability of g. In this section we assume only that the function g is continuous on $[0, R]$ and that h is such as previously. Then we do not express the solutions of equations (1.1) and (1.2) in the forms (2.1) or (3.1), respectively. Indeed, in this case we do not interchange the order of differentiation and integration in (1.4) and (1.3). Therefore, by (1.3) and (1.4), we define the approximate solution f_Δ of equations (1.1) and (1.2), respectively, in the forms

$$(4.1) \qquad f_\Delta(s) = \frac{\sin a\pi}{\pi} \frac{d}{ds} \int_0^s \frac{h'(t) g_\Delta(t)}{[h(s) - h(t)]^{1-a}}\, dt, \qquad s \in [0, R],$$

and

$$(4.2) \qquad f_\Delta(s) = -\frac{\sin a\pi}{\pi}\frac{d}{ds}\int\limits_s^R \frac{h'(t)g_\Delta(t)}{[h(t)-h(s)]^{1-a}}\,dt, \qquad s \in [0, R],$$

where g_Δ is defined by (2.3).

Here two cases are possible.

First, assume that $m > 1$ in (2.3). Since the function g_Δ is differentiable on $[0, R]$, we can interchange the order of differentiation and integration in (4.1) and (4.2) and we obtain approximate solutions f_Δ of equations (1.1) and (1.2) identical to the approximate solutions obtained in Sections 2 and 3 for the differentiable function g.

Second, assume that $m = 1$ in (2.3). Then the approximate solution $_\Delta$ of equation (1.1) defined by (4.1) can by expressed as follows:

$$f_\Delta(s) = \frac{\sin a\pi}{\pi} h'(s)\Big\{\alpha_0 h^{a-1}(s) + \alpha_1 \frac{h^a(s)}{a} + \\ + \sum_{j=1}^{n} \beta_j \theta[h(s), h(t_j)] \frac{[h(s)-h(t_j)]^a}{a}\Big\}.$$

The approximate solution f_Δ of equation (1.2) defined by (4.2) is equal to

$$f_\Delta(s) = \frac{\sin a\pi}{\pi}\Big[\alpha_0 a_0(s) + \alpha_1 a_1(s) + \sum_{j=1}^{n}\beta_j b_j(s)\Big],$$

where

$$a_0(s) = \frac{h'(s)}{a}[h(R)-h(s)]^{a-1}, \qquad a_1(s) = [h(s)-(1-a)h(R)]a_0(s),$$

$$b_j(s) = \begin{cases} a_1(s) - h(t_j)a_0(s) & \text{if } h(s) \geqslant h(t_j), \\ a_1(s) - h(t_j)a_0(s) - c(s) & \text{if } h(s) < h(t_j), \end{cases}$$

and

$$c(s) = \frac{h'(s)}{a}[h(t_j)-h(s)]^{a-1}[h(s)-(2-a)h(t_j)].$$

References

[1] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The theory of splines and their applications*, Academic Press, New York 1967.

[2] C. J. Cramers and R. C. Birkebak, *Application of the Abel integral equation to spectrographic data*, Appl. Math. Optim. 5 (1966), p. 1057-1064.

[3] C. A. Hall and W. W. Meyer, *Optimal error bounds for cubic spline interpolation*, J. Approximation Theory 16 (1976), p. 105-122.

[4] P. Linz, *A method for computing Bessel function integrals*, Math. Comput. 26 (1972), p. 509-513.
[5] H. Malinowski and R. Smarzewski, *A numerical method for solving the Abel integral equation*, Zastos. Mat. 16 (1978), p. 275-281.
[6] W. Pogorzelski, *Integral equations*, Vol. I, PWN, Warszawa 1953.
[7] C. H. Reinsch, *Smoothing by spline functions*, Numer. Math. 10 (1967), p. 177-183.
[8] R. Smarzewski and H. Malinowski, *On the numerical solution of an Abel integral equation*, Zastos. Mat. 16 (1979), p. 497-503.
[9] — *Numerical solutions of a class of Abel integral equations*, J. Inst. Math. Appl. 22 (1978), p. 159-170.
[10] I. N. Sneddon, *Mixed boundary value problems in potential theory*, North Holland Publ., Amsterdam 1966.

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NUMERYCZNE ROZWIĄZYWANIE UOGÓLNIONYCH RÓWNAŃ CAŁKOWYCH ABELA ZA POMOCĄ FUNKCJI SKLEJANYCH

STRESZCZENIE

W niniejszej pracy przedstawiono pewne metody numerycznego rozwiązywania uogólnionych równań całkowych Abela postaci (1.1) oraz (1.2), gdzie f jest szukanym rozwiązaniem, $0 < a < 1$, h jest ściśle rosnącą i ciągle różniczkowalną funkcją w przedziale $[0, R]$, a g jest daną funkcją ciągłą na przedziale $[0, R]$. Do konstrukcji przybliżonych rozwiązań tych równań wykorzystano funkcje sklejane (interpolujące lub wygładzające). Ponadto podano twierdzenia o zbieżności oraz trzy przykłady numerycznych obliczeń.
