ZASTOSOWANIA MATEMATYKI
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Z. ROMANOWICZ (Wrocław)

GEOMETRIC STRUCTURE OF POSITIVE BASES IN LINEAR SPACES

1. Introduction. A set $A \subset L$ *positively spans* a real linear space L if each element of L can be represented as a linear combination of elements of A with positive coefficients. The minimal set which positively spans a space L is called a *positive basis* of L (cf. [1], [4], [7]–[9]). It is well known ([2], [3], [5], [6], [8], [10]) that if B is a positive basis of a finite-dimensional space L, then

$$1+\dim L \leqslant \operatorname{card} B \leqslant 2\dim L, \tag{1}$$

and for each integer k, $1+\dim L \leqslant k \leqslant 2\dim L$, there exists a positive basis B of L such that $\operatorname{card} B = k$. Moreover, if $\operatorname{card} B = 1+\dim L$, then B is the set of vertices of a simplex containing in its interior the origin. This basis is called a *simplicial basis*. On the other hand, if $\operatorname{card} B = 2\dim L$, then

$$B = B_1 \cup B_2,$$

where

$$B_1 = \{b_1, b_2, \ldots, b_n\}$$

is a linear basis of L, and

$$B_2 = \{\beta_1 b_1, \beta_2 b_2, \ldots, \beta_n b_n\},$$
$$\beta_i < 0,\ i = 1, 2, \ldots, n,\ n = \dim L.$$

In this case, B is the so-called maximal basis of L. In the general case, however, the geometric structure of positive bases is not uniquely determined and its description is a rather complicated problem.

The purpose of this paper is to prove a theorem on decomposition of positive bases into the union of disjoint simplices. Such a representation elucidates entirely the geometric structure of positive bases. Moreover, it gives immediately the evaluation (1). Note that this theorem is essentially stronger than the one formulated in [8]. Additionally, there is given a characterization of the subsets of L, which can be extended to positive bases in L.

Let us first establish the notation and give the indispensable definitions. We denote by L^* the space of linear functionals defined on L (we assume that $\dim L < \infty$). For any subset A of L we denote by $\operatorname{lin} A$ ($\operatorname{pos} A$) the set of all linear combinations of elements of A with real (nonnegative) coefficients. The set $\operatorname{pos} A$ is a convex cone with vertex at the origin. A set A is *linearly* (*positively*) *independent* if

$$\operatorname{lin}(A \setminus \{a\}) \neq \operatorname{lin} A \quad (\operatorname{pos}(A \setminus \{a\}) \neq \operatorname{pos} A) \qquad \text{for every } a \in A.$$

Note that B is a positive basis of L iff B positively spans L, and B is positively independent. By $\operatorname{aff} A$ we denote a carrying flat of the set A:

$$\operatorname{aff} A = \{x\colon\ x = \sum_{i=1}^{k} \alpha_1 a_i,\ a_i \in A,\ \sum_{i=1}^{k} \alpha_i = 1\}$$

or

$$\operatorname{aff} A = \operatorname{lin}(A \setminus \{a\}) + a, \qquad a \in A.$$

The *simplex* is any set of affinely independent elements of L, i.e., the set D such that

$$\operatorname{aff}(D \setminus \{d\}) \neq \operatorname{aff} D \qquad \text{for every } d \in D.$$

By $\operatorname{conv} A$, $A \subset L$, we denote the convex hull of A. We write $f(A) = 0$ ($f(A) > 0$), $f \in L^*$, $A \subset L$, if $f(x) = 0$ ($f(x) > 0$) for every $x \in A$. For $f \in L^*$, $f \neq 0$, we adopt the following notation:

$$H_f = \{x \in L\colon f(x) = 0\},$$

$$H_f^+ = \{x \in L\colon f(x) > 0\} \quad \text{and} \quad H_f^- = \{x \in L\colon f(x) < 0\}.$$

Let S be a convex set in L. We denote by $\operatorname{relint} S$ ($\operatorname{int} S$) the set of points x, $x \in S$, such that

$$\forall y \in S\ \exists \varepsilon > 0\ \big(x + \varepsilon(x - y) \in S\big)$$

$$\big(\forall y \in L\ \exists \varepsilon > 0\ \big(x + \varepsilon(x - y) \in S\big)\big).$$

If $\dim S = \dim L$ ($\dim S \overset{\text{df}}{=} \dim \operatorname{aff} S$), then clearly

$$\operatorname{relint} S = \operatorname{int} S.$$

Note that $\operatorname{relint} S$ (relative interior of S) is an algebraic notion. If L is, however, a linear topological space, then $\operatorname{relint} S$ is the interior of S in the induced topology in the carrying flat of S.

A linear combination of elements of A (A is a finite subset of L) will be denoted by $\mathscr{L}(A)$ and, if all coefficients of $\mathscr{L}(A)$ are positive, by $\mathscr{L}^+(A)$.

In this paper we use the following properties of $\operatorname{pos} A$ and $\operatorname{relint}\operatorname{pos} A$:

LEMMA 1. *An element x belongs to* $\operatorname{relint}\operatorname{pos} A$ *iff for every finite set $M \subset A$ there exists M_1, $M \subset M_1 \subset A$, such that $x = \mathscr{L}^+(M_1)$.*

By this lemma we have

(2) $$x \in \operatorname{relint}\operatorname{pos} A,\ y \in \operatorname{pos} A \Rightarrow x+y \in \operatorname{relint}\operatorname{pos} A,$$

(3) $$x \in \operatorname{relint}\operatorname{pos} A,\ \lambda > 0 \Rightarrow \lambda x \in \operatorname{relint}\operatorname{pos} A.$$

LEMMA 2. *If $A \neq \emptyset$ and $B \neq \emptyset$, then*

(4) $$\operatorname{pos}(A \cup B) = \operatorname{pos} A + \operatorname{pos} B,$$

(5) $$\operatorname{relint}\operatorname{pos}(A \cup B) = \operatorname{relint}\operatorname{pos} A + \operatorname{relint}\operatorname{pos} B.$$

Proof. The property (4) is evident. In order to prove (5) it is sufficient to show that

$$\operatorname{relint}\operatorname{pos}(A \cup B) \subset \operatorname{relint}\operatorname{pos} A + \operatorname{relint}\operatorname{pos} B,$$

because the converse inclusion follows immediately from Lemma 1.

Let $x \in \operatorname{relint}\operatorname{pos}(A \cup B)$. Since $A \neq \emptyset$, there exists $y \in \operatorname{relint}\operatorname{pos} A$. Clearly, $y \in \operatorname{pos}(A \cup B)$. For a certain $\varepsilon > 0$ we have

$$z = x + \varepsilon(x-y) \in \operatorname{pos}(A \cup B),$$

and hence

$$x = y_1 + z_1, \qquad y_1 \in \operatorname{relint}\operatorname{pos} A,\ z_1 \in \operatorname{pos}(A \cup B).$$

Therefore, by (4) and (2),

$$x = y_2 + z_2, \qquad y_2 \in \operatorname{relint}\operatorname{pos} A,\ z_2 \in \operatorname{pos} B.$$

Analogously,

$$x = y_3 + z_3, \qquad y_3 \in \operatorname{pos} A,\ z_3 \in \operatorname{relint}\operatorname{pos} B,$$

and hence

$$x = \tfrac{1}{2}(y_2+z_2) + \tfrac{1}{2}(y_3+z_3) = y_4 + z_4,$$

where, by virtue of (2) and (3),

$$y_4 \in \operatorname{relint}\operatorname{pos} A \quad \text{and} \quad z_4 \in \operatorname{relint}\operatorname{pos} B,$$

which completes the proof.

2. Simplicial decomposition of a positive basis. A set $A \subset L$ is called a *positive basis* of L with respect to a subspace E (w.r.t. E) if

$$\operatorname{pos}(A \cup E) = L$$

and for each $a \in A$

$$\operatorname{pos}((A \setminus \{a\}) \cup E) \neq L.$$

The positive basis of L w.r.t. $E = \{0\}$ is the positive basis of L.

LEMMA 3. *Let E be a proper subspace of L. The following conditions are equivalent*:

(i) $\operatorname{pos}(A \cup E) = L$;

(ii) $0 \in \operatorname{int} \operatorname{conv}(A \cup E)$;

(iii) *for each $f \in L^*$, $f \neq 0$, and $E \subset H_f$, we have*

$$H_f^+ \cap A \neq \emptyset.$$

Proof. (i) $\Rightarrow$ (ii). Let $y \in L$. From (i) it follows that

$$-y = u + \sum_{i=1}^{k} \lambda_i a_i, \quad u \in E,\ \lambda_i > 0,\ a_i \in A.$$

Taking

$$\varepsilon = \frac{1}{1+\lambda}, \quad \lambda = \sum_{i=1}^{k} \lambda_i,$$

we obtain

$$-\varepsilon y = \frac{1}{1+\lambda} u + \sum_{i=1}^{k} \frac{\lambda_i}{1+\lambda} a_i \in \operatorname{conv}(A \cup E),$$

and therefore $0 \in \operatorname{int} \operatorname{conv}(A \cup E)$.

(ii) $\Rightarrow$ (iii). Let us suppose that $H_f^+ \cap A = \emptyset$ for a certain $f \in L^*$, $f \neq 0$, $E \subset H_f$. Then

$$H_f^+ \cap \operatorname{conv}(A \cup E) = \emptyset. \tag{6}$$

On the other hand, if (ii) holds, then for $y \in H_f^-$ there exists $\varepsilon > 0$ such that $-\varepsilon y \in \operatorname{conv}(A \cup E)$ and $-\varepsilon y \in H_f^+$, which contradicts (6).

(iii) $\Rightarrow$ (i). Let us assume that $\operatorname{pos}(A \cup E) \neq L$. Since $\operatorname{pos}(A \cup E)$ is a convex cone with the vertex at the origin, by virtue of the Support Theorem there exists $f \in L^*$, $f \neq 0$, such that

$$\operatorname{pos}(A \cup E) \subset H_f^- \cup H_f.$$

In this case, $E \subset H_f$ and $H_f^+ \cap A = \emptyset$, which contradicts (iii).

Now we notice some evident properties of positive bases and positive bases w.r.t. E. In the sequel, we restrict our considerations to the nontrivial case where $\dim E < \dim L$.

STATEMENT 1. *If B is a positive basis of L w.r.t. E, then the set*

$$B^* = \{b^* \colon b^* = b + u_b,\ b \in B,\ u_b \in E\}$$

is also a positive basis of L w.r.t. E and $\operatorname{card} B^* = \operatorname{card} B$.

STATEMENT 2. *Let* $L = L_1 + L_2$, $L_1 \cap L_2 = \{0\}$, *and let* B_2 *be the orthogonal projection of* B *onto* L_2. *Then the set* B *is a positive basis of* L *w.r.t.* L_1 *iff* B_2 *is a positive basis of* L_2.

STATEMENT 3. *If* B *is a positive basis of* L *w.r.t.* E, *then* $B \cap E = \emptyset$ *but* $E \cap \operatorname{relint} \operatorname{conv} B \neq \emptyset$.

A subset B_1 of a positive basis B is called a *subbasis* of B if it is a positive basis of the subspace $\operatorname{lin} B_1$. If $B_1 \neq B$, then B_1 is called a *proper subbasis* of B.

STATEMENT 4. *If* B *is a positive basis of* L *and a set* $B_1 \subset B$ *positively spans a subspace* L_1, *then* B_1 *is a positive basis of* L_1, $B_1 = L_1 \cap B$, *and* $B \setminus B_1$ *is a positive basis of* L *w.r.t.* L_1.

Let B be a positive basis of L. We say that $C \subset L$ is a *critical set* of B if $\operatorname{pos}((B \setminus \{b\}) \cup C) \neq L$ for each $b \in B$. Elements of C are called *critical vectors* of B.

Note that each subset of a critical set is also a critical one. Moreover, the origin is a critical vector for every positive basis. Let us notice some evident properties of critical sets of positive bases.

STATEMENT 5. *Let* B *be a positive basis of* L. *The following conditions are equivalent*:

(i) C *is a critical set for* B;

(ii) $\forall b \in B \; \exists f \in L^*, f \neq 0 \; \big(f((B \setminus \{b\}) \cup C) \leqslant 0\big)$;

(iii) $\forall b \in B \; \big(-\operatorname{pos} C \cap \operatorname{int} \operatorname{pos}(B \setminus \{b\}) = \emptyset\big)$;

(iv) $\forall b \in B \; \big(-\operatorname{pos} C \cap \operatorname{int} \operatorname{pos}((B \setminus \{b\}) \cup C) = \emptyset\big)$;

(v) $-\operatorname{pos} C \subset L \setminus \bigcup_{b \in B} \operatorname{int} \operatorname{pos}(B \setminus \{b\})$;

(vi) $-\operatorname{pos} C \subset L \setminus \bigcup_{b \in B} \operatorname{int} \operatorname{pos}((B \setminus \{b\}) \cup C)$.

STATEMENT 6. *If* C *is a critical set of a positive basis* B, *then* $\operatorname{pos} C$ *and* $\operatorname{cl} C$ *are also critical sets of* B.

STATEMENT 7. *If each element of a convex set* C *is a critical vector of a positive basis* B, *then* C *is a critical set of this basis.*

Let $C(B)$ denote the set of all critical vectors of a positive basis B of a space L. According to Statement 5 (v) we obtain

$$-C(B) = L \setminus \bigcup_{b \in B} \operatorname{int} \operatorname{pos}(B \setminus \{b\}). \tag{7}$$

In the particular case where $B = \{b_0, b_1, \ldots, b_n\}$ is a simplicial basis of L, $n = \dim L > 2$, we have

$$C(B) = -\bigcup_{i \neq j} \operatorname{pos}(B \setminus \{b_i, b_j\}).$$

For the maximal basis

$$C(B) = \{0\}. \tag{8}$$

Note that if $B = \{b_1, b_2, \dots, b_m\}$ is a positive basis of L, then $B^* = \{\beta_1 b_1, \beta_2 b_2, \dots, \beta_m b_m\}$, $\beta_i > 0$, is also a positive basis of L and $C(B^*) = C(B)$.

STATEMENT 8. *Let B_2 be a positive basis of L w.r.t. a subspace L_1 and let B_1 be a positive basis of L_1. Then the set $B = B_1 \cup B_2$ is a positive basis of L iff $C = \operatorname{pos} B_1 \cap \operatorname{pos} B_2$ is a critical set of B_1.*

Proof. Suppose that $B = B_1 \cup B_2$ is a positive basis of L. If $\operatorname{pos}((B_1 \setminus \{b\}) \cup C) = L_1$ for a certain $b \in B_1$ then, using $B_1 \cap B_2 = \emptyset$, the evident equality $\operatorname{pos}(M \cup C) = \operatorname{pos} M$ for $C \subset \operatorname{pos} M$ and (4), we obtain a contradiction:

$$\operatorname{pos}(B \setminus \{b\}) = \operatorname{pos}((B_1 \setminus \{b\}) \cup C \cup B_2) = L,$$

Now, let C be a critical set of B_1 and let

$$\operatorname{pos}(B \setminus \{b\}) = L$$

for a certain $b \in B$. Note that $b \in B_1$, because

$$\operatorname{pos}(L_1 \cup (B_2 \setminus \{b\})) \neq L \quad \text{for } b \in B_2.$$

Since $L_1 \subset \operatorname{pos}(B \setminus \{b\})$, each $x \in L_1$ can be represented as $\mathscr{L}^+(M_1) + \mathscr{L}^+(M_2)$, where $M_1 \subset B_1 \setminus \{b\}$, $M_2 \subset B_2$, and $\mathscr{L}^+(M_2) \in C$. In this way we get

$$\operatorname{pos}((B \setminus \{b\}) \cup C) = L,$$

which contradicts the definition of C.

STATEMENT 9. *Every positive basis different from a simplicial one contains a proper subbasis. If B_1 is a maximal proper subbasis of B, then the set $B \setminus B_1$ is a simplex and*

$$\operatorname{lin} B_1 \cap \operatorname{relint} \operatorname{conv}(B \setminus B_1) = \{c\},$$

where c is a critical vector of B_1.

Proof. Since $0 \in \operatorname{int} \operatorname{conv} B$ (Lemma 3), there exists a simplex $D \subset B$ such that $0 \in \operatorname{relint} \operatorname{conv} D$. Since B is not a simplicial basis, the set D is a proper subbasis of B.

Let B_1 be a maximal proper subbasis of B and let

$$L_1 = \operatorname{lin} B_1 = \operatorname{pos} B_1.$$

Since $B \setminus B_1$ is a positive basis of the space $\operatorname{lin} B$ w.r.t. L_1, we have

$$L_1 \cap \operatorname{relint} \operatorname{conv}(B \setminus B_1) \neq \emptyset.$$

We will show that this intersection is a one-element set and that the set $B \setminus B_1$ is a simplex. Otherwise, there would exist a simplex $\Delta \subset B \setminus B_1$,

$\Delta \neq B \backslash B_1$, such that

$$L_1 \cap \operatorname{relint} \operatorname{conv} \Delta \neq \emptyset.$$

Let Δ be a minimal simplex with this property. Hence

$$L_1 \cap \operatorname{relint} \operatorname{conv} \Delta = \{c\}$$

and it is evident that Δ is a positive basis of the space $\operatorname{lin}(L_1 \cup \Delta)$ w.r.t. L_1. Therefore, the set $B_1 \cup \Delta$ is a proper subbasis of B which contains B_1 as a proper subset. This fact leads to a contradiction with the definition of B_1.

Thus we have shown that $B \backslash B_1$ is a simplex and the set $\operatorname{relint} \operatorname{conv}(B \backslash B_1)$ cuts L_1 at the unique point c. From Statement 8 it follows directly that c is a critical vector of B_1, and the proof is completed.

Now, let $B = B_1 \cup (B \backslash B_1)$ be a decomposition of the basis B of the space L given by Statement 9. Taking

$$L_1 = \operatorname{pos} B_1, \quad \{c\} = L_1 \cap \operatorname{relint} \operatorname{conv}(B \backslash B_1),$$
$$\Delta = (B \backslash B_1) - c, \quad L_2 = \operatorname{aff} \Delta,$$

we obtain a decomposition of the space L into the direct sum a subspace L_1 and L_2, and a decomposition of the basis B into the union $B_1 \cup (\Delta + c)$, where B_1 is a positive basis of L_1, Δ is a simplicial basis of L_2, and c is a critical vector of B_1.

Note that if $L = L_1 + L_2$, $L_1 \cap L_2 = \{0\}$, B_1 is a positive basis of L_1, Δ is a simplicial basis of L_2 and c is a critical vector of B_1, then $B_1 \cup (\Delta + c)$ is a positive basis of L. Hence, using also Statement 9, we obtain

THEOREM 1. *The set $B \subset L$ is a positive basis of L iff B is a simplicial basis of L or B admits the partition*

$$B = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_r + c_{r-1}),$$

where $\Delta_1, \ldots, \Delta_r$ are simplicial bases of subspaces $L_1, \ldots, L_r$, $L = L_1 + \ldots + L_r$, $L_i \cap L_j = \{0\}$ for $i \neq j$, $\dim L_i \geqslant 1$, and c_j $(j = 1, 2, \ldots, r-1)$ are critical vectors of bases B_j of the spaces $L_1 + \ldots + L_j$, where

$$B_1 = \Delta_1 \quad \textit{and} \quad B_j = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_j + c_{j-1})$$

for $j = 2, \ldots, r$ $(r \geqslant 2)$.

This theorem characterizes completely the geometric structure of positive bases and provides a method for constructing positive bases in a given linear space.

Let us remark that if

$$\dim L = n, \quad \dim L_i = k_i, \ i = 1, 2, \ldots, r$$

(i.e., $\operatorname{card} \Delta_i = k_i+1$), then by virtue of the obvious equality $k_1+\ldots+k_r = n$, we obtain

$$n+1 \leqslant \operatorname{card} B = n+r \leqslant 2n,$$

because $1 \leqslant r \leqslant n$.

The equality $\operatorname{card} B = n+1$ takes place only for $r = 1$, i.e., if B is a simplicial basis. If $\operatorname{card} B = 2n$, then $r = n$ and $k_i = 1$ for $i = 1, 2, \ldots, n$. Moreover, by (8) we obtain $c_j' = 0$ for $j = 1, 2, \ldots, n-1$. Hence in the case $\operatorname{card} B = 2n$ the basis B has to be of the form

$$\{b_1, \ldots, b_n, b_{n+1}, \ldots, b_{2n}\},$$

where $b_{n+i} = \beta_i b_i$, $\beta_i < 0$, $i = 1, 2, \ldots, n$, and $\{b_1, \ldots, b_n\}$ is a linear basis of the space L.

3. Condition for extension of the set to a positive basis. It is well known that every linearly independent subset of the space L can be extended to a basis of L. The analogous property for positively independent sets is not so evident. This is easy to see if one considers in $\boldsymbol{R}^3$ the set of vertices of a regular pentagon whose carrying plane does not contain the origin.

In this section we formulate necessary and sufficient conditions for the extension of a set to a positive basis in L. Let A be an arbitrary subset of L and

$$Q(A) = \operatorname{relint} \operatorname{pos} A \setminus \bigcup_{a\in A} \operatorname{relint} \operatorname{pos}(A\setminus\{a\}).$$

It is easy to see that if A is a positive basis of L, then $-Q(A)$ is the set of critical vectors of A; thus $0 \in Q(A)$. We will show that A admits an extension to a positive basis iff $Q(A) \neq \emptyset$.

First, let us show a few simple statements.

STATEMENT 10. *If $Q(A) \neq \emptyset$, then for an arbitrary nonempty subset B of A we have also $Q(B) \neq \emptyset$.*

Proof. Suppose that for a certain set $B \subset A$, $B \neq A$, $B \neq \emptyset$, we have $Q(B) = \emptyset$. Then

$$\operatorname{relint} \operatorname{pos} B = \bigcup_{b\in B} \operatorname{relint} \operatorname{pos}(B\setminus\{b\}). \tag{9}$$

Let $x \in \operatorname{relint} A$. Then, by Lemma 2,

$$x = y+z, \quad y \in \operatorname{relint} \operatorname{pos} B,\ z \in \operatorname{relint} \operatorname{pos}(A\setminus B),$$

and by virtue of (9) there exists $b \in B$ such that

$$y \in \operatorname{relint} \operatorname{pos}(B\setminus\{b\}).$$

Let M be a finite subset of $A\setminus\{b\}$ and let $M_1 = M \cap B$, $M_2 = M\setminus M_1$. Then, using Lemma 1 we have

$$y = \mathscr{L}^+(M_1 \cup N_1), \quad N_1 \subset B\setminus\{b\},$$

as well as

$$z = \mathscr{L}^+(M_2 \cup N_2), \quad N_2 \subset A \setminus B,$$

and hence

$$x = \mathscr{L}^+(M \cup N), \quad \text{where } N \subset A \setminus \{b\}.$$

Taking into account again Lemma 1 we obtain

$$x \in \operatorname{relint} \operatorname{pos}(A \setminus \{b\}).$$

Due to the arbitrariness of the choice of x we have $Q(A) = \emptyset$, which contradicts the assumption.

STATEMENT 11. *If $Q(A) \neq \emptyset$, then the set A is positively independent.*

In order to show this statement it is sufficient to notice that if $\operatorname{pos} A = \operatorname{pos}(A \setminus \{a\})$ for a certain $a \in A$, then

$$\operatorname{relint} \operatorname{pos} A = \operatorname{relint} \operatorname{pos}(A \setminus \{a\}),$$

which gives $Q(A) = \emptyset$.

STATEMENT 12. $p \in \operatorname{relint} \operatorname{pos} A$ *iff*

$$0 \in \operatorname{relint} \operatorname{pos}(A \cup \{-p\}).$$

STATEMENT 13. $\operatorname{lin} A = \operatorname{pos} A$ *iff* $0 \in \operatorname{relint} \operatorname{pos} A$.

Proof. It is sufficient to notice that if $x \in \operatorname{lin} A$ and $x = \mathscr{L}(P)$, $P \subset A$, then in virtue of $0 \in \operatorname{relint} \operatorname{pos} A$ and Lemma 1 there exists a subset N of A such that

$$0 = \mathscr{L}^+(P \cup M \cup N),$$

where M is an arbitrary subset of A. Thus

$$x = t\mathscr{L}^+(P \cup M \cup N) + \mathscr{L}(P) = \mathscr{L}^+(P \cup M \cup N)$$

provided that t is a sufficiently large positive number.

STATEMENT 14. *If*

$$0 \in \operatorname{relint} \operatorname{pos} A \quad \text{and} \quad 0 \in \operatorname{relint} \operatorname{pos}(A \setminus \{a\})$$

for a certain $a \in A$, then $Q(A) = \emptyset$.

Proof. It follows from the assumption that for an arbitrary set $M \subset A \setminus \{a\}$ we have

$$0 = \mathscr{L}^+(M \cup \{a\} \cup N_1), \quad N_1 \subset A \setminus \{a\},$$

and

$$0 = \mathscr{L}^+(M \cup N_1 \cup N_2), \quad N_2 \subset A \setminus \{a\}.$$

Thus

$$a = t\mathscr{L}^+(M \cup N_1 \cup N_2) - \mathscr{L}^+(M \cup N_1) = \mathscr{L}^+(M \cup N_1 \cup N_2),$$
$$N_1 \cup N_2 \subset A\setminus\{a\},$$

provided that t is a sufficiently large positive number. This means that

$$a \in \operatorname{relint} \operatorname{pos}(A\setminus\{a\}),$$

and hence $a \in \operatorname{pos}(A\setminus\{a\})$. The equality $Q(A) = \emptyset$ follows now immediately from Statement 11.

STATEMENT 15. *A set A is a positive basis of* $\operatorname{lin} A$ iff $0 \in Q(A)$.

Proof. If A is a positive basis, then, according to the previous considerations, $0 \in Q(A)$. If $0 \in Q(A)$, then $0 \in \operatorname{relint} \operatorname{pos} A$ but $0 \notin \operatorname{relint} \operatorname{pos}(A\setminus\{a\})$ for each $a \in A$. By Statement 13 we have $\operatorname{lin} A = \operatorname{pos} A$ as well as

$$\operatorname{lin} A \neq \operatorname{pos}(A\setminus\{a\}) \quad \text{for each } a \in A.$$

Thus A is a positive basis of $\operatorname{lin} A$.

STATEMENT 16. *If $0 \notin Q(A)$, then $p \in Q(A)$ iff*

(10) $$0 \in Q(A \cup \{-p\}).$$

Proof. If $p \in Q(A)$, then clearly

$$p \in \operatorname{relint} \operatorname{pos} A \quad \text{and} \quad p \notin \operatorname{relint} \operatorname{pos}(A\setminus\{a\})$$

for each $a \in A$. Hence, by virtue of Statement 12, we have

$$0 \in \operatorname{relint} \operatorname{pos}(A \cup \{-p\}) \quad \text{and} \quad 0 \notin \operatorname{relint} \operatorname{pos}\big((A\setminus\{a\}) \cup \{-p\}\big)$$

for each $a \in A$. Moreover, since $0 \notin Q(A)$ and $Q(A) \neq \emptyset$, we have also (cf. Statement 13) $0 \notin \operatorname{relint} \operatorname{pos} A$. Hence we obtain (10).

Assume now that (10) holds. Clearly, $-p \notin A$. By virtue of Statement 12 we have

$$p \in \operatorname{relint} \operatorname{pos} A \quad \text{and} \quad p \notin \operatorname{relint} \operatorname{pos}(A\setminus\{a\}) \text{ for each } a \in A,$$

which gives us $p \in Q(A)$.

THEOREM 2. *A nonempty set $A \subset L$ can be extended to a positive basis of a space L iff $Q(A) \neq \emptyset$.*

Proof. If A is a subset of a positive basis B, then $Q(B) \neq \emptyset$ (cf. Statement 15), and hence $Q(A) \neq \emptyset$ (by Statement 10). Let now $Q(A) \neq \emptyset$. It is sufficient to show that A admits an extension to a positive basis of $\operatorname{lin} A$. If $0 \in Q(A)$, then by Statement 15 the set A is a positive basis of $\operatorname{lin} A$. If $0 \notin Q(A)$, then there exists $p \in Q(A)$, $p \neq 0$, such that the set $B_1 = A \cup \{-p\}$ is a positive basis of $\operatorname{lin} A = \operatorname{pos} B_1$ (cf. Statement 16), which completes the proof.

Remark 1. If $Q(A) \neq \emptyset$ and A is a subset of an n-dimensional space, then $\operatorname{card} A \leqslant 2n$.

Remark 2. If $0 \notin Q(A)$, i.e., A is not a positive basis of $\operatorname{lin} A$, then $A \cup \{-p\}$ is a positive basis of $\operatorname{lin} A$ iff $p \in Q(A)$.

References

[1] W. Bonnice and V. L. Klee, *The generation of convex hulls*, Math. Ann. 152 (1963), pp. 1–29.

[2] W. Bonnice and J. R. Reay, *Interior points of convex hulls*, Israel J. Math. 4 (1966), pp. 243–248.

[3] – *Relative interiors of convex hulls*, Proc. Amer. Math. Soc. 20 (1969), pp. 246–250.

[4] Ch. Davis, *Theory of positive linear dependence*, Amer. J. Math. 76 (1954), pp. 733–746.

[5] L. L. Dines and N. H. McCoy, *On linear inequalities*, Trans. Roy. Soc. Canada, Sect. 3, 27 (1933), pp. 37–70.

[6] D. Gale, *Linear combination of vectors with non-negative coefficients*, Amer. Math. Monthly 59 (1952), p. 47.

[7] R. L. McKinney, *Positive bases for linear spaces*, Trans. Amer. Math. Soc. 103 (1962), pp. 131–148.

[8] J. R. Reay, *Generalization of a theorem of Carathéodory*, Mem. Amer. Math. Soc. 54 (1969).

[9] – *Unique minimal representations with positive bases*, Amer. Math. Monthly 73 (1966), pp. 253–261.

[10] F. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, *1*, *2*, *3*, J. Reine Angew. Math. 143 (1913), pp. 128–175; 144 (1914), pp. 1–40; 146 (1916), pp. 1–52.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW