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ON A LINEARIZATION OF AN EQUATION OF AN ELASTIC ROD (II)

1. In a previous paper [1] we discussed a method of approximate integration of the system of equations

$$z''+\cos z = 0, \quad x' = \cos z, \quad y' = \sin z \tag{1.1}$$

for the unknown functions x, y, z defined in the interval $0 \leqslant l \leqslant l_0$ and satisfying the boundary conditions

$$z(0) = z'(l_0) = 0, \quad x(0) = y(0) = 0. \tag{1.2}$$

It is well known how these equations are connected with the problem of large deformations of an elastic rod.

In the paper mentioned above some approximating formulas for the unknown function $z(l)$ as well as the estimates of the error of the approximation were given. It was also indicated how these formulas may be used to get the approximating formulas for the functions $x(l)$ and $y(l)$. In the present paper our attention is concentrated on the function $y(l)$ exclusively and our purpose is to describe a direct way of finding its approximate form without using approximate formulas for $z(l)$. Let us remind that the function $y(l)$ represents, in suitable units, deflection of a rod and that the conditions (1.2) imply

$$0 < z(l_0) < \pi/2 \quad \text{for} \quad l_0 > 0.$$

2. Let us introduce a new unknown function $v(t)$ of an argument t running through the interval $\langle 0, 1\rangle$ by means of the formulas

$$v(t) = \frac{\sin z(l)}{\sin z_0}, \quad t = l/l_0, \tag{2.1}$$

where $z_0 = z(l_0)$. On account of (1.1) we have

$$y(l) = l_0 \sin z_0 \int_0^t v(t)\,dt \tag{2.2}$$

and the function $v(t)$ satisfies the following equation

$$v^{\cdot\cdot}+\frac{l_0^2}{u_0}[1+u_0^2(2v-3v^2)] = 0 \tag{2.3}$$

playing a basic role in this paper. In this equation $u_0 = \sin z_0$ and a dot denotes the differentiation with respect to t. To show this we differentiate twice the equality (2.1) which leads to

$$v^{\cdot\cdot} = \frac{l_0^2}{u_0}(z''\cos z - z'\sin z),$$

then we replace z'' by $-\cos z$, according to (1.1), and eliminate z'^2 by using the first integral of the equation $z''+\cos z = 0$ which for the given boundary conditions imposed on $z(l)$ has the form $\frac{1}{2}z'^2+\sin z = \sin z_0$. Finally, replacing $\sin z$ by $v(t)\sin z_0$, in accordance with the formula (2.1), we get the desired equation (2.3).

The function $v(t)$ satisfies, besides of (2.3), the following boundary conditions

$$v(0) = 0, \quad v(1) = 1, \quad v^{\cdot}(1) = 0. \tag{2.4}$$

If we can find some approximation $v_{\mathrm{ap}}(t)$ for $v(t)$, then the formula

$$y_{\mathrm{ap}}(l) = l_0 u_0 \int_0^t v_{\mathrm{ap}}(t)\,dt \tag{2.5}$$

gives an approximation for the function $y(l)$. Moreover, if the error of the approximation v_{ap}, defined by L^2-norm of $v-v_{\mathrm{ap}}$, i.e. by $\|v-v_{\mathrm{ap}}\|$, is known then the estimate of the error, corresponding to the approximation y_{ap}, may be obtained from the formula

$$|y(l)-y_{\mathrm{ap}}(l)| \leqslant u_0(l_0 l)^{1/2}\|v-v_{\mathrm{ap}}\| \tag{2.6}$$

which results simply from (2.2), (2.5) by applying Schwarz's inequality. The whole problem is thus reduced to finding a suitable function v_{ap} and an estimate of $\|v-v_{\mathrm{ap}}\|$. This will be done in the next sections. The equation (2.3) with the boundary conditions (2.4) will be the starting point of the procedure proposed there. Moreover, the procedure enables us to make the estimate of $\|v-v_{\mathrm{ap}}\|$, in a sense, the best one.

3. The desired function v_{ap} will be the solution, subject to the conditions

$$v_{\mathrm{ap}}(0) = 0, \quad v_{\mathrm{ap}}(1) = 1, \tag{3.1}$$

of a linear equation

$$\ddot{v}_{\mathrm{ap}}+\frac{l_0^2}{u_0}(av_{\mathrm{ap}}+b) = 0 \tag{3.2}$$

with constant coefficients a, b not yet known. If we denote by $\eta(t)$ the difference $v(t)-v_{\rm ap}(t)$ then the subtraction of the equations (3.2) and (2.3) one from the other yields the relation

$$\eta^{\cdot\cdot}+\frac{l_0^2}{u_0}[a\eta-3u_0^2(v^2-\alpha v-\beta)]=0, \tag{3.3}$$

where α and β denote some constants connected with a and b by the formulas

$$a=u_0^2(2-3\alpha),\qquad b=1-3u_0^2\beta. \tag{3.4}$$

Multiplying both sides of (3.3) by η, then integrating the result over $(0,1)$, we get, due to the boundary conditions $\eta(0)=\eta(1)=0$, the equality

$$\|\eta^{\cdot}\|^2+r^2\|\eta\|^2=3l_0^2u_0(\alpha v+\beta-v^2,\eta), \tag{3.5}$$

where (,) has its ordinary meaning of the scalar product in $L^2(0,1)$ and where we put, for the brevity sake, $r^2=-al_0^2/u_0$. It will be shown later that if a "good" choice of the parameter a is wished then the non-positive values of it are only to be considered. Therefore the right-hand side of the last relation defining r^2 is non-negative.

From the formula (3.5) we derive now an estimate of $\|\eta\|=\|v-v_{\rm ap}\|$ as follows: applying Schwarz's inequality to its right-hand side we get

$$\|\eta^{\cdot}\|^2+r^2\|\eta\|^2\leqslant 3l_0^2u_0\|v^2-\alpha v-\beta\|\,\|\eta\|$$

and hence, using Steklov's inequality $\pi\|\eta\|\leqslant\|\eta^{\cdot}\|$, we are led to the relation

$$\|\eta\|\leqslant\frac{3l_0^2u_0}{\pi^2+r^2}\|v^2-\alpha v-\beta\| \tag{3.6}$$

which is true under the assumption $\pi^2+r^2>0$. The only thing to do is to estimate from above the quantity

$$\delta=\|v^2-\alpha v-\beta\|. \tag{3.7}$$

It is possible to get a better estimation than (3.6), following the procedure similar to that one used in [1]. Nevertheless, we shall content ourselves with (3.6).

4. According to the initial conditions $v(1)=1$, $v^{\cdot}(1)=0$, satisfied by the function $v(t)$, we can write the first integral of the equation (2.3) in the form

$$v^{\cdot 2}=\frac{2l_0^2}{u_0}(1-v)(1-u_0^2v^2).$$

This follows simply by multiplying (2.3) by v and integrating over $(t, 1)$ with respect to t. By means of this integral we may rewrite δ^2 in the form

$$\delta^2 = \int_0^1 (v^2 - av - \beta)^2 dt = \frac{1}{l_0}\left(\frac{u_0}{2}\right)^{1/2} \int_0^1 \frac{(v^2 - av - \beta)^2}{\sqrt{(1-v)(1-u_0^2 v^2)}} dv$$

which gives, when we replace $(1-u_0^2 v^2)^{-1/2}$ by $(1-u_0^2)^{-1/2}$, the inequality

$$\delta^2 \leqslant \frac{1}{l_0}\left(\frac{2u_0}{1-u_0^2}\right)^{1/2} I^2(a, \beta), \tag{4.1}$$

where

$$I^2(a, \beta) = \frac{1}{2}\int_0^1 \frac{(v^2 - av - \beta)^2}{\sqrt{1-v}} dv. \tag{4.2}$$

Collecting the formulas (3.6) and (4.1) we get finally the estimate of $\|\eta\|$ in the form

$$\|\eta\| \leqslant \frac{3 l_0^{3/2} u_0}{\pi^2 + r^2}\left(\frac{2u_0}{1-u^2}\right)^{1/4} I(a, \beta). \tag{4.3}$$

Thus the problem of the estimation of $\|\eta\|$ has been reduced to an evaluation of the integral I. If a and β will be chosen now so that I, which depends upon them, attains its minimal value $I_{\min}$, the method used here will give the best estimate of δ. Let us denote

$$C_n = \int_0^1 v^n (1-v)^{-1/2} dv.$$

The equations which result from the condition $I = I_{\min}$ are

$$\frac{\partial}{\partial a} I^2 = C_2 a + C_1 \beta - C_3 = 0,$$

$$\frac{\partial}{\partial \beta} I^2 = C_1 a + C_0 \beta - C_2 = 0,$$

with $C_0 = 2$, $C_1 = 4/3$, $C_2 = 16/15$, $C_3 = 32/35$. Solving them we get for the optimal choice of the parameters a, β the values

$$a = \frac{8}{7}, \quad \beta = -\frac{8}{35}, \tag{4.4}$$

and using then the formula (4.2) we calculate the minimal value of $I(a, \beta)$ which results in

$$I_{\min} = \frac{8}{105}.$$

We are now able to determine the values of the unknown constants a and b which appear in (3.2). We find namely from (3.4) and (4.4)

$$a = -\tfrac{10}{7}u_0^2, \quad b = 1+\tfrac{24}{35}u_0^2, \tag{4.5}$$

and a has in fact a non-positive value as it was mentioned before. Taking into account the values of a and $I_{\min}$ calculated above we get from (4.3) the estimate of $\|\eta\|$ in the form

$$\|\eta\| \leqslant \frac{8}{35}\cdot\frac{l_0^{3/2}u_0}{\pi^2+\frac{10}{7}u_0^2}\left(\frac{2u_0}{1-u_0^2}\right)^{1/4} \tag{4.6}$$

Instead of minimizing I, which is only a part of the expression appearing on the right-hand side of (4.3), this may be done with this whole expression. Nevertheless the result seems to be insufficient to justify the complication. A similar situation was discussed in [2] and [3].

Gathering all the estimates obtained till now we may write the inequality

$$|y(l)-y_{\mathrm{ap}}(l)| \leqslant \frac{8}{35}\cdot\frac{l_0^2u_0^2}{\pi^2+\frac{10}{7}u_0^2}\left(\frac{2u_0}{1-u_0^2}\right)^{1/4} l^{1/2}. \tag{4.7}$$

This formula represents the desired estimate of the error of the proposed approximation $y(l) \approx y_{\mathrm{ap}}(l)$ where the function y_{ap} is defined by (2.5) and v_{ap} appearing there is the solution of the problem (3.1), (3.2) with a and b given by (4.5).

5. A simple calculation gives us

$$v_{\mathrm{ap}}(t) = \operatorname{sh}\varrho t/\operatorname{sh}\varrho+\lambda\big(1-\operatorname{ch}\varrho(t-\tfrac{1}{2})/\operatorname{ch}\tfrac{1}{2}\varrho\big), \tag{5.1}$$

where ϱ^2 and λ are equal to $\frac{10}{7}l_0^2u_0$ and $\frac{7}{10}(u_0^{-2}+\frac{24}{35})$ respectively. Consequently the formula (2.5) yields

$$y_{\mathrm{ap}}(l) = l_0u_0\left[\frac{\operatorname{ch}\varrho t-1}{\varrho\operatorname{sh}\varrho}+\lambda\left(t-\frac{\operatorname{sh}\varrho(t-\frac{1}{2})+\operatorname{sh}\frac{1}{2}\varrho}{\varrho\operatorname{ch}\frac{1}{2}\varrho}\right)\right] \tag{5.2}$$

with $t = l/l_0$. This is the wanted formula for calculating the approximate values of $y(l)$.

In particular, the deflection of the end of rod is approximately given by

$$y_{\mathrm{ap}}(l_0) = \frac{2}{3}l_0u_0\left\{\frac{3}{4}\left(\frac{2}{\varrho}\right)\operatorname{th}\frac{\varrho}{2}+\frac{1}{4}\left(\frac{l_0^2}{2u_0}\right)\left(1+\frac{24}{35}u_0^2\right)\frac{1-(2/\varrho)\operatorname{th}\frac{1}{2}\varrho}{\frac{1}{3}(\varrho/2)^2}\right\}. \tag{5.3}$$

From this formula the asymptotic behaviour of $y_{\mathrm{ap}}(l_0)$ for $z_0 \to 0$ is clearly seen. In this case we have $u_0 \to 0$ and, on account of the inequality

$$\left(\frac{z_0}{\sin z_0}\right)^{1/2} \leqslant \frac{l_0}{\sqrt{2z_0}} \leqslant \left(\frac{1}{\cos z_0}\right)^{1/2} \tag{5.4}$$

given in [1] (formula (4.5))(¹), the expression $l_0^2/2u_0$ tends to 1. Moreover, we have $\varrho \to 0$. All these facts combined together with the formula (5.3) imply

$$y_{\mathrm{ap}}(l_0) \sim \tfrac{2}{3} l_0 u_0 \sim \tfrac{1}{3} l_0^3, \qquad \text{for} \quad z_0 \to 0 .$$

This formula suggests to choose $\frac{1}{3}l_0^3$ as a value which the error of the approximation should be compared with. An especially simple form for the estimation will be obtained by using the inequality

$$\frac{1}{l_0}\left(\frac{2u_0}{1-u_0^2}\right)^{1/2} \leqslant \frac{\operatorname{tg} z_0}{z_0}$$

which results simply from (5.4). Taking into account all the above considerations we obtain from the formula (4.7) the following estimate

$$\frac{|y(l)-y_{\mathrm{ap}}(l)|}{\frac{1}{3}l_0^3} \leqslant \frac{24}{35} \cdot \frac{u_0^2}{\pi^2 + \frac{10}{7}u_0^2} \left(\frac{\operatorname{tg} z_0}{z_0}\right)^{1/2} \left(\frac{l}{l_0}\right)^{1/2} \tag{5.5}$$

It may be of interest to see on few examples the numerical values of the estimates of errors of the proposed approximations. These examples are given in the table.

z_0	0,25	0,50	0,75	1,00
$\dfrac{\lvert y-y_{\mathrm{ap}}\rvert}{\frac{1}{3}l_0^3} \leqslant$	$0{,}43 \cdot 10^{-2}$	1,62	3,4	5,5

Finally, using l_0 evaluated for $z_0 = 1$, we get for $y(l_0)$ the approximate value $y(l_0) = 2{,}0 \pm 0{,}1$, which gives some indication on the accuracy of the approximation.

6. When the deflection of the rod is small, the evaluations outlined in the preceding sections may be significantly simplified. We may in this case replace the equation (3.2) by a more simple equation

$$\ddot{v}_{\mathrm{ap}} + l_0^2 b / u_0 = 0 \tag{6.1}$$

leaving unchanged the boundary conditions (3.1). Now the new function $\eta = v - v_{\mathrm{ap}}$ will satisfy the equation

$$\ddot{\eta} + \frac{l_0^2}{u_0}[1 - b + u_0^2(2v - 3v^2)] = 0,$$

(¹) In (4.5) of [1] the exponent $\frac{1}{2}$ on the left has been accidentally omitted.

and instead of (3.5) we shall have the identity

$$\|\eta'\|^2 = \frac{l_0^2}{u_0}\left(1-b+u_0^2(2v-3v^2),\, \eta\right)$$

which, by using Schwarz's and Steklov's inequalities, yields

$$\|\eta\| \leqslant \frac{l_0^2}{\pi^2 u_0}\,\|1-b+u_0^2(2v-3v^2)\|.$$

The problem of estimating the error $\|v-v_{ap}\|$ is thus reduced to the problem of estimating the right-hand side of the last inequality and the needed procedure may be carried out as it was done in section 3. We obtain in this way

$$b = 1-\tfrac{4}{15}u_0^2,$$

$$\|\eta\| \leqslant 0{,}147\, l_0^{3/2} u_0\left(\frac{2u_0}{1-u_0^2}\right)^{1/4}$$

From the last formula we get the desired estimate of $|y(l)-y_{ap}(l)|$ in the form

$$|y(l)-y_{ap}(l)| \leqslant 0{,}147\, l_0^2 u_0^2\left(\frac{2u_0}{1-u_0^2}\right)^{1/4} l^{1/2}. \tag{6.2}$$

The solution v_{ap} of the equation (6.1), satisfying the boundary conditions (3.1), is now

$$v_{ap}(t) = \frac{l_0^2}{2u_0}\left(1-\frac{4}{15}u_0^2\right)(t-t^2)+t$$

and the function y_{ap} is given by

$$y_{ap}(l) = \frac{l_0^3}{12}\left(1-\frac{4}{15}u_0^2\right)(3t^2-2t^3)+\frac{l_0 u_0}{2}t^2, \qquad t = \frac{l}{l_0}. \tag{6.3}$$

Therefore we have for small values of l_0, as before, $y_{ap}(l_0) \sim \frac{1}{3}l_0^2$, and the same quantity $\frac{1}{3}l_0^2$ may be used for a comparison with the error of the approximation (6.3). We thus get

$$\frac{|y(l)-y_{ap}(l)|}{\frac{1}{3}l_0^3} \leqslant 0{,}147u_0^2\left(\frac{\operatorname{tg} z_0}{z_0}\right)^{1/2}\left(\frac{l}{l_0}\right)^{1/2} \tag{6.4}$$

The approximation proposed here gives the estimate which is only ca twice worse than the more complicated approximation (5.2), as may be seen from the formulas (6.4) and (5.4). Confronting the exactness of the approximations obtained here with that of the approximations of $z(l)$ given in [1] we see that a better approximation for $y(l)$ would be wished. We hope to return later to this question.

References

[1] A. Krzywicki and A. Rybarski, *On a linearization of an equation of an elastic rod*, Zastosow. Mat. 6 (1962), pp. 321-332.

[2] A. Rybarski, *Über eine gewisse Linearisationsmethode der Differentialgleichungen von Pendeltypus*, Bull. Acad. Polon. Sci., Cl. III, 6 (1958), pp. 175-179.

[3] — *Pewna metoda linearyzacji równań różniczkowych typu równania wahadła*, Zastosow. Mat. 5 (1960), pp. 247-259.

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LINEARYZACJA RÓWNAŃ PRĘTA SPRĘŻYSTEGO (II)

STRESZCZENIE

W pracy tej, będącej kontynuacją pracy [1], podano pewne przybliżenia dla funkcji $y(l)$, określonej w przedziale $\langle 0, l_0\rangle$ wzorem

$$y(l) = \int_0^l \sin z(l)\,dl, \quad \text{gdzie} \quad z''+\cos z = 0, \quad z(0) = z'(l_0) = 0.$$

Podano również oszacowania błędów tych przybliżeń. Funkcja $y(l)$ określa odchylenie od położenia równowagi pręta sprężystego, obciążonego na jednym końcu siłą prostopadłą do osi pręta nieodkształconego i zamocowanego sztywno na drugim końcu.

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ЛИНЕАРИЗАЦИЯ УРАВНЕНИЙ УПРУГОГО СТЕРЖНЯ (II)

РЕЗЮМЕ

В этой статье, являющейся продолжением статьи [1] даются некоторые приближения функции $y(l)$, определенной на отрезке $\langle 0, l_0\rangle$ формулой

$$y(l) = \int_0^l \sin z(l)\,dl \quad \text{где} \quad z''+\cos z = 0, \quad z(0) = z'(l_0) = 0.$$

Даны также оценки погрешности этих приближений. Функция $y(l)$ определяет отклонение упругого стержня от положения равновесия если на одном его конце действует сила перпендикулярна к оси ненагруженного стержня, а другой его конец жестко закреплён.
